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INVARIANT SUBMANIFOLDS OF NORMAL CONTACT METRIC MANIFOLDS

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Introduction. Yano and Ishihara [8] have obtained conditions for an invariant submanifold of a normal contact metric manifold to be totally geodesic in the case of codimension 2.

In this connection, the purpose of the present note is to obtain some conditions for an invariant submanifold of codimension $p \geqq 2$ in a normal contact metric manifold to be totally geodesic. In §1, we shall recall notations and formulas for submanifolds and, in §2, definitions and some properties of a normal contact metric manifold. In §3, we shall give basic formulas for later use and obtain conditions for an invariant submanifold of a normal contact metric manifold to be totally geodesic under some additional conditions. In the last section, invariant submanifolds satisfying the condition $\tilde{R}(X, \xi)\cdot\alpha=0$ will be studied in a normal contact metric manifold. For tensor calculus we follow to notations employed in [1].

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§1. Submanifolds.

Let M be a manifold immersed in a Riemannian manifold $\bar{M}$. Because we shall describe only local properties, we may assume that M is small enough to be imbedded in $\bar{M}$ as a proper submanifold. Let $\mathfrak{X}(M)$ be the Lie algebra of vector fields on M and $\mathfrak{X}(M)^{\perp}$ the set of all vector fields perpendicular to M. We denote by $\bar{g}$ the metric tensor field on $\bar{M}$ and g the metric induced on M. $\bar{\nabla}$ denotes the covariant differentiation in $\bar{M}$ and ∇ the covariant differentiation in M determined by the induced metric g.

Let α be the second fundamental form of M. Then the formulas of Gauss and Weingarten are given by

$$(1.1)\qquad \bar{\nabla}_X Y=\nabla_X Y+\alpha(X, Y), \qquad X, Y\in\mathfrak{X}(M),$$

$$(1.2)\qquad \bar{\nabla}_X N=-A_N(X)+D_X N, \qquad X\in\mathfrak{X}(M), \qquad N\in\mathfrak{X}(M)^{\perp},$$

where $\bar{g}(A_N(X), Y)=\bar{g}(\alpha(X, Y), N)$ and $D_X N$ denotes the covariant derivative of

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a cross section N of the normal bundle $T(M)^{\perp}$ in the direction of X with respect to the connection in $T(M)^{\perp}$.

For a normal bundle valued 2-form β, we define the covariant derivative, denoted by $\tilde{\nabla}_X\beta$, to be

$$(\tilde{\nabla}_X\beta)(Y, Z)=D_X(\beta(Y, Z))-\beta(\nabla_X Y, Z)-\beta(Y, \nabla_X Z), \tag{1.3}$$

$X, Y, Z\in\mathfrak{X}(M)$ (cf. [2] and [3]).

A submanifold M of $\bar{M}$ is, by definition, totally geodesic if and only if its second fundamental form α is identically zero.

The curvature transformations of M and $\bar{M}$ will be denoted by $R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$, $X, Y\in\mathfrak{X}(M)$ and $\bar{R}(\bar{X}, \bar{Y})=[\bar{\nabla}_{\bar{X}}, \bar{\nabla}_{\bar{Y}}]-\bar{\nabla}_{[\bar{X},\bar{Y}]}$, $\bar{X}, \bar{Y}\in\mathfrak{X}(\bar{M})$, respectivelly.

Using the formulas (1. 1) and (1. 2) of Gauss and Weingarten, we obtain for any vector fields X, Y and Z tangent to M

$$\bar{R}(X, Y)Z=R(X, Y)Z-A_{\alpha(Y,Z)}(X)+A_{\alpha(X,Z)}(Y)+(\tilde{\nabla}_X\alpha)(Y, Z)-(\tilde{\nabla}_Y\alpha)(X, Z). \tag{1.4}$$

Thus, if W is tangent to M, then we get the equation of Gauss

$$\bar{g}(\bar{R}(X, Y)Z, W)=g(R(X, Y)Z, W)+\bar{g}(\alpha(Y, W), \alpha(X, Z))-\bar{g}(\alpha(X, W), \alpha(Y, Z)). \tag{1.5}$$

§2. Normal contact metric manifolds.

Now we recall definitions and some properties of a normal contact metric manifold (cf. [5]).

