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A CHARACTERIZATION OF CONVEX SUBSETS OF NORMED SPACES

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1. Introduction.

There are several ways in which one can introduce a notion of convexity in a metric space (see e.g. [1], [2], [4] and [5]). This paper deals with that of W-convexity as introduced by Takahashi [8].

W-spaces are natural set-ups for certain generalizations of fixed-point theorems for nonexpansive mappings in Banach spaces ([3], [6] and others), some of these generalizations appear in [7]. Thus it seems desirable to stablish the relationship between W-convexity and the usual notion of (linear) convexity. Theorems 1 and 2, below, give an answer to this problem in terms of simple geometrical conditions. Roughly speaking, Theorem 1 tells us when a W-space X is, essentially, a convex subset of some normed space E. Theorem 2 says that the space E is essentially unique.

2. Basic properties.

In [8] Takahashi introduced the notion of convexity in a metric space X by means of an operator W from $X^2\times[0, 1]$ into X, such that

$$d[z, W(x, y, \alpha)]\leqq\alpha d[z, x]+(1-\alpha)d[z, y] \tag{A}$$

for all x, y and z in X and every α in the interval $I=[0,1]$. Here we refer to such a space as a W-space and, for simplicity, write (x, y, α) instead of $W(x, y, \alpha)$. It is clear that the usual notion of (linear) convexity in a normed space is of this type with $(x, y, \alpha)=\alpha x+(1-\alpha)y$. As in the linear case, we say that a subset Y of a W-space X is W-convex if (x, y, α) belongs to Y whenever x and y are elements of Y, and α is in the unit interval.

A nontrivial example of a W-space is obtained as follows: consider a closed subset X of the unit ball $S_1=\{||x||=1\}$ in a Hilbert space H, such that X has diameter $\delta(X)\leqq\sqrt{2}$ and is geodesically connected, i.e., the point

$$W(x, y, \alpha)=\frac{\alpha x+(1-\alpha)y}{||\alpha x+(1-\alpha y)||}$$

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lies in X whenever $x, y \in X$ and $\alpha \in I$. The metric space we obtain by measuring distances in X through central angles, i.e., with the metric

$$d_1[x, y] = \cos^{-1}\langle x, y\rangle, \qquad \forall\, x, y \in X,$$

turns out to be a W-space (whose convex sets are exactly the geodesically connected subsets of X).

The following properties are direct consequences of (A):

(1) $d[x, (x, y, \alpha)] = (1-\alpha)d[x, y]$.

(2) $d[y, (x, y, \alpha)] = \alpha d[x, y]$.

(3) $(x, x, \alpha) = (x, y, 1) = (y, x, 0) = x$.

(4) $\alpha \in [0, 1] \to (x, y, \alpha) \in X$ is an injective mapping.

(5) balls (open or closed) in X are W-convex sets.

(6) the union of a directed family of W-convex sets and the intersection of W-convex sets are W-convex.

Motivated by the linear case we define multiple convex combinations, inductively: if $x_1, \cdots, x_n \in X$ and $\alpha_1, \cdots, \alpha_n \in I$, with $\sum_1^n \alpha_i = 1$, we set

$$(x_1, \cdots, x_n;\ \alpha_1, \cdots, \alpha_n) = \left(\left(x_1, \cdots, x_{n-1};\ \frac{\alpha_1}{1-\alpha_n}, \cdots, \frac{\alpha_{n-1}}{1-\alpha_n}\right), x_n, 1-\alpha_n\right) \qquad \text{if} \quad \alpha_n \neq 1,$$

and

$$(x_1, \cdots, x_n;\ 0, \cdots, 1) = x_n.$$

We now introduce two other conditions (both are satisfied in normed spaces):

(B) $(x, y, z;\ \alpha, \beta, \gamma) = (y, z, x;\ \beta, \gamma, \alpha), \quad \forall\, x, y, z \in X, \quad \alpha, \beta, \gamma \in I, \quad \alpha+\beta+\gamma = 1.$

(C) $d[(x, z, \alpha), (y, z, \alpha)] = \alpha d[x, y], \quad \forall\, x, y, z \in X$ and $\alpha \in I$.

Their geometrical meaning is clear: the first says that convex combinations do not depend of the order they are carried out, the second says that the distance is homothetic.

At this point we can state our main result:

THEOREM 1. *If X is a W-convex metric space satisfying conditions* (B) *and* (C) *above, then there is an isometry $\mathcal{J}$ from X onto a convex subset of some normed space E, which preserves convex combinations, i.e., for every $x, y \in X$ and $\alpha \in I$ we have that*

$$\mathcal{J}((x, y, \alpha)) = \alpha \mathcal{J}(x) + (1-\alpha)\mathcal{J}(y).$$

If X is a metric space with an operator W: $X^2 \times [0, 1] \to X$, it is easy to see that (A), (B) and (C) are necessary conditions for the existence of an isometric embedding, preserving convex combinations, of X in some normed space E. Theorem 1 tells us that the three conditions are also sufficient, i.e. they characterize those W-spaces which are, essentially, convex subsets of normed spaces, since then the metric and convexity structures coincide. In this way Theorem 1 gives a characterization of convex subsets in normed spaces.

