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ON HARMONIC DIFFERENCE FORMS ON A MANIFOLD

Dedicated to Professor Yûsaku Komatu on his 60th birthday

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Introduction.

In the present note we aim to obtain an orthogonal decomposition theorem of difference forms on a *polyangulation* of a 3-dimensional manifold which is analogous to de Rham-Kodaira's theory on a Riemannian manifold.

In the previous paper [6], we concerned ourselves with the problem of constructing a theory of discrete harmonic and analytic differences on a polyhedron and the problem of approximating harmonic and analytic differentials on a Riemann surface by harmonic and analytic differences respectively, where our definition of a polyhedron differs from the ordinary one based on a triangulation and admits also a polygon and a lune as 2-simplices (cf. § 1. **1** of [6]). In order to set the definitions of a conjugate difference, we introduced concepts of a conjugate polyhedron and a complex polyhedron. In the present note, we shall also introduce similar concepts of a conjugate polyhedron and a complex polyhedron (cf. § 1. **3**) on a 3-dimensional manifold, and we shall show that on such a complex polyhedron a theory of harmonic difference forms analogous to de Rham-Kodaira's theory on Riemannian manifold is obtained.

§ 1. Foundation of topology.

1. Polyangulation. Let E^3 be the 3-dimensional euclidean space. By a *euclidean* 0-*simplex* we mean a point on E^3. By a *euclidean* 1-*simplex* we mean a closed line segment or a closed circular arc. By a *euclidean* 2-*simplex* we mean a closed polygon on a hyperplane or a convex surface, surrounded by a finite number ($\geqq 2$) of segments and circular arcs. A lune (biangle) and a triangle are also admitted as a euclidean 2-simplex. By a *euclidean* 3-*simplex* we mean a closed convex polyhedron surrounded by a finite number ($\geqq 2$) of such polygons (euclidean 2-simplices). A dihedron and a trihedron (closed convex polyhedra surrounded by two polygons and three ones respectively) are also admitted as a euclidean 3-simplex.

Let F be a 3-dimensional orientable manifold. By an *n-simplex* s^n ($n=0, 1, 2, 3$) on F we mean a pair of a euclidean n-simplex e^n and a one-to-one bi-

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continuous mapping ϕ of e^n into F. We shall write $s^n=[e^n, \phi]$ $(n=0, 1, 2, 3)$. The image of e^n under ϕ is called the *carrier* of s^n, and is denoted by $|s^n|$; that is, $\phi(e^n)=|s^n|$. The images of the faces, edges and vertices of a euclidean 3-simplex e^3 by ϕ are called *faces, edges* and *vertices* of $s^3=[e^3, \phi]$. Each face, each edge and each vertex of s^3 is a 2-simplex, a 1-simplex and a 0-simplex respectively. We say that a point p *belongs* to s^n when $p\in|s^n|$ $(n=0, 1, 2, 3)$.

Let us suppose that a collection K of 3-simplices is defined on F in such a way that each point p on F belongs to at least one 3-simplex in K and such that the following conditions (i), (ii), (iii) and (iv) are satisfied:

(i) if p belongs to a 3-simplex s^3 of K but is not on a face of s^3, then s^3 is the only 3-simplex containing p and $|s^3|$ is a neighborhood of p;

(ii) if p belongs to a face s^2 of a 3-simplex s_1^3 in K but does not belong to an edge of s_1^3, there is exactly one other 3-simplex s_2^3 in K such that $|s^2|\subset|s_1^3|\cap|s_2^3|$, s_1^3 and s_2^3 are the only 3-simplices containing p, and $|s_1^3|\cup|s_2^3|$ is a neighborhood of p;

(iii) if p belongs to an edge s^1 of a 3-simplex s_1^3 in K but is not a vertex of s_1^3, there are a finite number of 3-simplices $s_1^3, \cdots, s_\kappa^3$ $(\kappa\geqq2)$ such that each successive pair of 3-simplices s_j^3, s_{j+1}^3 $(j=1, \cdots, \kappa;\ s_{\kappa+1}^3=s_1^3)$ have at least one face in common, $s_1^3, \cdots, s_\kappa^3$ are the only 3-simplices containing p, and $|s_1^3|\cup\cdots\cup|s_\kappa^3|$ forms a neighborhood of p, where it is permitted that some pair of 3-simplices have two or more faces in common;

(iv) if p is a vertex of s_1^3, there are a finite number of 3-simplices $s_1^3, \cdots, s_\nu^3$, $(\nu\geqq2)$, each having p as a vertex, $s_1^3, \cdots, s_\nu^3$ are the only 3-simplices containing p, and $|s_1^3|\cup\cdots\cup|s_\nu^3|$ forms a neighborhood of p.

Then, K is called a *polyangulation* of F or a *polyhedron*[1], and F on which a polyangulation is defined, is called a *polyangulated manifold.*

Let Ω be a compact bordered subregion of F whose boundary consists of faces (2-simplices) of a polyangulation K. Then the collection of 3-simplices of K having their carriers in Ω is called a *compact bordered polyhedron.* If F is closed (open resp.), then K is said to be *closed* (*open* resp.).

Let K and L be two polyhedra. If every 3-simplex of L is a 3-simplex of K, then L is called a *subpolyhedron* of K and K is said to *contain* L.

2. Homology. On a polyhedron we can define a homology in the same manner as the case of a triangulated polyhedron. An *ordered* n-simplex $(n=0, 1, 2, 3)$ is defined in a similar way. An ordered n-simplex $(n=0, 1, 2, 3)$ is denoted by the same notation s^n as an n-simplex. The orientation of simplices induces an orientation of the manifold F.

For a fixed dimension n $(n=0, 1, 2, 3)$ a free Abelian group $C_n(K)$ is defined by the following conditions (i) and (ii):

(i) all ordered n-simplices are generators of $C_n(K)$;

(ii) each element c^n of $C_n(K)$ can be represented in the form of finite sum

1) Throughout the present paper, the terminology "polyhedron" will be taken in this sense.

$$c^n=\sum_j x_j s_j^n,$$

where x_j are integers. Each element of $C_n(K)$ is called an *n-dimensional chain* or an *n-chain*.