Let M be a C^∞-manifold and ϕ a tensor field of type (1, 1) on M such that

$$\phi^2=-I+\eta\otimes\xi, \tag{2.1}$$

where I denotes the identity tensor of type (1, 1), ξ a vector field and η a 1-form on M satisfying

$$\phi\xi=0, \qquad \eta(\phi X)=0, \qquad \eta(\xi)=1, \qquad X\in\mathfrak{X}(M). \tag{2.2}$$

Then M is said to have an almost contact structure (ϕ, ξ, η). The almost contact structure is said to be normal if $N+d\eta\otimes\xi=0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y), \qquad \eta(X)=g(X, \xi). \tag{2.3}$$

Then (ϕ, ξ, η, g)-structure is called an almost contact metric structure. Define a tensor field Φ of type (0, 2) by $\Phi(X, Y)=g(\phi X, Y)$. If $d\eta=\Phi$, an almost contact metric structure is called a contact metric structure. If moreover $N+d\eta\otimes\xi=0$, a contact metric structure is said to be a normal contact metric structure (Sasakian structure), which satisfies

(2. 4) $$(\nabla_X\phi)Y=\eta(Y)X-g(X, Y)\xi,$$

where ∇ indicates the Riemannian connection for g.

§3. Invariant submanifolds in normal contact metric manifolds.

A submanifold $M=M^{2n+1}$ of a normal contact metric manifold $\bar{M}=\bar{M}^{2r+1}$ $(r-n=p>0)$ with structure tensors $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is said to be invariant if

(i) $\bar{\xi}$ is tangent to M everywhere on M,

(ii) $\bar{\phi}(X)$ is tangent to M for any tangent vector X to M.

An invariant submanifold M has the induced structure tensors (ϕ, ξ, η, g). An invariant submanifold $M(\phi, \xi, \eta, g)$ of a normal contact metric manifold $\bar{M}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also normal [7].

The formulas given in the following lemma have been proved by Yano and Ishihara [7].

LEMMA 3. 1. *Let M be an invariant submanifold of a normal contact metric manifold $\bar{M}$. Then we have*

(3. 1) $$\alpha(X, \phi Y)=\bar{\phi}(\alpha(X, Y)),$$

(3. 2) $$\alpha(\phi X, \phi Y)=-\alpha(X, Y),$$

(3. 3) $$\alpha(X, \xi)=0$$

for any vector fields X and Y on M.

Let $M=M^{2n+1}$ be an invariant submanifold of a normal contact metric manifold $\bar{M}=\bar{M}^{2r+1}$ and R, $\bar{R}$ be the Riemannian curvature tensors of M and $\bar{M}$ respectively. Then, using (1. 5), we have

(3. 4) $$R(X, \phi X, X, \phi X)=\bar{R}(X, \phi X, X, \phi X)-2\bar{g}(\alpha(X, X), \alpha(X, X))$$

for any vector field X on M.

Let K (resp. $\bar{K}$) be a ϕ (resp. $\bar{\phi}$)-sectional curvature of M (resp. $\bar{M}$). Then as the immediate consequence of (3. 4), we get

PROPOSITION 3. 2. *Let M be an invariant submanifold of $\bar{M}$. Then the ϕ and $\bar{\phi}$-sectional curvature satisfy the inequality $K\leqq\bar{K}$, with equality holding if and only if M is totally geodesic.*

Let S be the Ricci tensor of M. Since M is an invariant submanifold of $\bar{M}$, we can choose a $\bar{\phi}$-basis $(V_1, \cdots, V_r, \bar{\phi} V_1, \cdots, \bar{\phi} V_r, \bar{\xi})$ such that $V_1, \cdots, V_n$ are tangent to M. Then $(V_1, \cdots, V_n, \phi V_1, \cdots, \phi V_n, \xi)$ is an orthonormal basis in $T_x(M)$.

If $\bar{M}$ is of constant $\bar{\phi}$-sectional curvature $\bar{k}$, then for any $\bar{X}$, $\bar{Y}$, $\bar{Z}\in\mathfrak{X}(\bar{M})$,

$$
\begin{aligned}
4\bar{R}(\bar{X}, \bar{Y})\bar{Z}=&(\bar{k}+3)\{\bar{g}(\bar{Y}, \bar{Z})\bar{X}-\bar{g}(\bar{X}, \bar{Z})\bar{Y}\}+(\bar{k}-1)\{\bar{\eta}(\bar{X})\bar{\eta}(\bar{Z})\bar{Y} \\
&-\bar{\eta}(\bar{Y})\bar{\eta}(\bar{Z})\bar{X}+\bar{g}(\bar{X}, \bar{Z})\bar{\eta}(\bar{Y})\bar{\xi}-\bar{g}(\bar{Y}, \bar{Z})\bar{\eta}(\bar{X})\bar{\xi} \\
&+\bar{g}(\bar{\phi}\bar{Y}, \bar{Z})\bar{\phi}\bar{X}+\bar{g}(\bar{\phi}\bar{Z}, \bar{X})\bar{\phi}\bar{Y}-2\bar{g}(\bar{\phi}\bar{X}, \bar{Y})\bar{\phi}\bar{Z}\},
\end{aligned} \tag{3. 5}
$$

[4]. Thus, if $(\bar{M}, \bar{g})$ is of constant $\bar{\phi}$-sectional curvature 1, it is of constant curvature 1.