The construction of the supporting space E is somewhat technical but simple to outline: heuristically the first step involves the construction of the cone $C=R^+X$ of all "positive multiples" of elements of X, the normed space E is then defined as the set $C-C$ of all "differences" of elements in C.

To prove Theorem 1 we will need some auxiliary identities – if X is a W-convex space satisfying conditions (B) and (C) we have:

(7) $((x, y, \alpha), z, \beta)=\left(\left(y, z, \dfrac{\beta(1-\alpha)}{1-\alpha\beta}\right), x, 1-\alpha\beta\right)$ if $\alpha\beta\neq 1$.

(8) $(x, y, \alpha)=(y, x, 1-\alpha)$.

(9) $((x, y, \alpha), y, \beta)=(x, y, \alpha\beta)$.

(10) if $(x, z, \alpha)=(y, z, \alpha)$ and $\alpha\neq 0$ then $x=y$.

(11) $d[(x, y, \alpha), (x, y, \beta)]=|\alpha-\beta| d[x, y]$.

(12) if σ is a permutation of the set $\{1, 2, \cdots, n\}$ then

$$(x_{\sigma(1)}, \cdots, x_{\sigma(n)};\ \alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)})=(x_1, \cdots, x_n;\ \alpha_1, \cdots, \alpha_n)$$

Proofs. For (7) just consider the identity

$$(x, y, z;\ \alpha\beta, \beta(1-\alpha), 1-\beta)=(y, z, x;\ \beta(1-\alpha), 1-\beta, \alpha\beta).$$

If $\beta\neq 0$, we get, by the definition of multiple convex combinations, formula (7). If $\beta=0$, (7) follows immediately from (3).

(8) Assuming $\alpha\neq 1$ we have, by (7), that

$$(x, y, \alpha)=((x, y, \alpha), x, 1)=((y, x, 1), x, 1-\alpha)=(y, x, 1-\alpha)$$

If $\alpha=1$, (8) follows from (3).

(9) is a direct consequence of (7) and (8).

(10) follows from condition (C).

(11) if $\beta\leqq\alpha\neq 0$ we have by (10) that

$$d[(x, y, \beta), (x, y, \alpha)]=d\left[\left(\left(x, y, \frac{\beta}{\alpha}\right), y, \alpha\right), (x, y, \alpha)\right]$$

$$=\alpha d\left[\left(x, y, \frac{\beta}{\alpha}\right), x\right]=\alpha\left(1-\frac{\beta}{\alpha}\right)d[x, y]=(\alpha-\beta)d[x, y].$$

The other case, $\alpha=\beta=0$ is trivial.

(12) it is enough to consider the case when σ is a transposition, the proof by induction is straightforward and because of that is ommited.

3. The cone extension.

In order to define "positive multiples" of elements of X, we select an element x_0 as a reference point, or origin, in X and start by defining fractional products αx where $x\in X$ and $\alpha\in[0,1]$ by the formula $\alpha x=(x, x_0, \alpha)$. The following properties are then valid:

(13) $0x=\alpha x_0=x_0$ and $1x=x$.

(14) $\alpha(\beta x)=(\alpha\beta)x$.

(15) $d[\alpha x, \alpha y]=\alpha d[x, y]$, in particular if $\alpha x=\alpha y$ and $\alpha\neq 0$, then $x=y$ (cancellation law).

(16) $\beta(x, y, \alpha)=\left(\frac{\alpha\beta}{1+\alpha\beta-\beta}x, y, 1+\alpha\beta-\beta\right)$ if $\alpha, \beta\in(0,1)$.

Next we consider in the cartesian product $R^+\times X$ (as before, R^+ denotes the set of positive real numbers), the identification given by

$$(\lambda, x)\sim(\delta, y) \text{ if and only if } \frac{\lambda}{\lambda+\delta}x=\frac{\delta}{\lambda+\delta}y.$$

This is an equivalence relation, to prove the transitivity, for instance, we notice that if $(\lambda, x)\sim(\delta, y)$ and $(\delta, y)\sim(\gamma, z)$, i.e.

$$\frac{\lambda}{\lambda+\delta}x=\frac{\delta}{\lambda+\delta}y, \qquad \frac{\delta}{\delta+\gamma}y=\frac{\gamma}{\delta+\gamma}z$$

we get, after appropriate multiplication,

$$\frac{\lambda}{\lambda+\delta+\gamma}x=\frac{\delta}{\lambda+\delta+\gamma}y=\frac{\gamma}{\lambda+\delta+\gamma}z$$

hence

$$\frac{\lambda+\delta}{\lambda+\delta+\gamma}\left(\frac{\lambda}{\lambda+\gamma}x\right)=\frac{\lambda+\delta}{\lambda+\delta+\gamma}\left(\frac{\gamma}{\lambda+\gamma}z\right)$$

so that, by cancellation, we have that $(\lambda, x)\sim(\gamma, z)$ as wished.