The *boundary* ∂ of an n-simplex s^n $(n=1, 2, 3)$ is defined by

$$\partial s^n=s_1^{n-1}+\cdots+s_\kappa^{n-1} \qquad (\kappa=2 \text{ if } n=1;\ \kappa\geqq 2 \text{ if } n=2, 3),$$

where $s_1^{n-1}, \cdots, s_\kappa^{n-1}$ are vertices, edges and faces of s^n in the cases of $n=1, 2, 3$, respectively, with the orientation induced by the orientation of s^n. The boundary ∂s^0 of a 0-simplex s^0 is defined as 0; $\partial s^0=0$. The boundary of an n-chain $c^n=\sum_j x_j s_j^n$ $(n=0, 1, 2, 3)$ is defined by

$$\partial c^n=\sum_j x_j \partial s_j^n.$$

An n-chain whose boundary is zero, is called a *cycle*.

3. Complex polyhedron. If two open or closed polyangulations K and K^* of a common manifold F satisfy the following conditions (i) and (ii), then K^* (K resp.) is called the *conjugate polyhedron* of K (K^* resp.):

(i) To each 0-simplex s^0 of K and K^*, there is exactly one 3-simplex s^3 of K^* and K respectively such that $|s^0|\in|s^3|$. Then, s^3 and s^0 are said to be *conjugate* to s^0 and s^3 respectively, and the conjugate simplices of s^0 and s^3 are denoted by $*s^0$ and $*s^3$ respectively;

(ii) To each 1-simplex s^1 of K and K^*, there is exactly one 2-simplex s^2 of K^* and K respectively such that $|s^1|$ intersects $|s^2|$ at only one point. If the oriented 1-simplex s^1 runs through the oriented 2-simplex s^2 from the reverse side to the front side, then s^2 and s^1 are said to be *conjugate* to s^1 and s^2 respectively, and the conjugate simplices of s^1 and s^2 are denoted by $*s^1$ and $*s^2$ respectively.

By the definition, we have always $**s^n=*(*s^n)=s^n$ for $n=0, 1, 2, 3$.

The pair of K and K^* is called a *complex polyangulation* of F or a *complex polyhedron*, and is denoted by $\boldsymbol{K}=\langle K, K^*\rangle$. A manifold F on which a complex polyangulation is defined, is called a *complex polyangulated manifold.* If F is open or closed, then $\boldsymbol{K}=\langle K, K^*\rangle$ is said to be *open* or *closed* respectively. Let L be a compact bordered subpolyhedron of K and L^* be the sum of 3-simplices of K^* having their carriers in $|L|$. Let us suppose that L^* is not vacuous and is connected. Then L^* is the maximal compact bordered subpolyhedron of K^* under the condition $|L^*|\subset|L|$. The pair $\boldsymbol{L}=\langle L, L^*\rangle$ is called a *compact bordered complex polyhedron.*

Let $\boldsymbol{K}=\langle K, K^*\rangle$ and $\boldsymbol{L}=\langle L, L^*\rangle$ be two complex polyhedra. If L and L^* are subpolyhedra of K and K^* respectively, then $\boldsymbol{L}$ is called a *complex subpolyhedron* of $\boldsymbol{K}$.

By an *n-chain* X $(n=0, 1, 2, 3)$ of a complex polyhedron $\boldsymbol{K}$, we mean a formal sum $X=X_1+X_2$ of an n-chain X_1 of K and an n-chain X_2 of K^*. Here we agree that if $\boldsymbol{K}$ is compact bordered then the conjugate 2-simplex $*s^1$ of each 1-simplex $s^1 \in \partial K$ and the conjugate 1-simplex $*s^2$ of each 2-simplex $s^2 \in \partial K$ is admitted as a generator of $C_2(K^*)$ and that of $C_1(K^*)$ respectively, and thus X_2 is precisely an n-chain of $K^* + \{*s^1, *s^2 | s^1, s^2 \in \partial K\}$. The *boundary* ∂X is defined by $\partial X = \partial X_1 + \partial X_2$. X is said to be *homologous to zero*, denoted by $X \sim 0$, if and only if $X_1 \sim 0$ and $X_2 \sim 0$.

4. Complex boundary. Let $\boldsymbol{K} = \langle K, K^* \rangle$ be a compact bordered complex polyhedron. Now we shall try to define a new polyhedron K^{*+} such that $K^* \subset K^{*+}$ and $|K^{*+}| = |K|$. Let s^2 be an arbitrary 2-simplex of ∂K. Then the carrier $|*s^2|$ of the conjugate 1-simplex $*s^2$ is divided into two portions by the point $p = |s^2| \cap |*s^2|$. We divide $*s^2$ into two 1-simplices s_1^1 and s_2^1 whose carriers are the portions of $|*s^2|$ lying on the reverse side and the front side of s^2 respectively. Then s_1^1 is called the *conjugate half* 1-*simplex of* s^2 *with respect to* ∂K and is denoted by $※s^2$. The terminal vertex of s_1^1, whose carrier lies on $|s^2|$, is called the *conjugate* 0-*simplex of* s^2 *on* ∂K and is denoted by $*s^2(\partial K)$.

Let s^1 be an arbitrary oriented 1-simplex of ∂K. Then there exist exactly two oriented 2-simplices σ_1^2 and σ_2^2 of ∂K such that s^1 is a common edge of σ_1^2 and σ_2^2, where s^1 is assumed to have the orientation induced by the orientation of σ_2^2 and thus of $-\sigma_1^2$. Let $s_1^3, \cdots, s_\kappa^3$ ($\kappa \geqq 1$) be the collection of 3-simplices of K having s^1 as their common edge such that σ_1^2 and σ_2^2 are the faces of s_1^3 and s_κ^3 respectively, and such that each successive pair s_j^3, s_{j+1}^3 of 3-simplices has a common face s_j^2 with the edge s^1, where s_j^2 is assumed to be oriented so that the orientation of s_j^2 induces that of s^1. Here if $\kappa = 1$, then σ_1^2 and σ_2^2 are the faces of the common 3-simplex $s_1^3 = s_\kappa^3$, and $\{s_j^2\}_{j=1}^{\kappa-1} = \emptyset$. Let σ_1^0 and σ_2^0 be the terminal vertices of $※\sigma_1^2$ and $※\sigma_2^2$ lying on σ_1^2 and σ_2^2 respectively. We define a new 1-simplex σ^1 with $\partial\sigma^1 = \sigma_2^0 - \sigma_1^0$ whose carrier $|\sigma^1|$ is a line segment lying on $|\sigma_2^2| \cup |\sigma_1^2|$ and intersects $|s^1|$ at only one point. The 1-simplex σ^1 is said to be *conjugate to* s^1 *on* ∂K and is denoted by $*s^1(\partial K)$. Furthermore we define a new 2-simplex σ^2 such that

$$\partial\sigma^2 = -*s^1(\partial K) - ※\sigma_1^2 + \sum_{j=1}^{\kappa-1} *s_j^2 + ※\sigma_2^2 . \tag{1.1}$$

The 2-simplex σ^2 is called the *conjugate half* 2-*simplex of* s^1 *with respect to* ∂K and is denoted by $※s^1$.

