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ON ALMOST CONTACT AFFINE 3-STRUCTURES

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The almost quaternion structure has been studied by Ako [10], Bonan [1], Obata [6, 7] and one of the present authors [10]. The purpose of the present paper is to study almost contact affine 3-structures [2, 3, 4, 5, 8, 9] induced on hypersurfaces of an almost quaternion or quaternion manifold.

§1. Hypersurfaces of an almost quaternion manifold.

Let M^{4n} be an almost quaternion manifold, that is, a $4n$-dimensional differentiable manifold which admits a set of three tensor fields $\tilde{F}, \tilde{G}, \tilde{H}$ of type (1, 1) satisfying

$$\tag{1.1} \begin{gathered} \tilde{F}^2=-I, \qquad \tilde{G}^2=-I, \qquad \tilde{H}^2=-I, \\ \tilde{F}=\tilde{G}\tilde{H}=-\tilde{H}\tilde{G}, \qquad \tilde{G}=\tilde{H}\tilde{F}=-\tilde{F}\tilde{H}, \qquad \tilde{H}=\tilde{F}\tilde{G}=-\tilde{G}\tilde{F}, \end{gathered}$$

I denoting the identity tensor.

We first prove

LEMMA 1.1. *There exists an almost Hermitian metric $\tilde{g}$ for the almost quaternion structure $\tilde{F}, \tilde{G}, \tilde{H}$, that is, a Riemannian metric $\tilde{g}$ satisfying*

$$\tag{1.2} \begin{aligned} \tilde{g}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y})&=\tilde{g}(\tilde{X}, \tilde{Y}), \\ \tilde{g}(\tilde{G}\tilde{X}, \tilde{G}\tilde{Y})&=\tilde{g}(\tilde{X}, \tilde{Y}), \\ \tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y})&=\tilde{g}(\tilde{X}, \tilde{Y}) \end{aligned}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ of M^{4n}.

Proof. Take an arbitrary Riemannian metric $\tilde{a}$ in M^{4n} and put

$$\tilde{b}(\tilde{X}, \tilde{Y})=\tilde{a}(\tilde{X}, \tilde{Y})+\tilde{a}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}),$$

then we easily see that

$$\tilde{b}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y})=\tilde{b}(\tilde{X}, \tilde{Y})$$

since $\tilde{F}^2=-I$. We next put

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$$\tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{b}(\tilde{X}, \tilde{Y}) + \tilde{b}(\tilde{G}\tilde{X}, \tilde{G}\tilde{Y}),$$

then we see that

$$\tilde{g}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}),$$

$$\tilde{g}(\tilde{G}\tilde{X}, \tilde{G}\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}),$$

$$\tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}).$$

Suppose that a $(4n-1)$-dimensional orientable differentiable manifold M^{4n-1} is immersed differentiably in M^{4n} by the immersion

$$i: M^{4n-1} \longrightarrow M^{4n}$$

and denote by B the differential of i. We denote by C the unit normal to $i(M^{4n-1})$ with respect to the Hermitian metric $\tilde{g}$ introduced above. Then the transform $\tilde{F}BX$ of a vector field BX tangent to $i(M^{4n-1})$ by $\tilde{F}$ can be expressed as

$$\tilde{F}BX = BFX + u(X)C,$$

where F is a tensor field of type (1, 1), u a 1-form, and X an arbitrary vector field of M^{4n-1}.

Replacing $\tilde{Y}$ by $\tilde{F}\tilde{Y}$ in

$$\tilde{g}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}),$$

we find

$$\tilde{g}(\tilde{F}\tilde{X}, \tilde{Y}) = -\tilde{g}(\tilde{X}, \tilde{F}\tilde{Y}),$$

from which, putting $\tilde{X} = C$, $\tilde{Y} = C$,

$$\tilde{g}(\tilde{F}C, C) = -\tilde{g}(C, \tilde{F}C) = 0,$$

and consequently $\tilde{F}C$ is tangent to $i(M^{4n-1})$. Thus we can put

$$\tilde{F}C = -BU,$$

U being a vector field of M^{4n-1}.