The quotient space $(R^+\times X)/\sim$ will be denoted by C. For the sake of simplicity the equivalence class of the pair (λ, x) and the pair itself will be denoted in the same way, thus, for instance, we have that $(1, x_0)=\{(\lambda, x_0), \lambda>0\}$.

We now introduce in C an addition, a product by nonnegative scalars and a metric as follows:

(i) $\delta(\lambda, x)=(\delta\lambda, x)$ if $\delta>0$ and $0(\lambda, x)=(1, x_0)$.

(ii) $(\lambda, x)+(\delta, y)=\left(\lambda+\delta, \left(x, y, \frac{\lambda}{\lambda+\delta}\right)\right)$.

(iii) $d'[(\lambda, x), (\delta, y)]=(\lambda+\delta)d\left[\frac{\lambda}{\lambda+\delta}x, \frac{\delta}{\lambda+\delta}y\right]$.

The product as given by (i) is well defined (recall that we are dealing with classes), furthermore we have that

$$\Delta(\Delta'(\lambda, x))=(\Delta\Delta')(\lambda, x),$$

$$1(\lambda, x)=(\lambda, x) \quad \text{and} \quad \lambda(1, x_0)=(1, x_0).$$

To see that the addition is well defined, consider the pairs (λ, x), (δ, y) and (Δ, z) where $(\lambda, x)\sim(\delta, y)$. As

$$\frac{\lambda}{\lambda+\delta}x=\frac{\delta}{\lambda+\delta}y$$

we have after multiplication by $(\lambda+\delta)/(\lambda+\delta+\Delta)$, and convex combination with z, that

$$\left(\frac{\lambda}{\lambda+\delta+\Delta}x, z, \frac{\lambda+\delta+\Delta}{\lambda+\delta+2\Delta}\right)=\left(\frac{\delta}{\lambda+\delta+\Delta}y, z, \frac{\lambda+\delta+\Delta}{\lambda+\delta+2\Delta}\right)$$

hence, by (16),

$$\frac{\lambda+\Delta}{\lambda+\delta+2\Delta}\left(x, z, \frac{\lambda}{\lambda+\Delta}\right)=\frac{\delta+\Delta}{\lambda+\delta+2\Delta}\left(y, z, \frac{\delta}{\delta+\Delta}\right)$$

so that

$$\left(\lambda+\Delta, \left(x, z, \frac{\lambda}{\lambda+\Delta}\right)\right)\sim\left(\delta+\Delta, \left(y, z, \frac{\delta}{\delta+\Delta}\right)\right)$$

and, as wanted,

$$(\lambda, x)+(\Delta, z)=(\delta, y)+(\Delta, z).$$

The addition is also commutative and has a zero element, the class $(1, x_0)$; to prove that it is an associative operation, consider the identity in X

$$\left(x, y, z; \frac{\lambda}{s}, \frac{\delta}{s}, \frac{\Delta}{s}\right)=\left(y, z, x; \frac{\delta}{s}, \frac{\Delta}{s}, \frac{\lambda}{s}\right),$$

where $s=\lambda+\delta+\Delta$. From it we obtain

$$\left(\left(x, y, \frac{\lambda}{\lambda+\delta}\right), z, \frac{\lambda+\delta}{s}\right)=\left(\left(y, z, \frac{\delta}{\delta+\Delta}\right), x, \frac{\delta+\Delta}{s}\right)$$
$$=\left(x, \left(y, z, \frac{\delta}{\delta+\Delta}\right), \frac{\lambda}{s}\right).$$

which, in turn, gives

$$\left(\lambda+\delta,\left(x, y, \frac{\lambda}{\lambda+\delta}\right)\right)+(\Delta, z)=(\lambda, x)+\left(\delta+\Delta,\left(y, z, \frac{\delta}{\delta+\Delta}\right)\right)$$

thus,

$$[(\lambda, x)+(\delta, y)]+(\Delta, z)=(\lambda, x)+[(\delta, y)+(\Delta, z)]$$

as we wanted. The two distributive laws are easily verified:

$$(\Delta+\Delta')(\lambda, x)=\Delta(\lambda, x)+\Delta'(\lambda, x),$$

$$\Delta[(\lambda, x)+(\delta, y)]=\Delta(\lambda, x)+\Delta(\delta, y).$$

To prove now that d', as given by (iii), is well defined, notice that if $(\lambda, x)=(\delta, y)$ we have, after appropriate multiplication, that