Let s^0 be an arbitrary 0-simplex of ∂K. Let $s_1^1, \cdots, s_\nu^1$ ($\nu \geqq 2$) be the collection of 1-simplices of K whose common initial vertex is s^0, and let $s_1^1, \cdots, s_\mu^1$ ($\mu \leqq \nu$) be the collection of those lying on ∂K. Then we define a new 2-simplex σ^2 with $|\sigma^2| \subset |\partial K|$ such that

$$\partial\sigma^2 = \sum_{j=1}^{\mu} *s_j^1(\partial K) .$$

The 2-simplex σ^2 is said to be *conjugate to* s^0 *on* ∂K and is denoted by $*s^0(\partial K)$. Furthermore we define a new 3-simplex σ^3 such that

$$\partial\sigma^3 = *s^0(\partial K) + \sum_{j=1}^{\mu} ※s_j^1 + \sum_{j=\mu+1}^{\nu} *s_j^1 , \tag{1.2}$$

where if $\mu=\nu$, then the last term of (1.2) is vacuous. The 3-simplex σ^3 is called the *conjugate half* 3*-simplex of* s^0 *with respect to* ∂K and is denoted by $※s^0$.

The (*simple*) *boundary* $\partial\boldsymbol{K}=\langle\partial K, \partial K^*\rangle$ of $\boldsymbol{K}$ is defined by the sum of the 1-chains ∂K and ∂K^*. Next, by K^{*+} we denote the new polyhedron defined as the sum of all 3-simplices of K^* and the conjugate half 3-simplices of all 0-simplices $s^0\in\partial K$ with respect to ∂K. Then $|K^{*+}|=|K|$. The sum of ∂K and ∂K^{*+} is called the *complex boundary* of $\boldsymbol{K}$ and denoted by $\boldsymbol{\partial K}=\langle\partial K, \partial K^{*+}\rangle$, where ∂K^{*+} is the 2-chain defined as the sum of $*s^0(\partial K)$ for all $s^0\in\partial K$. Throughout the present paper we shall preserve these notations.

§ 2. Differences on a polyhedron.

1. Difference calculus. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be an arbitrary complex polyhedron.

By an *n-th order difference* or *n-difference* φ^n on $\boldsymbol{K}$ ($n=0, 1, 2, 3$), we mean the complex valued function φ^n on the set of oriented n-simplices of $\boldsymbol{K}$ such that φ^n has a value $\varphi^n(s^n)$ for each oriented n-simplex s^n and $\varphi^n(-s^n)=-\varphi^n(s^n)$. A zero order difference φ^0 on $\boldsymbol{K}$ is also called a *function* on $\boldsymbol{K}$.

We assume that differences of arbitrary order satisfy the linearity property:

$$(c_1\varphi^n+c_2\psi^n)(s^n)=c_1\cdot\varphi^n(s^n)+c_2\cdot\psi^n(s^n) \qquad (n=0, 1, 2, 3),$$

where φ^n and ψ^n are n-differences on $\boldsymbol{K}$, and c_1 and c_2 are complex constants.

The multiplication of a 2-difference ψ^2 with a 0-difference φ^0 is defined as a 2-difference satisfying the condition

$$\varphi^0\psi^2(s^2)=\psi^2\varphi^0(s^2)=\frac{1}{2}\{\varphi^0(s_1^0)+\varphi^0(s_2^0)\}\psi^2(s^2)$$

for each 2-simplex $s^2\in\boldsymbol{K}$, where s_1^0 and s_2^0 are the 0-simplices such that $\partial*s^2=s_2^0-s_1^0$. The multiplication of a 3-difference ψ^3 with a 0-difference φ^0 is defined as a 3-difference satisfying the condition

$$\varphi^0\psi^3(s^3)=\varphi^0(*s^3)\psi^3(s^3) \qquad \text{for each 3-simplex } s^3\in\boldsymbol{K}.$$

The *exterior product* of a 1-difference φ^1 and a 2-difference ψ^2 is defined as a 3-difference satisfying the condition

$$\varphi^1\psi^2(s^3)=\psi^2\varphi^1(s^3)=\frac{1}{2}\sum_{j=1}^{\kappa}\varphi^1(*s_j^2)\psi^2(s_j^2)$$

for each 3-simplex $s^3\in\boldsymbol{K}$, where $s_1^2, \cdots, s_\kappa^2$ are the 2-simplices such that $\partial s^3=s_1^2+\cdots+s_\kappa^2$.

The *complex conjugate* $\bar{\varphi}^n$ of an n-difference φ^n ($n=0, 1, 2, 3$) is defined by $\bar{\varphi}^n(s^n)=\overline{\varphi^n(s^n)}$.

The *difference* of an n-difference φ^n ($n=0, 1, 2$) is defined as an $(n+1)$-difference $\Delta\varphi^n$ satisfying the condition

$$\Delta\varphi^n(s^{n+1})=\sum_{j=1}^{\kappa}\varphi^n(s_j^n) \qquad \text{for each } (n+1)\text{-simplex } s^{n+1}\in\boldsymbol{K},$$

where $s_1^n, \cdots, s_\kappa^n$ are the n-simplices such that $\partial s^{n+1}=s_1^n+\cdots+s_\kappa^n$. The difference of a 3-difference φ^3 is defined as 0; $\varDelta\varphi^3=0$. If $\varDelta\varphi^n=0$ $(n=0,1,2,3)$, then φ^n is said to be *closed*. If for an n-difference φ^n $(n=1,2,3)$ there exists an $(n-1)$-difference ϕ^{n-1} such that $\varphi^n=\varDelta\phi^{n-1}$, then φ^n is said to be *exact*. Obviously, if φ^n is exact, then φ^n is closed. We can easily verify that the partial difference formula

$$\varDelta(\varphi^0\phi^2)=(\varDelta\varphi^0)\phi^2+\varphi^0\varDelta\phi^2 \tag{2.1}$$

holds for a 0-difference φ^0 and a 2-difference ϕ^2.