In this way, we have formulas of the form

$$(1.3)\qquad \begin{aligned} &\text{(i)} && \tilde{F}BX = BFX + u(X)C, && \tilde{F}C = -BU, \\ &\text{(ii)} && \tilde{G}BX = BGX + v(X)C, && \tilde{G}C = -BV, \\ &\text{(iii)} && \tilde{H}BX = BHX + w(X)C, && \tilde{H}C = -BW, \end{aligned}$$

where F, G, H are tensor fields of type (1, 1), U, V, W vector fields and u, v, w 1-forms of M^{4n-1}.

Applying $\tilde{F}$ to (1. 3) (i) and taking account of (1. 3) (i), we find

$$(1.4)\qquad F^2 = -I + u \otimes U, \qquad u \circ F = 0, \qquad FU = 0, \qquad u(U) = 1,$$

which show that M^{4n-1} admits an almost contact affine structure (F, U, u).

Similarly, we can prove

$$G^2=-I+v\otimes V, \qquad v\circ G=0, \qquad GV=0, \qquad v(V)=1 \tag{1.5}$$

and

$$H^2=-I+w\otimes W, \qquad w\circ H=0, \qquad HW=0, \qquad w(W)=1, \tag{1.6}$$

which show that M^{4n-1} admits another affine almost contact structures (G, V, v) and (H, W, w).

On the other hand, from

$$\tilde{G}\tilde{H}BX=\tilde{F}BX$$

and (1. 3), we have

$$\tilde{G}(BHX+w(X)C)=BFX+u(X)C,$$

$$BGHX+v(HX)C-w(X)BV=BFX+u(X)C,$$

from which

$$GH=F+w\otimes V, \qquad v\circ H=u.$$

Also, from

$$\tilde{G}\tilde{H}C=\tilde{F}C$$

and (1. 3), we have

$$\tilde{G}(-BW)=-BU,$$

$$-BGW-v(W)C=-BU,$$

from which

$$GW=U, \qquad v(W)=0.$$

Thus

$$GH=F+w\otimes V, \qquad v\circ H=u, \qquad GW=U, \qquad v(W)=0. \tag{1.7}$$

Similarly, we can prove

$$HF=G+u\otimes W, \qquad w\circ F=v, \qquad HU=V, \qquad w(U)=0 \tag{1.8}$$

and

$$FG=H+v\otimes U, \qquad u\circ G=w, \qquad FV=W, \qquad u(V)=0. \tag{1.9}$$

Also, from

$$(\tilde{G}\tilde{H}+\tilde{H}\tilde{G})BX=0$$

and (1. 3), we have

$$\tilde{G}(BHX+w(X)C)+\tilde{H}(BGX+v(X)C)=0,$$

$$BGHX+v(HX)C-w(X)BV+BHGX+w(GX)C-v(X)BW=0,$$

from which,

$$(GH+HG)X=v(X)W+w(X)V$$

and

$$v(HX)+w(GX)=0.$$

Also, from

$$(\tilde{G}\tilde{H}+\tilde{H}\tilde{G})C=0$$

and (1. 3), we have

$$-\tilde{G}BW-\tilde{H}BV=0,$$

$$BGW+v(W)C+BHV+w(V)C=0,$$

from which,

$$GW+HV=0, \qquad v(W)+w(V)=0.$$

Thus

$$\begin{aligned}(1.10)\qquad & GH+HG=v\otimes W+w\otimes V,\\ & v\circ H+w\circ G=0, \qquad GW+HV=0, \qquad v(W)+w(V)=0.\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}(1.11)\qquad & HF+FH=w\otimes U+u\otimes W,\\ & w\circ F+u\circ H=0, \qquad HU+FW=0, \qquad w(U)+u(W)=0\end{aligned}$$

and

$$\begin{aligned}(1.12)\qquad & FG+GF=u\otimes V+v\otimes U,\\ & u\circ G+v\circ F=0, \qquad FV+GU=0, \qquad u(V)+v(U)=0.\end{aligned}$$