$$\frac{\lambda}{\lambda+\delta+\Delta} x=\frac{\delta}{\lambda+\delta+\Delta} y$$

thus

$$d\left[\frac{\lambda}{\lambda+\delta+\Delta} x, \frac{\Delta}{\lambda+\delta+\Delta} z\right]=d\left[\frac{\delta}{\lambda+\delta+\Delta} y, \frac{\Delta}{\lambda+\delta+\Delta} z\right]$$

hence, by (15),

$$\frac{\lambda+\Delta}{\lambda+\Delta+\delta} d\left[\frac{\lambda}{\lambda+\Delta} x, \frac{\Delta}{\lambda+\Delta} z\right]=\frac{\delta+\Delta}{\lambda+\delta+\Delta} d\left[\frac{\delta}{\delta+\Delta} y, \frac{\Delta}{\delta+\Delta} z\right]$$

multiplication by $(\lambda+\delta+\Delta)$ now, leads to

$$d'[(\lambda, x), (\Delta, z)]=d'[(\delta, y), (\Delta, z)]$$

as wanted.

The function d' is seen to satisfy all properties of a metric, only the proof of the triangle inequality is not entirely trivial: from

$$d\left[\frac{\lambda}{s} x, \frac{\delta}{s} y\right] \leqq d\left[\frac{\lambda}{s} x, \frac{\Delta}{s} z\right]+d\left[\frac{\Delta}{s} y, \frac{\delta}{s} z\right]$$

where $s=\lambda+\delta+\Delta$, we get, using (15) that

$$\frac{\lambda+\delta}{s} d\left[\frac{\lambda}{\lambda+\delta} x, \frac{\delta}{\lambda+\delta} y\right] \leqq \frac{\lambda+\Delta}{s} d\left[\frac{\lambda}{\lambda+\Delta} x, \frac{\Delta}{\lambda+\Delta} z\right]+\frac{\Delta+\delta}{s} d\left[\frac{\Delta}{\Delta+\delta} z, \frac{\delta}{\Delta+\delta} y\right]$$

multiplication by s now gives, as wanted,

$$d'[(\lambda, x), (\delta, y)] \leqq d'[(\lambda, x), (\Delta, z)]+d'[(\Delta, z), (\delta, y)].$$

For the sake of reference we now list all properties just proved:

(17) the addition in C is commutative and associative, there is a zero element $(1, x_0)$ which we will indicate, from now on, by c_0.

(18) the product by nonnegative scalars is distributive and associative, we also have that, if $c \in C$ and $\lambda \geqq 0$, then

$$1c=c, \qquad \lambda c_0 = c_0, \qquad 0c=c_0.$$

(19) d' makes C into a metric space.

4. The special character of C.

As a transition between the W-space X and the final normed space E, still to be constructed, the cone C should not only contain a copy of the first, but have, as well, most characteristics of the second. That this is so, is stablished by the next three properties:

(20) the metric d' in C is homothetic, i.e. if c and c' belong to C and $\lambda \geqq 0$, then we have that

$$d'[\lambda c, \lambda c'] = \lambda d'[c, c'].$$

(21) d' is translation invariant, i.e. for c, c' and c'' in C,

$$d'[c+c'', c'+c''] = d'[c, c'].$$

(22) the mapping Π: $X \to C$ given by $\Pi(x) = (1, x)$ is an isometry, $\Pi(x_0) = c_0$ and preserves convex combinations, i.e. for all x and y in X and $\alpha \in [0, 1]$ we have that

$$\Pi((x, y, \alpha)) = \alpha \Pi(x) + (1-\alpha)\Pi(y)$$

Proofs. (20) is proved in a straightforward manner from the definitions.

(21) By condition (C) we have that

$$td\left[\left(\frac{\lambda}{s}x, z, \frac{s}{t}\right), \left(\frac{\delta}{s}y, z, \frac{s}{t}\right)\right] = sd\left[\frac{\lambda}{s}x, \frac{\delta}{s}y\right]$$

where $s = \lambda + \delta + \Delta$ and $t = \lambda + \delta + 2\Delta$. Using now property (16) and (C), on the left and right members respectively, we have

$$td\left[\frac{\lambda+\Delta}{t}\left(x, z, \frac{\lambda}{\lambda+\Delta}\right), \frac{\delta+\Delta}{t}\left(y, z, \frac{\delta}{\delta+\Delta}\right)\right] = s\frac{\lambda+\delta}{s}d\left[\frac{\lambda}{\lambda+\delta}x, \frac{\delta}{\lambda+\delta}y\right]$$

that is,

$$d'\left[\left(\lambda+\Delta, \left(x, z, \frac{\lambda}{\lambda+\Delta}\right)\right), \left(\lambda+\Delta, \left(y, z, \frac{\delta}{\delta+\Delta}\right)\right)\right] = d'[(\lambda, x), (\delta, y)]$$

hence, as wanted,

$$d'[(\lambda, x)+(\Delta, z), (\delta, y)+(\Delta, z)] = d'[(\lambda, x), (\delta, y)].$$