2. Summation of differences. We can define the *sum* of an n-difference $(n=0,1,2,3)$ over an n-chain. Let $c^n=\sum_j x_j s_j^n$ be an n-chain $(n=0,1,2,3)$ of a complex polyhedron $\boldsymbol{K}$. The *sum of an n-difference* φ^n *over* c^n is defined by

$$\mathop{S}_{c^n}\varphi^n=\sum_j x_j\varphi^n(s_j^n) \qquad (n=0,\,1,\,2,\,3)\,.$$

The basic duality between a chain and a difference

$$\mathop{S}_{c^n}\varDelta\varphi^{n-1}=\mathop{S}_{\partial c^n}\varphi^{n-1} \qquad (n=1,\,2,\,3) \tag{2.2}$$

is obvious, where c^n is an n-chain and φ^n is an n-difference. The formula for partial summation

$$\mathop{S}_{c^3}(\varDelta\varphi^0)\phi^2=\mathop{S}_{\partial c^3}\varphi^0\phi^2-\mathop{S}_{c^3}\varphi^0\varDelta\phi^2 \tag{2.3}$$

follows from (2.1) and (2.2).

The following two criteria are also obvious:

An n-difference φ^n $(n=0,1,2)$ is closed if and only if $\mathop{S}_{c^n}\varphi^n=0$ for every cycle c^n that is homologous to 0;

An n-difference φ^n $(n=1,2,3)$ is exact if and only if $\mathop{S}_{c^n}\varphi^n=0$ for every cycle c^n.

If an n-difference φ^n $(n=0,1,2)$ is closed, then the *period of* φ^n *along an n-cycle* c^n is defined by $\mathop{S}_{c^n}\varphi^n$, which depends only on the homology class of c^n.

Now we shall define the *sum* of 3-difference over a complex polyhedron $\boldsymbol{K}=\langle K, K^*\rangle$. If $\boldsymbol{K}$ is compact bordered or closed, then the sum of a 3-difference φ^3 over $\boldsymbol{K}$

$$\mathop{S}_{\boldsymbol{K}}\varphi^3$$

is defined as the sum of φ^3 over the 3-chain $\boldsymbol{K}$ because $\boldsymbol{K}$ is itself a 3-chain. If $\boldsymbol{K}$ is open, then we can set

$$\mathop{S}_{\boldsymbol{K}}\varphi^3=\lim_{c^3\to\boldsymbol{K}}\mathop{S}_{c^3}\varphi^3 \tag{2.4}$$

provided that the limit exists, where c^3 is a 3-chain of $\boldsymbol{K}$ such that $c^3\subset\boldsymbol{K}$.

3. Conjugate differences. Let φ^n $(n=0,1,2,3)$ be an n-difference on a complex polyhedron $\boldsymbol{K}$. Then the *conjugate difference* $*\varphi^n$ of φ^n is defined as a $(3-n)$-difference satisfying the condition

$$*\varphi^n(*s^n)=\varphi^n(s^n) \qquad (n=0, 1, 2, 3)$$

for each n-simplex $s^n \in \boldsymbol{K}$. Then we can easily see that

$$**\varphi^n=\varphi^n \qquad (n=0, 1, 2, 3), \tag{2.5}$$

$$*\varphi^n * \psi^{3-n}=\varphi^n \psi^{3-n} \qquad (n=0, 1, 2, 3). \tag{2.6}$$

An n-difference φ^n $(n=1, 2)$ is said to be *harmonic* if φ^n and $*\varphi^n$ are both closed. By (2.5) and the definition, φ^n and $*\varphi^n$ are simultaneously harmonic. Let u be a function (0-difference) on $\boldsymbol{K}$. u is called a *harmonic function on* $\boldsymbol{K}$ if the difference Δu is harmonic. A function u is harmonic on $\boldsymbol{K}$ if and only if

$$u(s^0)=\frac{1}{\kappa}\sum_{j=1}^{\kappa} u(s_j^0)$$

for every 0-simplex s^0 of $\boldsymbol{K}$ whose carrier $|s^0|$ is in the interior of $|K|$, where $\partial s_j^1 = s_j^0 - s^0$ $(j=1, \cdots, \kappa)$ and $s_1^1, \cdots, s_\kappa^1$ are all 1-simplices having s^0 as a vertex.

§ 3. The Hilbert space of differences.

1. The inner product. Let φ^n and ψ^n $(n=0,1,2,3)$ be two n-differences on a complex polyhedron $\boldsymbol{K}=\langle K, K^*\rangle$. We shall define the *inner product* $(\varphi^n, \psi^n)=(\varphi^n, \psi^n)_{\boldsymbol{K}}$ of φ^n and ψ^n. If $\boldsymbol{K}$ is closed, then it is defined by

$$(\varphi^n, \psi^n)_{\boldsymbol{K}}=\sum_{s^n \in \boldsymbol{K}} \varphi^n(s^n)\overline{\psi^n(s^n)} \qquad (n=0, 1, 2, 3).$$

If $\boldsymbol{K}$ is compact bordered, then it is defined by

$$(\varphi^0, \psi^0)_{\boldsymbol{K}}=\sum_{s^3 \in \boldsymbol{K}} \varphi^0(*s^3)\overline{\psi^0(*s^3)},$$

$$\begin{aligned}(\varphi^n, \psi^n)_{\boldsymbol{K}}=&\sum_{s^n \in K-\partial K} \varphi^n(s^n)\overline{\psi^n(s^n)}+\frac{1}{2}\sum_{s^n \in \partial K} \varphi^n(s^n)\overline{\psi^n(s^n)} \\ &+\sum_{s^{3-n} \in K-\partial K} \varphi^n(*s^{3-n})\overline{\psi^n(*s^{3-n})} \\ &+\frac{1}{2}\sum_{s^{3-n} \in \partial K} \varphi^n(*s^{3-n})\overline{\psi^n(*s^{3-n})} \qquad (n=1, 2),\end{aligned}$$

$$(\varphi^3, \psi^3)_{\boldsymbol{K}}=\sum_{s^3 \in \boldsymbol{K}} \varphi^3(s^3)\overline{\psi^3(s^3)}.$$

If $\boldsymbol{K}$ is open, then it is defined by the limit process

$$(\varphi^n, \psi^n)_{\boldsymbol{K}}=\lim_{L \to \boldsymbol{K}} (\varphi^n, \psi^n)_L \qquad (n=0, 1, 2, 3),$$

provided that the limit exists, where $\boldsymbol{L}=\langle L, L^*\rangle$ is a compact bordered complex polyhedron such that $\boldsymbol{L}\subset\boldsymbol{K}$.