Using (1. 5), (3. 1), (3. 2) and (3. 5), we have by a straitforward calculation

PROPOSITION 3. 3. *Let $\bar{M}^{2r+1}$ be a normal contact metric manifold of constant $\bar{\phi}$-sectional curvature $\bar{k}$ and M^{2n+1} be an invariant submanifold of $\bar{M}$. Then we get*

$$
\begin{aligned}
S(X, Y)=&\frac{n(\bar{k}+3)+(\bar{k}-1)}{2}g(X, Y)-\frac{(n+1)(\bar{k}-1)}{2}\eta(X)\eta(Y) \\
&-2\sum_{i=1}^{n}\bar{g}(\alpha(V_i, X), \alpha(V_i, Y))
\end{aligned} \tag{3. 6}
$$

for any $X, Y\in T_x(M)$, where $(V_1, \cdots, V_r, \bar{\phi}V_1, \cdots, \bar{\phi}V_r, \bar{\xi})$ is a $\bar{\phi}$-basis for $T_x(\bar{M})$ such that $V_1, \cdots, V_n\in T_x(M)$.

Now, we prove

THEOREM 3. 4. *If there is an invariant Einstein submanifold M in $\bar{M}$ of constant $\bar{\phi}$-sectional curvature $\bar{k}$, then $\bar{k}\geqq 1$. When and only when $\bar{k}=1$, M is totally geodesic.*

Proof. By the assumption, we have $S(X, Y)=2ng(X, Y)$ for any $X, Y\in T_x(M)$, from which, $S(V_j, V_j)=2n$. On the other hand, by means of (3. 6), we obtain

$$
S(V_j, V_j)=\frac{n(\bar{k}+3)+(\bar{k}-1)}{2}-2\sum_{i=1}^{n}\bar{g}(\alpha(V_i, V_j), \alpha(V_i, V_j)) \tag{3. 7}
$$

for any V_j $(j=1, \cdots, n)$. Hence we get

$$
(n+1)(\bar{k}-1)=4\sum_{i=1}^{n}\bar{g}(\alpha(V_i, V_j), \alpha(V_i, V_j)),
$$

which proves our assertion. Q.E.D.

Theorem 3. 4 has been given by Yano and Ishihara [8] in the case of codimension 2. If $\bar{M}$ is of constant curvature and if M is Einsteinian, then, by Theorem 3. 4, M is of constant curvature.

Let Sc denote the scalar curvature of an invariant submanifold M^{2n+1} of a normal contact metric manifold $\bar{M}^{2r+1}$. Then, using (3. 7), we obtain

PROPOSITION 3. 5. *Let $\bar{M}^{2r+1}$ be a normal contact metric manifold of constant $\bar{\phi}$-sectional curvature $\bar{k}$ and M^{2n+1} be an invariant submanifold of $\bar{M}$. Then the scalar curvature* Sc *of M is given by*

$$\mathrm{Sc}=n^2(\bar{k}+3)+n(\bar{k}+1)-4\sum_{i,j=1}^{n}\bar{g}(\alpha(V_i, V_j), \alpha(V_i, V_j)).$$

From Proposition 3. 5, we have

THEOREM 3. 6. *Let $\bar{M}^{2r+1}$ be a normal contact metric manifold of constant $\bar{\phi}$-sectional curvature $\bar{k}$ and M^{2n-1} be an invariant submanifold of $\bar{M}$. Then M is totally geodesic if and only if*

$$\mathrm{Sc}\geqq n^2(\bar{k}+3)+n(\bar{k}+1).$$

If M is of constant ϕ-sectional curvature k, then $\mathrm{Sc}=n^2(k+3)+n(k+1)$, [4]. Therefore, we see that, if M is of constant ϕ-sectional curvature k, by Proposition 3. 5,

$$\bar{k}-k=\frac{4}{n^2+n}\sum_{i,j=1}^{n}\bar{g}(\alpha(V_i, V_j), \alpha(V_i, V_j)).$$

§4. Invariant submanifold satisfying $\tilde{R}(X, Y)\cdot\alpha=0$.