(22) follows from the fact that

$$d'[(1, x), (1, y)]=2d\left[\frac{1}{2}x, \frac{1}{2}y\right]=d[x, y]$$

and

$$(1, (x, y, \alpha))=(\alpha, x)+(1-\alpha, y)=\alpha(1, x)+(1-\alpha)(1, y)$$

for $\alpha\in(0, 1)$, while that if $\alpha=0$ or 1 the equality in (22) is trivial.

As direct consequences of (20), (21) and (22) we have that

(23) if $c+c''=c'+c''$, then $c=c'$ (cancellation);

(24) if $\lambda c=\lambda c'$ and $\lambda>0$ then $c=c'$;

(25) if $\lambda c=\lambda' c$ and $c\neq c_0$ then $\lambda=\lambda'$;

(26) $\Pi(x_1, \cdots, x_n; \alpha_1, \cdots, \alpha_n)=\sum_1^n \alpha_i \Pi(x_i)$;

(27) $(\lambda, x)=\lambda(1, x)$, $\forall \lambda>0$, $x\in X$, thus if we identify X and $\Pi(X)$ we can write that $C=R^+X$, that is, the cone spanned by X.

Finally we remark that given a metric space X with an operator W: $X^2\times I\to X$, and a cone C, that is, some other metric space with an algebraic structure satisfying properties (17) - (21), then the existence of an isometry Π: $X\to C$ preserving convex combinations as in (22), implies that the operator W satisfies the conditions (A), (B) and (C), i.e., they are necessary and sufficient conditions for the possibility of an imbedding as stablished here.

5. The supporting normed space E.

To formalize the construction of the normed space E suggested in section 1 we follow a standard procedure, because of that most proofs will be omitted. We start by introducing in $C\times C$ an equivalence relation: if c, d, e and f are elements of C we say that

$$(c, d)\sim(e, f) \quad \text{if and only if} \quad c+f=e+d.$$

Now we define in the quotient space $(C\times C)/\sim$, denoted by E, an addition, a product by scalars and a metric: using the some symbol for the pair (c, d) and the corresponding equivalence class, we set

(i′) $(c, d)+(e, f)=(c+e, d+f)$.

(ii′) $\lambda(c, d)=(\lambda c, \lambda d)$ if $\lambda\geqq 0$,

$\lambda(c, d)=|\lambda|(d, c)$ if $\lambda<0$.

(iii′) $d''[(c, d), (e, f)]=d'[c+f, e+d]$.

The addition of classes is well defined, associative and commutative. The equivalence class of the pair (c_0, c_0), i.e. the set $\{(c, c),\ c \in C\}$ is a zero element for the addition and every element (c, d) has a symmetric, namely, (d, c).

The product by real scalars is well defined and satisfies the following identities

(28) $\lambda(\delta(c, d)) = (\lambda\delta)(c, d)$.

(29) $\lambda[(c, d)+(e, f)] = \lambda(c, d)+\lambda(e, f)$.

(30) $(\lambda+\delta)(c, d) = \lambda(c, d)+\delta(c, d)$.

(31) $1(c, d) = (c, d)$.

The mapping d'': $E \times E \rightarrow R$ is also well defined—it does not depend on the particular pairs used to represent the equivalence classes. Indeed, if $(c, d) \sim (c_1, d_1)$, that is $c+d_1 = c_1+d$, then we have, by property (21), that

$$d''[(c_1, d_1),\ (e, f)] = d'[c_1+f,\ e+d_1]$$

$$= d'[c_1+f+c+d,\ e+d_1+c+d]$$

$$= d'[c+f+c_1+d,\ e+d+c+d_1]$$

$$= d'[c+f,\ e+d] = d''[(c, d),\ (e, f)].$$

as wanted. In addition, d'' is a metric on E which is translation-invariant and homothetic, i.e.

$$d''[(c, d)+(g, h),\ (e, f)+(g, h)] = d''[(c, d),\ (e, f)].$$

$$d''[\lambda(c, d),\ \lambda(e, f)] = |\lambda| d''[(c, d),\ (e, f)].$$

All of these properties, put together, imply that E is a normed vector space with norm given by the formula

$$||v|| = d''[v, 0],$$

where $v = (c, d)$ is a generic element of E and $0 = (c_0, c_0)$ is the zero element.