If $\boldsymbol{K}$ is closed or open, then we can easily see that

$$(\varphi^n, \phi^n)_{\boldsymbol{K}}=\int_{\boldsymbol{K}}\varphi^n*\bar{\phi}^n \qquad (n=0, 1, 2, 3).$$

If $\boldsymbol{K}$ is compact bordered, then we can easily verify that

$$(\varphi^n, \phi^n)_{\boldsymbol{K}}=\int_{\boldsymbol{K}}\varphi^n*\bar{\phi}^n \qquad (n=0, 3),$$

$$(\varphi^1, \phi^1)_{\boldsymbol{K}}=\int_{\boldsymbol{K}}\varphi^1*\bar{\phi}^1+\frac{1}{2}\sum_{s^1\in\partial K}\varphi^1(s^1)\overline{\phi^1(s^1)}$$
$$+\frac{1}{2}\sum_{s^2\in\partial K^*}\varphi^1(*s^2)\overline{\phi^1(*s^2)},$$

$$(\varphi^2, \phi^2)_{\boldsymbol{K}}=\int_{\boldsymbol{K}}\varphi^2*\bar{\phi}^2+\frac{1}{2}\sum_{s^1\in\partial K}\varphi^2(*s^1)\overline{\phi^2(*s^1)}$$
$$+\frac{1}{2}\sum_{s^2\in\partial K^*}\varphi^2(s^2)\overline{\phi^2(s^2)}.$$

By the definition of the inner product, for every case of $\boldsymbol{K}$ and for $n=0, 1, 2, 3$, we have

$$(*\varphi^n, *\phi^n)=(\varphi^n, \phi^n), \tag{3.1}$$

$$(\varphi^n, \phi^n)=(\bar{\phi}^n, \bar{\varphi}^n). \tag{3.2}$$

Let φ^n be an n-difference $(n=0, 1, 2, 3)$ on a complex polyhedron $\boldsymbol{K}$. Then the norm $\|\varphi^n\|=\|\varphi^n\|_{\boldsymbol{K}}$ of φ^n is defined by

$$\|\varphi^n\|_{\boldsymbol{K}}=(\varphi^n, \varphi^n)_{\boldsymbol{K}}^{1/2} \qquad (n=0, 1, 2, 3). \tag{3.3}$$

Let us denote the Hilbert space of all n-differences φ^n on $\boldsymbol{K}$ with $\|\varphi^n\|<\infty$ by Γ, for a fixed $n=1$ or $n=2$. Furthermore, we define the closed subspaces of Γ as follows:

$$\Gamma_c=\{\varphi^n|\varphi^n \text{ is closed, } \varphi^n\in\Gamma\},$$

$$\Gamma_e=\{\varphi^n|\varphi^n \text{ is exact, } \varphi^n\in\Gamma\},$$

$$\Gamma_h=\{\varphi^n|\varphi^n \text{ is harmonic, } \varphi^n\in\Gamma\},$$

$$\Gamma_c^*=\{\varphi^n|*\varphi^n \text{ is closed, } \varphi^n\in\Gamma\},$$

$$\Gamma_e^*=\{\varphi^n|*\varphi^n \text{ is exact, } \varphi^n\in\Gamma\},$$

$$\Gamma_h^*=\{\varphi^n|*\varphi^n \text{ is harmonic, } \varphi^n\in\Gamma\}.$$

Then it is obvious that $\Gamma_h^*=\Gamma_h$, $\Gamma_e\subset\Gamma_c$, $\Gamma_h=\Gamma_c\cap\Gamma_c^*$.

2. The definition of φ^n on ∂K^{*+}. Let $\partial\boldsymbol{K}=\langle\partial K, \partial K^{*+}\rangle$ be a complex boundary of a compact bordered complex polyhedron $\boldsymbol{K}=\langle K, K^*\rangle$. We shall define an n-difference ($n=0,1,2$) on ∂K^{*+}.

Let φ^0 be a 0-difference on $\boldsymbol{K}$. Then φ^0 is defined on ∂K^{*+} by

$$\varphi^0(\sigma^0)=\frac{1}{2}\{\varphi^0(s_1^0)+\varphi^0(s_2^0)\} \qquad \text{for each 0-simplex } \sigma^0 \text{ of } \partial K^{*+}, \tag{3.4}$$

where $\partial * s^2=s_2^0-s_1^0$, s^2 is the 2-simplex of ∂K with $\sigma^0=*s^2(\partial K)$ and φ^0 is assumed to be defined at s_2^0.

Let φ^1 be a 1-difference on $\boldsymbol{K}$. Then φ^1 is defined on ∂K^{*+} by

$$\varphi^1(\sigma^1)=-\frac{1}{2}\varphi^1(*\sigma_1^2)+\sum_{j=1}^{\kappa-1}\varphi^1(*s_j^2)+\frac{1}{2}\varphi^1(*\sigma_2^2)$$

for each 1-simplex σ^1 of ∂K^{*+}, where

$$\partial\sigma^2=-\sigma^1-※\sigma_1^2+\sum_{j=1}^{\kappa-1}*s_j^2+※\sigma_2^2,$$

σ^2 is the conjugate half 2-simplex of s^1 with respect to ∂K, s^1 is the 1-simplex of ∂K with $\sigma^1=*s^1(\partial K)$, and σ_1^2, σ_2^2 and s_j^2 ($j=1,\cdots,\kappa-1$) is the notations defined in (1.1).

Let φ^2 be a 2-difference on $\boldsymbol{K}$. Then φ^2 is defined on ∂K^{*+} by

$$\varphi^2(\sigma^2)=-\frac{1}{2}\sum_{j=1}^{\mu}\varphi^2(*s_j^1)-\sum_{j=\mu+1}^{\nu}\varphi^2(*s_j^1)$$

for each 2-simplex σ^2 of ∂K^{*+}, where

$$\partial\sigma^3=\sigma^2+\sum_{j=1}^{\mu}※s_j^1+\sum_{j=\mu+1}^{\nu}*s_j^1,$$

σ^3 is the conjugate half 3-simplex of s^0 with respect to ∂K, s^0 is the 0-simplex of ∂K with $\sigma^2=*s^0(\partial K)$ and s_j^1 ($j=1,\cdots,\nu$) is the notations defined in (1.2).