A set $(F, G, H; U, V, W; u, v, w)$ of tensor fields F, G, H of type (1, 1), vector fields U, V, W and 1-forms u, v, w satisfying (1. 4), (1. 5), (1. 6); (1. 7), (1. 8), (1. 9) and (1. 10), (1. 11), (1. 12) is called an almost contact affine 3-structure. Thus, we have proved

THEOREM 1.1. *An orientable hypersurface of an almost quaternion manifold admits an almost contact affine 3-structure.*

Equations (1.4)~(1.12) can also be written as follows

$$
\begin{aligned}
&F^2=-I+u\otimes U, && G^2=-I+v\otimes V, && H^2=-I+w\otimes W,\\
&GH=F+w\otimes V, && HF=G+u\otimes W, && FG=H+v\otimes U,\\
&HG=-F+v\otimes W, && FH=-G+w\otimes U, && GF=-H+u\otimes V,
\end{aligned} \tag{1.13}
$$

$$
\begin{aligned}
&u\circ F=0, && u\circ G=w, && u\circ H=-v,\\
&v\circ F=-w, && v\circ G=0, && v\circ H=u,\\
&w\circ F=v, && w\circ G=-u, && w\circ H=0,
\end{aligned} \tag{1.14}
$$

$$
\begin{aligned}
&FU=0, && FV=W, && FW=-V,\\
&GU=-W, && GV=0, && GW=U,\\
&HU=V, && HV=-U, && HW=0,
\end{aligned} \tag{1.15}
$$

$$
\begin{aligned}
&u(U)=1, && u(V)=0, && u(W)=0,\\
&v(U)=0, && v(V)=1, && v(W)=0,\\
&w(U)=0, && w(V)=0, && w(W)=1.
\end{aligned} \tag{1.16}
$$

Suppose that there is given a Hermitian metric $\tilde{g}$ with respect to $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$. In this case, we put

$$\tilde{g}(BX, BY)=g(X, Y)$$

which gives the Riemannian metric induced on the hypersurface $i(M^{4n-1})$.

From

$$\tilde{g}(\tilde{F}BX, \tilde{F}BY)=\tilde{g}(BX, BY)=g(X, Y),$$

we find

$$\tilde{g}(BFX+u(X)C, BFY+u(Y)C)=g(X, Y),$$

$$g(FX, FY)+u(X)u(Y)=g(X, Y),$$

or

$$g(FX, FY)=g(X, Y)-u(X)u(Y).$$

We have also

$$\tilde{g}(BX, \tilde{F}C)=\tilde{g}(BX, -BU)=-g(X, U)$$

and on the other hand

$$\begin{aligned}\tilde{g}(BX, \tilde{F}C) &= \tilde{g}(\tilde{F}BX, \tilde{F}^2C)\\ &= \tilde{g}(BFX+u(X)C, -C)\\ &= -u(X),\end{aligned}$$

and consequently

$$g(X, U)=u(X).$$

Thus

$$\begin{aligned}&g(FX, FY)=g(X, Y)-u(X)u(Y),\\ &g(X, U)=u(X), \qquad g(U, U)=1.\end{aligned} \tag{1.17}$$

Similarly, we have

$$\begin{aligned}&g(GX, GY)=g(X, Y)-v(X)v(Y),\\ &g(X, V)=v(X), \qquad g(V, V)=1\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}&g(HX, HY)=g(X, Y)-w(X)w(Y),\\ &g(X, W)=w(X), \qquad g(W, W)=1.\end{aligned} \tag{1.19}$$

An almost contact affine 3-structure with a Riemannian metric g satisfying (1.17), (1.18) and (1.19) is called an almost contact metric 3-structure. Thus we have proved

THEOREM. 1.2. *An orientable hypersurface of an almost quaternion manifold with a Hermitian metric admits an almost contact metric 3-structure.*

Equations

$$g(X, U)=u(X), \qquad g(X, V)=v(X), \qquad g(X, W)=w(X)$$

and

$$v(W)=0, \qquad w(U)=0, \qquad u(V)=0$$

show that U, V, W are mutually orthogonal unit vectors.

§ 2. Hypersurfaces of a quaternion manifold.