We now proceed to show the existence of an imbedding of X into the normed space E as described in Theorem 1. First we consider the mapping $\mathcal{C}$ from the cone C to the space E defined by $\mathcal{C}(c) = (c, c_0)$. This is easily seen to be an isometry which preserves addition and multiplication by nonnegative scalars, in particular $\mathcal{C}$ preserves convex combinations and takes the origin $c_0 \in C$ into the origin of E. The conclusion is that E contains a copy of C, namely the set $\mathcal{C}(C)$; in fact with this identification in mind we can write that $E = C - C$ in the sense that any vector $v = (c, d)$ in E can be split as follows:

$$v = (c, d) = (c, c_0) - (d, c_0) = \mathcal{C}(c) - \mathcal{C}(d).$$

We are now able to prove Theorem 1, as a matter of fact the proof, after the preceeding development, is now rather simple.

Proof of Theorem 1. Consider the composition $\mathcal{J}=\mathcal{C}\circ\Pi$ of the two mappings Π: $X\to C$ and $\mathcal{C}$: $C\to E$,

$$\mathcal{J}(x)=\mathcal{C}((1,x))=((1,x),\ (1,x_0)),\qquad \forall\, x\in X.$$

As both, $\mathcal{C}$ and Π, are isometries and preserve convex combinations, the same is true for $\mathcal{J}$; the image $\mathcal{J}(X)$ is then a convex subset of the normed space E which cannot be distinguished from the original W-space X as far as the metric and convexity structures are concerned.

The point x_0, selected as reference in X, gets in this way identified with the zero vector of the normed space E. After the identification $x\to\mathcal{J}(x)$ we can write that $E=R^+X-R^+X$ or still, $E=\text{span } X$, indeed if $v=(c,d)\in E$ and, say, $c=(\lambda,x)$, $d=(\delta,y)$ then we have

$$\begin{aligned} v=(c,d)&=\mathcal{C}(c)-\mathcal{C}(d)=\mathcal{C}(\lambda(1,x))-\mathcal{C}(\delta(1,y))\\ &=\lambda\mathcal{C}(1,x)-\delta\mathcal{C}(1,y)=\lambda\mathcal{J}(x)-\delta\mathcal{J}(y), \end{aligned}$$

so that we can think of the normed space E as the minimal extension of X.

6. The uniqueness of the extension.

Theorem 1 stablished that conditions (A), (B) and (C) are necessary and sufficient for the existence of an imbedding into some normed space. The uniqueness of the solution of the imbedding problem is guaranteed by the following result:

THEOREM 2. *Let X be a W-space satisfying* (B) *and* (C). *Consider any two imbeddings $\mathcal{J}$: $X\to E$ and $\mathcal{J}_1$: $X\to E_1$ as described in Theorem* 1 *and assume, for simplicity, that $\mathcal{J}(x_0)=0$ and $\mathcal{J}_1(x_0)=0_1$ where* 0 *and* 0_1 *are the zero vectors of the normed spaces E and E_1 respectively. Then there is a linear isometry T between the minimal extensions*, span $\mathcal{J}(X)$ *and* span $\mathcal{J}_1(X)$, *which makes the following diagram commutative*

$$\begin{array}{ccc} & \overset{\mathcal{J}}{\nearrow} & \text{span } \mathcal{J}(X)\subset E \\ X & & \big\downarrow T \\ & \underset{\mathcal{J}_1}{\searrow} & \text{span } \mathcal{J}_1(X)\subset E_1 \end{array}$$

Proof of Theorem 2. Observing that span $\mathcal{J}(X)=R^+\mathcal{J}(X)-R^+\mathcal{J}(X)$ we put, tentatively,

$$T(\lambda\mathcal{J}(x)-\delta\mathcal{J}(y))=\lambda\mathcal{J}_1(x)-\delta\mathcal{J}_1(y),$$

now, if

$$\lambda\mathcal{J}(x)-\delta\mathcal{J}(y)=\lambda'\mathcal{J}(x')-\delta'\mathcal{J}(y')$$

we have that

$$\lambda\mathcal{J}(x)+\delta'\mathcal{J}(y)=\lambda'\mathcal{J}(x')+\delta\mathcal{J}(y).$$

hence, setting $s=(\lambda+\delta+\lambda'+\delta')^{-1}$, we have

$$\lambda s\mathcal{J}(x)+\delta' s\mathcal{J}(y')+\lambda' s\mathcal{J}(x_0)+\delta s\mathcal{J}(x_0)$$
$$=\lambda' s\mathcal{J}(x')+\delta s\mathcal{J}(y)+\lambda s\mathcal{J}(x_0)+\delta' s\mathcal{J}(x_0).$$

that is

$$\mathcal{J}((x, y', x_0, x_0;\ \lambda s, \delta' s, \lambda' s, \delta s))=\mathcal{J}((x', y, x_0, x_0;\ \lambda' s, \delta s, \lambda s, \delta' s)).$$

and, as $\mathcal{J}$ is $1-1$,

$$(x, y', x_0, x_0;\ \lambda s, \delta' s, \lambda' s, \delta s)=(x', y, x_0, x_0;\ \lambda' s, \delta s, \lambda s, \delta' s).$$