The multiplication of a 2-difference ψ^2 with a 0-difference φ^0 on $\partial\boldsymbol{K}=\langle\partial K, \partial K^{*+}\rangle$ is defined as a 2-difference on $\partial\boldsymbol{K}$ satisfying the condition

$$\varphi^0\psi^2(s^2)=\psi^2\varphi^0(s^2)=\varphi^0(s^0)\psi^2(s^2) \qquad \text{for each 2-simplex } s^2\in\partial\boldsymbol{K},$$

where if $s^2\in\partial K$ then $s^0=*s^2(\partial K)$ and if $s^2\in\partial K^{*+}$ then $s^2=*s^0(\partial K)$.

The *exterior product* of two 1-differences φ^1 and ψ^1 on $\partial\boldsymbol{K}=\langle\partial K, \partial K^*\rangle$ is defined as a 2-difference $\varphi^1\psi^1$ satisfying the condition

$$\varphi^1\psi^1(s^2)=-\frac{1}{2}\sum_{j=1}^{\kappa}\varphi^1(\sigma_j^1)\psi^1(s_j^1) \qquad \text{for each 2-simplex } s^2\in\partial\boldsymbol{K},$$

where $\partial s^2=s_1^1+\cdots+s_\kappa^1$, and if $s^2\in\partial K$ then $\sigma_j^1=*s_j^1(\partial K)$ and if $s^2\in\partial K^{*+}$ then $s_j^1=-*\sigma_j^1(\partial K)$.

For an arbitrary 1-difference φ^1, we shall agree to define

$$\varDelta\varphi^1(*s^1)=0 \qquad \text{for each 1-simplex } s^1\in\partial K. \tag{3.5}$$

3. Fundamental theorem.

THEOREM 3.1. *If a complex polyhedron* $\boldsymbol{K}$ *is compact bordered or closed, then we have*

$$(3.6)\qquad (\Delta\varphi^{n-1}, \psi^n)_K = \int_{\partial K} \varphi^{n-1} * \bar{\psi}^n + (\varphi^{n-1}, \delta\psi^n)_K \qquad (n=1, 2, 3),$$

where δ *is the operator* $(-1)^n * \Delta *$ *for an n-difference, and if* $\boldsymbol{K}$ *is closed then the first term of the right-hand side vanishes.*

Proof. The case of $n=1$: By the definition of the inner product and (2.3), we see that

$$\begin{aligned}(\Delta\varphi^0, \psi^1)_K &= \int_K \Delta\varphi * \bar{\psi} + \frac{1}{2}\sum_{s^1\in\partial K} \Delta\varphi(s^1)\overline{\psi(s^1)} + \frac{1}{2}\sum_{s^2\in\partial K^*} \Delta\varphi(*s^2)\overline{\psi(*s^2)} \\ &= \Big(\int_{\partial K} \varphi * \bar{\psi} + \frac{1}{2}\sum_{s^1\in\partial K} \Delta\varphi(s^1)\overline{\psi(s^1)} + \frac{1}{2}\sum_{s^2\in\partial K^*} \Delta\varphi(*s^2)\overline{\psi(*s^2)}\Big) \\ &\quad -\int_K \varphi\Delta*\bar{\psi} \\ &= \Big(\int_{\partial K} \varphi * \bar{\psi} + \frac{1}{2}\sum_{s^2\in\partial K^*} \{\varphi(s_1^0)+\varphi(s_2^0)\} * \overline{\psi(s^2)} \\ &\qquad + \frac{1}{2}\sum_{s^1\in\partial K} \Delta\varphi(s^1)\overline{*\psi(*s^1)} + \frac{1}{2}\sum_{s^2\in\partial K^*} \Delta\varphi(*s^2)\overline{*\psi(s^2)}\Big) \\ &\quad + (\varphi, \delta\psi)_K,\end{aligned}$$

where $\varphi=\varphi^0$ and $\psi=\psi^1$, and $\partial * s^2 = s_2^0 - s_1^0$. Here if we note that

$$\begin{aligned}\int_{\partial K^{*+}} \varphi * \bar{\psi} &= \sum_{s^0\in\partial K} \varphi(s^0)\overline{*\psi(*s^0(\partial K))} \\ &= \sum_{s^2\in\partial K^*} \varphi(s_2^0)\overline{*\psi(s^2)} + \sum_{s^1\in\partial K} \Delta\varphi(s^1)\cdot\frac{1}{2}\overline{*\psi(*s^1)},\end{aligned}$$

then we obtain (3.6).

The case of $n=3$ can be easily reduced to the case of $n=1$.

The case of $n=2$: By the definition of the inner product, we see that

$$(3.7)\qquad \begin{aligned}(\Delta\varphi^1, \psi^2)_K &= \sum_{s^2\in K-\partial K} \Delta\varphi(s^2)\overline{*\psi(*s^2)} + \frac{1}{2}\sum_{s^2\in\partial K} \Delta\varphi(s^2)\overline{*\psi(*s^2)} \\ &\quad + \sum_{s^1\in K-\partial K} \Delta\varphi(*s^1)\overline{*\psi(s^1)} + \frac{1}{2}\sum_{s^1\in\partial K} \Delta\varphi(*s^1)\overline{*\psi(s^1)},\end{aligned}$$

where $\varphi=\varphi^1$ and $\psi=\psi^2$. By the definition (3.5) the last term of the right-hand side of (3.7) is equal to zero, and further we have

$$\begin{aligned}&\sum_{s^2\in K-\partial K} \Delta\varphi(s^2)\overline{*\psi(*s^2)} + \frac{1}{2}\sum_{s^2\in\partial K} \Delta\varphi(s^2)\overline{*\psi(*s^2)} \\ &= \sum_{s^1\in K-\partial K} \varphi(s^1)\overline{\Delta*\psi(*s^1)} + \sum_{s^1\in\partial K} \varphi(s^1)\overline{*\psi(*s^1(\partial K))}.\end{aligned}$$

Similarly, we have

$$(\varphi^1, \delta\psi^2)_K = \sum_{s^2\in K-\partial K}\varphi(*s^2)\overline{\Delta*\psi(s^2)} + \frac{1}{2}\sum_{s^2\in\partial K}\varphi(*s^2)\overline{\Delta*\psi(s^2)}$$

$$+\sum_{s^1\in K-\partial K}\varphi(s^1)\overline{\Delta*\psi(*s^1)} + \frac{1}{2}\sum_{s^1\in\partial K}\varphi(s^1)\overline{\Delta*\psi(*s^1)}$$

$$= \sum_{s^1\in K-\partial K}\overline{*\psi(s^1)}\Delta\varphi(*s^1) + \sum_{s^1\in\partial K}\overline{*\psi(s^1)}\varphi(*s^1(\partial K))$$

$$+\sum_{s^1\in K-\partial K}\varphi(s^1)\overline{\Delta*\psi(*s^1)}.$$

Hence we find that

$$(\Delta\varphi^1, \psi^2)_K - (\varphi^1, \delta\psi^2)_K = \sum_{s^1\in\partial K}\varphi(s^1)\overline{*\psi(*s^1(\partial K))} - \sum_{s^1\in\partial K}\overline{*\psi(s^1)}\varphi(*s^1(\partial K))$$

$$= \int_{\partial K}\varphi^1\overline{*\psi^2}.$$

4. Orthogonal projection on a compact polyhedron. In 4~5, we shall briefly state the method of orthogonal projection of the Hilbert space of differences which is analogous to de Rham-Kodaira's orthogonal decomposition theorem for differential forms on a Riemannian manifold.