In this section, we shall give some conditions for an invariant submanifold of a normal contact metric manifold to be totally geodesic. First, we prove

PROPOSITION 4. 1. *Let M be an invariant submanifold of a normal contact metric manifold $\bar{M}$. Then M is totally geodesic if and only if $(\tilde{\nabla}_X\tilde{\nabla}_Y\alpha)(\xi, \xi)=0$ for any vector fields X and Y on M.*

Proof. Taking account of (1. 3) and Lemma 3. 1, we have

$$\begin{aligned}\alpha(X, \phi Y)&=D_Y(\alpha(X, \xi))-(\tilde{\nabla}_Y\alpha)(X, \xi)-\alpha(\nabla_YX, \xi)\\&=-(\tilde{\nabla}_Y\alpha)(X, \xi).\end{aligned}\tag{4. 1}$$

Using (4. 1), we obtain

$$\begin{aligned}\alpha(\phi X, \phi Y)&=-(\tilde{\nabla}_Y\alpha)(\phi X, \xi)\\&=-D_X((\tilde{\nabla}_Y\alpha)(\xi, \xi))+(\tilde{\nabla}_X\tilde{\nabla}_Y\alpha)(\xi, \xi)+(\tilde{\nabla}_Y\alpha)(\nabla_X\xi, \xi)\\&=D_X(\alpha(\xi, \phi Y))+(\tilde{\nabla}_X\tilde{\nabla}_Y\alpha)(\xi, \xi)-\alpha(\phi X, \phi Y)\\&=(\tilde{\nabla}_X\tilde{\nabla}_Y\alpha)(\xi, \xi)-\alpha(\phi X, \phi Y).\end{aligned}\tag{4. 2}$$

Thus, by Lemma 3. 1,

$$\alpha(X, Y)=-\frac{1}{2}(\tilde{\nabla}_X\tilde{\nabla}_Y\alpha)(\xi, \xi)\qquad\text{for any}\quad X, Y\in\mathfrak{X}(M),$$

which proves our assertion. Q.E.D.

We now put $\tilde{R}(X, Y)\cdot\beta=[\tilde{\nabla}_X, \tilde{\nabla}_Y]\beta-\tilde{\nabla}_{[X,Y]}\beta$ for a normal bundle valued symmetric 2-form β, where $\tilde{\nabla}_X$ being defined by (1. 3). Then we have

(4. 3) $(\tilde{R}(X, Y)\cdot\alpha)(V, W)=R^{\perp}(X, Y)(\alpha(V, W))-\alpha(R(X, Y)V, W)-\alpha(V, R(X, Y)W)$

for any vector fields X, Y, V and W on M, where $R^{\perp}(X, Y)=[D_X, D_Y]-D_{[X,Y]}$.

PROPOSITION 4. 2. *Let M be an invariant submanifold of a normal contact metric manifold $\bar{M}$. Then M is totally geodesic if and only if $\tilde{R}(X, \xi)\cdot\alpha=0$ for any vector field X on M.*

Proof. If M is totally geodesic, then, by (4. 3), $\tilde{R}(X, \xi)\cdot\alpha=0$ for any $X\in\mathfrak{X}(M)$. Conversely, if $\tilde{R}(X, \xi)\cdot\alpha=0$ for any $X\in\mathfrak{X}(M)$, then we have

$$R^{\perp}(X, \xi)(\alpha(V, W))=\alpha(R(X, \xi)V, W)+\alpha(V, R(X, \xi)W).$$

If we put $V=\xi$, then, from Lemma 3. 1, we get

$$\alpha(\xi, W)=0, \qquad \alpha(\xi, R(X, \xi)W)=0.$$

That is, we get

$$\alpha(R(X, \xi)\xi, W)=0. \tag{4. 4}$$

On the other hand, M being a normal contact metric manifold, we have

$$R(X, \xi)\xi=X-g(X, \xi)\xi. \tag{4. 5}$$

From (4. 4) and (4. 5), we conclude that

$$\alpha(X, W)=g(X, \xi)\alpha(\xi, W)=0,$$

which shows that M is totally geodesic. Q.E.D.

THEOREM 4. 3. *Let M be an invariant submanifold of a normal contact metric manifold $\bar{M}$. Then the following conditions are equivalent*:

(i) *M is totally geodesic*, (ii) $(\tilde{\nabla}_X\tilde{\nabla}_Y\alpha)(\xi, \xi)=0$,

(iii) $\tilde{R}(X, \xi)\cdot\alpha=0$, (iv) $\tilde{R}(X, Y)\cdot\alpha=0$,

X and Y being arbitrary vector fields on M.

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