If we now reverse the process, using $\mathcal{J}_1$ instead of $\mathcal{J}$, we conclude that

$$\lambda\mathcal{J}_1(x)-\delta\mathcal{J}_1(y)=\lambda'\mathcal{J}_1(x')-\delta'\mathcal{J}_1(y')$$

proving that T is a well defined mapping from span $\mathcal{J}(X)$ to span $\mathcal{J}_1(X)$. The same reasoning shows that T is one-to-one while it is obvious that T is onto.

To prove the additivity of T consider any two points in span $\mathcal{J}(X)$, say, $a=\lambda\mathcal{J}(x)-\delta\mathcal{J}(y)$ and $a'=\lambda'\mathcal{J}(x')-\delta'\mathcal{J}(y')$; then

$$a+a'=(\lambda+\lambda')\left(\frac{\lambda}{\lambda+\lambda'}\mathcal{J}(x)+\frac{\lambda'}{\lambda+\lambda'}\mathcal{J}(x')\right)-(\delta+\delta')\left(\frac{\delta}{\delta+\delta'}\mathcal{J}(y)+\frac{\delta'}{\delta+\delta'}\mathcal{J}(y')\right)$$
$$=(\lambda+\lambda')\mathcal{J}\left(\left(x, x', \frac{\lambda}{\lambda+\lambda'}\right)\right)-(\delta+\delta')\mathcal{J}\left(\left(y, y', \frac{\delta}{\delta+\delta'}\right)\right).$$

so that

$$T(a+a')=(\lambda+\lambda')\mathcal{J}_1\left(\left(x, x', \frac{\lambda}{\lambda+\lambda'}\right)\right)-(\delta+\delta')\mathcal{J}_1\left(\left(y, y', \frac{\delta}{\delta+\delta'}\right)\right).$$

If we start with $T(a)$ and $T(a')$ instead, we arrive to the same thing, the conclusion is thus, that $T(a+a')=T(a)+T(a')$, i.e. that T is additive.

If $\alpha>0$ we have that $\alpha a=\alpha\lambda\mathcal{J}(x)-\alpha\delta\mathcal{J}(y)$, so that

$$T(\alpha a)=\alpha\lambda\mathcal{J}_1(x)-\alpha\delta\mathcal{J}_1(y)=\alpha T(a);$$

on the other hand,

$$T(-a)=T(\delta\mathcal{J}(y)-\lambda\mathcal{J}(x))=\delta\mathcal{J}_1(y)-\lambda\mathcal{J}_1(x)=-T(a),$$

while

$$T(0)=T(a+(-a))=T(a)+T(-a)=0_1.$$

so that we can conclude that T is homogeneous as well. To see that T is an isometry consider that

$$\|a\|=d''[\lambda\mathcal{J}(x)-\delta\mathcal{J}(y),\ 0]=d''[\lambda\mathcal{J}(x),\ \delta\mathcal{J}(y)]$$

$$=(\lambda+\delta)d''\left[\frac{\lambda}{\lambda+\delta}\mathcal{J}(x)+\frac{\delta}{\lambda+\delta}\mathcal{J}(x_0),\ \frac{\delta}{\lambda+\delta}\mathcal{J}(y)+\frac{\lambda}{\lambda+\delta}\mathcal{J}(x_0)\right]$$

$$=(\lambda+\delta)d''\left[\mathcal{J}\left(\left(x,x_0,\frac{\lambda}{\lambda+\delta}\right)\right),\ \mathcal{J}\left(\left(y,x_0,\frac{\delta}{\lambda+\delta}\right)\right)\right]$$

$$=(\lambda+\delta)d\left[\left(x,x_0,\frac{\lambda}{\lambda+\delta}\right),\ \left(y,x_0,\frac{\delta}{\lambda+\delta}\right)\right],$$

similar development for $\|T(a)\|$ leads to the same expression so that $\|T(a)\|=\|a\|$ as wanted. Finally, if $a=\mathcal{J}(x)\in\mathcal{J}(X)$, we have that $T(a)=\mathcal{J}_1(x)$ so that $T\circ\mathcal{J}=\mathcal{J}_1$, i.e. the diagram is commutative. This completes the proof of Theorem 2.