THEOREM 3.2. *Let* K *be a closed complex polyhedron. Then the orthogonal decomposition*

$$\Gamma = \Gamma_c \dotplus \Gamma_e^* = \Gamma_c^* \dotplus \Gamma_e$$

holds for the Hilbert space Γ *of n-differences* ($n=1, 2$).

Proof. By Theorem 3.1 we see that

$$(\psi^n, *\Delta\varphi^{2-n}) = (-1)^{3-n}(\Delta\psi^n, *\varphi^{2-n}) \qquad (n=1, 2).$$

Hence $\Delta\psi^n=0$ implies that $(\psi^n, *\Delta\varphi^{2-n})=0$, and thus ψ^n is orthogonal to every element of Γ_e^*.

Conversely, if

$$(\Delta\psi^n, *\varphi^{2-n})=0$$

holds for all $(2-n)$-differences φ^{2-n} on K, then we can easily verify that $\Delta\psi^n=0$ on K. Hence on a closed complex polyhedron K, Γ_c is the orthogonal complement of Γ_e^*. Then by the general theory, we have the orthogonal decomposition $\Gamma=\Gamma_c\dotplus\Gamma_e^*$. The orthogonal decomposition $\Gamma=\Gamma_c^*\dotplus\Gamma_e$ for n-differences immediately follows from the decomposition $\Gamma=\Gamma_c\dotplus\Gamma_e^*$ for $(3-n)$-differences.

COROLLARY. (de Rham-Kodaira's decomposition theorem.)

$$\Gamma = \Gamma_h \dotplus \Gamma_e \dotplus \Gamma_e^* \qquad (n=1, 2).$$

Let K be a compact bordered complex polyhedron. An n-difference φ^n

$(n=0,1,2)$ on $\boldsymbol{K}$ is said to *vanish on the complex boundary* $\partial\boldsymbol{K}$ if $\varphi^n(s^n)=0$ for every n-simplex s^n of $\partial\boldsymbol{K}=\langle\partial K, \partial K^{*+}\rangle$. A closed n-difference φ^n $(n=1,2)$ is said to belong to the subspace Γ_{c0} if φ^n vanishes on $\partial\boldsymbol{K}$. Similarly, an exact n-difference $\varphi^n=\Delta\phi^{n-1}$ $(n=1,2)$ is said to belong to the subspace Γ_{e0} if $\phi^{n-1}=0$ on the complex boundary $\partial\boldsymbol{K}$.

By Theorem 3.1 we have the formula

$$(3.8)\qquad (\phi^n, *\Delta\varphi^{2-n})=\int_{\partial K}\overline{\varphi^{2-n}}\phi^n+(-1)^{3-n}(\Delta\phi^n, *\varphi^{2-n})\qquad (n=1,2).$$

By making use of (3.8) and the similar argument to the theorem 3.2, for the Hilbert space Γ of n-differences $(n=1,2)$ on a compact bordered complex polyhedron $\boldsymbol{K}$ we have the orthogonal decompositions

$$\Gamma=\Gamma_{c0}\dotplus\Gamma_e^*=\Gamma_{c0}^*\dotplus\Gamma_e,$$

$$\Gamma=\Gamma_c\dotplus\Gamma_{e0}^*=\Gamma_c^*\dotplus\Gamma_{e0}$$

and hence we have immediately the orthogonal decomposition

$$\Gamma=\Gamma_h\dotplus\Gamma_{e0}\dotplus\Gamma_{e0}^*.$$

5. Orthogonal projection on a generic polyhedron. Let us suppose that $\boldsymbol{K}$ is an open or closed complex polyhedron. An n-difference φ^n $(n=0,1,2,3)$ on $\boldsymbol{K}$ is said to have *compact support* if $\varphi^n(s^n)=0$ for all n-simplex $s^n\in\boldsymbol{K}$ except for a finite number of n-simplices of $\boldsymbol{K}$.

Let Γ'_{e0} be the subclass of Γ_e consisting of the n-differences φ^n such that $\varphi^n=\Delta\phi^{n-1}$ for an $(n-1)$-difference ϕ^{n-1} with compact support. We define the subspace Γ_{e0} of Γ as the closure in Γ of Γ'_{e0}. From the definition it follows that $\Gamma_{e0}=\Gamma_e$ for a closed complex polyhedron $\boldsymbol{K}$.

On an arbitrary complex polyhedron $\boldsymbol{K}$ we can prove that the following orthogonal decompositions for the Hilbert spaces of n-differences $(n=1,2)$ hold:

$$\Gamma=\Gamma_{e0}\dotplus\Gamma_c^*=\Gamma_{e0}^*\dotplus\Gamma_c,$$

$$\Gamma=\Gamma_h\dotplus\Gamma_{e0}\dotplus\Gamma_{e0}^*,$$

$$\Gamma_c=\Gamma_h\dotplus\Gamma_{e0},$$

$$\Gamma_e=\Gamma_{he}\dotplus\Gamma_{e0},$$

where $\Gamma_{he}=\Gamma_h\cap\Gamma_e$.

§ 4. Network flow problem.

1. ρ^n-harmonic differences. Let $\boldsymbol{K}=\langle K, K^*\rangle$ be an arbitrary complex polyhedron.

By an *n-th order density* or *n-density* ρ^n on $\boldsymbol{K}$ $(n=0,1,2,3)$ we mean the positive valued function defined on the set of n-simplices of $\boldsymbol{K}$ such that ρ^n has a positive value $\rho^n(s^n)$ for each n-simplex s^n of $\boldsymbol{K}$.