7. Final remarks.

We have settled the problem of existence, and uniqueness, of the imbedding of W-spaces into normed spaces. It is natural, at this point, to ask whether the (minimal) extension is necessarily complete, i.e. a Banach space. The answer is, in general, negative even when the W-space is itself a complete metric space, as the following example shows:

COUNTEREXAMPLE. Consider the subset of l^2, defined by

$$X=\{x\in l^2,\ |x_n|\leqq 1/2^n\ \ \forall n\}$$

clearly X is convex, closed, symmetric and contains the origin, in particular X is a complete W-convex metric space. We now prove that the linear subspace span X (the obvious normed space extension of X) is a *proper dense subset* of l^2. Indeed it is easy to check that

$$\text{span } X=R^+X=\{x\in l^2,\ |2^n x_n|\leqq M\ \ \forall n,\ \text{where } M=M(x)\}$$

Thus the point $e_n=(0,\cdots,0,1,0,\cdots)$ lies in the span X, so that this set is dense in l^2. Consider now the point $x=(x_n)_n$ where $x_n=2^{-n/2}$. While it is true that $x\in l^2$, we have

$$\lim_{n\to\infty}|2^n x_n|=+\infty,$$

so that x does not belong to span X. This settles the problem posed above.

It should also be remarked the fact (used several times in the proofs) that a mapping which preserves convex combinations of pairs, will necessarily preserve multiple convex combinations, i.e. the following is true:

LEMMA. *Let X and Y be two W-spaces——the convex combinations in X and Y will be denoted by (x, y, α) and $\{x, y, \alpha\}$ respectively. If the mapping f: $X \to Y$ is such that*

$$f((x, y, \alpha)) = \{f(x), f(y), \alpha\} \qquad \forall x, y \in X, \qquad \alpha \in I,$$

then f preserves multiple combinations, i.e.

$$f((x_1, \cdots, x_n; \alpha_1, \cdots, \alpha_n)) = \{f(x_1), \cdots, f(x_n); \alpha_1, \cdots, \alpha_n\}$$

for $x_1, \cdots, x_n$ in X and $\alpha_1, \cdots, \alpha_n \in I$, $\Sigma\alpha_i = 1$.

Finally we remark the possibility of replacing the system of conditions (A), (B) and (C) by simpler or geometrically more appealing equivalent systems. Thus, for instance, condition (A) may be replaced by

(A′) $(z, z, \alpha) = z, \qquad \forall z \in X, \quad \alpha \in I.$

Property (3), section 2, states that (A′) is implied by (A), (B) and (C); we must now prove that (A′), (B) and (C), together, imply condition (A). To do this we first observe that for all $x, y, z \in X$,

$$z = (x, y, z; 0, 0, 1) = (y, z, x; 0, 1, 0) = ((y, z, 0), x, 1)$$

so that

$$((y, z, 0), x, 1) = ((z, z, 0), x, 1) = (z, x, 1)$$

thus, by condition (C),

$$(y, z, 0) = z, \qquad \forall y, z \in X$$

and, going back, we also conclude that

$$(z, x, 1) = z, \qquad \forall x, z \in X.$$

Using these facts we have, for $\alpha \neq 1$,

$$(x, y, \alpha) = ((x, y, \alpha), x, 1) = (x, y, x; \alpha, 1-\alpha, 0)$$

$$= (y, x, x; 1-\alpha, 0, \alpha) = ((y, x, 1), x, 1-\alpha) = (y, x, 1-\alpha)$$

while that, for $\alpha = 1$, we have

$$(x, y, 1) = x = (y, x, 0)$$

so that in general,

$$(x, y, \alpha) = (y, x, 1-\alpha).$$

Now we can prove (A): from the previous considerations we have that

$$d[z,(x,y,\alpha)] \leqq d[z,(z,y,\alpha)] + d[(z,y,\alpha),(x,y,\alpha)]$$
$$= d[(z,z,1-\alpha),\ (y,z,1-\alpha)] + \alpha d[z,x]$$
$$= (1-\alpha)d[z,y] + \alpha d[z,x]$$

which is exactly condition (A).

REFERENCES

[1] BLUMENTHAL, L. M., Theory and applications of distance geometry. Oxford (1953).
[2] BLUMENTHAL, L. M., AND K. MENGER, Studies in geometry. San Francisco (1970).
[3] BROWDER, F. E., Nonexpansive nonlinear operators in Banach spaces. Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.
[4] DUNFORD, N., AND J. T. SCHWARTZ, Linear operators. New York (1955).
[5] KIJIMA, Y., AND W. TAKAHASHI, A fixed-point theorem for nonexpansive mappings in metric space. Kōdai Math. Sem. Rep. **21** (1969), 326–330.
[6] KIRK, W. A., A fixed-point theorem for mappings which do not increase distance. Amer. Math. Monthly **72** (1965), 1004–1006.
[7] MACHADO, H. V., Fixed-point theorems for nonexpansive mappings in metric spaces with normal structure. Ph. D. thesis, Chicago (1971).
[8] TAKAHASHI, W., A convexity in metric space and nonexpansive mappings, I. Kōdai Math. Rep. **22** (1970), 142–149.

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