A product of an n-difference φ^n with an n-density ρ^n is defined as an n-difference $\rho^n\varphi^n$ satisfying the condition

$$\rho^n\varphi^n(s^n)=\rho^n(s^n)\varphi^n(s^n) \qquad \text{for each } n\text{-simplex } s^n\in\boldsymbol{K}.$$

If $\rho^n\varphi^n$ is closed, i.e. $\Delta(\rho^n\varphi^n)=0$, then the n-difference φ^n is said to be *closed with respect to the density* ρ^n or ρ^n*-closed.* If $\rho^n\varphi^n$ is exact, then the n-difference φ^n is said to be *exact with respect to the density* ρ^n or ρ^n*-exact.*

The *conjugate density* $*\rho^n$ of an n-density ρ^n is defined as a $(3-n)$-density satisfying the condition

$$*\rho^n(*s^n)=\rho^n(s^n) \qquad \text{for each } n\text{-simplex } s^n\in\boldsymbol{K}.$$

An n-difference φ^n is said to be *harmonic with respect to a density* ρ^n or ρ^n*-harmonic* if φ^n is closed and $*\varphi^n$ is $*\rho^n$-closed. By the definition, an n-difference φ^n is ρ^n-harmonic if and only if the $(3-n)$-difference $*(\rho^n\varphi^n)$ is $*(1/\rho^n)$-harmonic.

2. The inner product with a density and orthogonal projection. Let ρ^n $(n=0,1,2,3)$ be a fixed n-density on $\boldsymbol{K}$, and let φ^n and ψ^n be arbitrary n-differences on $\boldsymbol{K}$. Then the *inner product* $(\varphi^n, \psi^n)_\rho=(\varphi^n, \psi^n)_{\rho,\boldsymbol{K}}$ *of* φ^n *and* ψ^n *with the density* ρ^n is defined by

$$(\varphi^n, \psi^n)_\rho=(\sqrt{\rho^n}\varphi^n, \sqrt{\rho^n}\psi^n)_{\boldsymbol{K}}=(\rho^n\varphi^n, \psi^n)_{\boldsymbol{K}} \qquad (n=0, 1, 2, 3), \tag{4.1}$$

where $(\sqrt{\rho^n}\varphi^n, \sqrt{\rho^n}\psi^n)$ is the inner product of $\sqrt{\rho^n}\varphi^n$ and $\sqrt{\rho^n}\psi^n$ defined in in § 3. **1**.

By the definitions (4.1), (3.2) and (3.1), we have

$$(\psi^n, \varphi^n)_\rho=(\bar{\varphi}^n, \bar{\psi}^n)_\rho, \tag{4.2}$$

$$(*\varphi^n, *\psi^n)_{*\rho}=(\varphi^n, \psi^n)_\rho. \tag{4.3}$$

The *norm* $\|\varphi^n\|_\rho=\|\varphi^n\|_{\rho,\boldsymbol{K}}$ *of* φ^n *with the density* ρ^n is defined by

$$\|\varphi^n\|_\rho=\sqrt{(\varphi^n, \varphi^n)_\rho}=\sqrt{(\rho^n\varphi^n, \varphi^n)} \qquad (n=0, 1, 2, 3). \tag{4.4}$$

Let us denote the Hilbert space of all n-differences φ^n on $\boldsymbol{K}$ with $\|\varphi^n\|_\rho<\infty$ by Γ^ρ, for a fixed $n=1$ or 2. Furthermore we define the closed subspaces of Γ^ρ as follows:

$$\Gamma^\rho_c=\{\varphi^n \mid \varphi^n \text{ is closed, } \varphi^n\in\Gamma^\rho\},$$

$$\Gamma^\rho_e=\{\varphi^n \mid \varphi^n \text{ is exact, } \varphi^n\in\Gamma^\rho\},$$

$$\Gamma^{\rho*}_c=\{\varphi^n \mid *\varphi^n \text{ is closed, } \varphi^n\in\Gamma^\rho\},$$

$$\Gamma^{\rho*}_e=\{\varphi^n \mid *\varphi^n \text{ is exact, } \varphi^n\in\Gamma^\rho\},$$

$$\Gamma_{\rho c}=\{\varphi^n \mid \varphi^n \text{ is } \rho^n\text{-closed, } \varphi^n\in\Gamma^\rho\},$$

$$\Gamma_{\rho e}=\{\Gamma^n | \varphi^n \text{ is } \rho^n\text{-exact, } \varphi^n \in \Gamma^\rho\},$$

$$\Gamma^*_{\rho c}=\{\varphi^n | *\varphi^n \text{ is } *\rho^n\text{-closed, } \varphi^n \in \Gamma^\rho\},$$

$$\Gamma^*_{\rho e}=\{\varphi^n | *\varphi^n \text{ is } *\rho^n\text{-exact, } \varphi^n \in \Gamma^\rho\},$$

$$\Gamma_{\rho h}=\{\varphi^n | \varphi^n \text{ is } \rho^n\text{-harmonic, } \varphi^n \in \Gamma^\rho\}.$$

Then it is obvious that $\Gamma^\rho_e \subset \Gamma^\rho_c$, $\Gamma_{\rho e} \subset \Gamma_{\rho c}$ and $\Gamma_{\rho h}=\Gamma^\rho_c \cap \Gamma^*_{\rho c}$.

Let $\boldsymbol{K}$ be a closed complex polyhedron. Then, by an argument similar to Theorem 3.2 we can prove the orthogonal decompositions

$$\Gamma^\rho=\Gamma_{\rho c} \dotplus \Gamma^{\rho *}_e=\Gamma^*_{\rho c} \dotplus \Gamma^\rho_e,$$

$$\Gamma^\rho=\Gamma^\rho_c \dotplus \Gamma^*_{\rho e}=\Gamma^{\rho *}_c \dotplus \Gamma_{\rho e}$$

for the Hilbert space Γ^ρ of n-differences ($n=1, 2$). Hence we obtain the orthogonal decompositions

$$\Gamma^\rho=\Gamma_{\rho h} \dotplus \Gamma^\rho_e \dotplus \Gamma^*_{\rho e},$$

$$\Gamma^\rho_c=\Gamma_{\rho h} \dotplus \Gamma^\rho_e.$$

Similarly, on a compact bordered or an open complex polyhedron $\boldsymbol{K}$, we can also show the orthogonal decompositions for the Hilbert space Γ^ρ which are analogous to those in § 3. **4** and § 3. **5**.

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