Ako and one of the present authors [10] proved following theorems:

THEOREM A. *Let $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ define an almost quaternion structure. If two of six Nijenhuis tensors*:

$$[\tilde{F}, \tilde{F}],\ [\tilde{G}, \tilde{G}],\ [\tilde{H}, \tilde{H}],\ [\tilde{G}, \tilde{H}],\ [\tilde{H}, \tilde{F}],\ [\tilde{F}, \tilde{G}]$$

vanish, then the others vanish too.

If there exists a coordinate system with respect to which components of the tensor fields $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ are all constants, the almost quaternion structure $(\tilde{F}, \tilde{G}, \tilde{H})$ is integrable and the almost quaternion structure is called a quaternion structure.

THEOREM B. *In order that there exists, in an almost quaternion manifold, a symmetric affine connection $\tilde{\nabla}$ such that*

$$\tilde{\nabla}\tilde{F}=0, \qquad \tilde{\nabla}\tilde{G}=0, \qquad \tilde{\nabla}\tilde{H}=0,$$

it is necessary and sufficient that two of Nijenhuis tensors

$$[\tilde{F}, \tilde{F}],\ [\tilde{G}, \tilde{G}],\ [\tilde{H}, \tilde{H}],\ [\tilde{G}, \tilde{H}],\ [\tilde{H}, \tilde{F}],\ [\tilde{F}, \tilde{G}]$$

vanish.

THEOREM C. *A necessary and sufficient condition that an almost quaternion structure $(\tilde{F}, \tilde{G}, \tilde{H})$ be integrable is that two of Nijenhuis tensors*

$$[\tilde{F}, \tilde{F}],\ [G, \tilde{G}],\ [\tilde{H}, \tilde{H}],\ [\tilde{G}, \tilde{H}],\ [\tilde{H}, \tilde{F}],\ [\tilde{F}, \tilde{G}]$$

vanish and

$$\tilde{R}=0,$$

where $\tilde{R}$ is the curvature tensor of the affine connection $\tilde{\nabla}$ appearing in Theorem B.

We assume in this section that the almost quaternion structure $(\tilde{F}, \tilde{G}, \tilde{H})$ is integrable and denote by $\tilde{\nabla}$ the symmetric affine connection with respect to which $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ are covariantly constant.

We now cover M^{4n} by a system of coordinate neighborhoods $\{U;\ x^h\}$ and denote by $\tilde{F}_i{}^h$, $\tilde{G}_i{}^h$, $\tilde{H}_i{}^h$ components of $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ respectively and by $\tilde{\nabla}_j$ the operator of covariant differentiation with respect to the symmetric affine connection $\tilde{\nabla}$, then

$$\tilde{\nabla}_j\tilde{F}_i{}^h=0, \qquad \tilde{\nabla}_j\tilde{G}_i{}^h=0, \qquad \tilde{\nabla}_j\tilde{H}_i{}^h=0. \tag{2.1}$$

We represent $i(M^{4n-1})$ by

$$x^h=x^h(y^a), \tag{2.2}$$

$\{y^a\}$ being local coordinates on M^{4n-1} and put $B_b{}^h=\partial_b x^h$ $(\partial_b=\partial/\partial y^b)$ and denote by C^h components of C used in §1. Then equations of Gauss and Weingarten are

$$\nabla_c B_b{}^h = h_{cb} C^h,$$
(2. 3)
$$\nabla_c C^h = -h_c{}^a B_a{}^h + l_c C^h$$

respectively, where h_{cb} and $h_c{}^a$ are the second fundamental tensors with respect to the affine normal C^h and l_c the third fundamental tensor.

We write the first equation of (1. 3) (i) in the form

$$\tilde{F}_i{}^h B_b{}^i = F_b{}^a B_a{}^h + u_b C^h,$$

where $F_b{}^a$ and u_b are components of F and u respectively and differentiate this covariantly along $i(M^{4n-1})$. Then we get

$$\tilde{F}_i{}^h (h_{cb} C^i) = (\nabla_c F_b{}^a) B_a{}^h + F_b{}^e h_{ce} C^h + (\nabla_c u_b) C^h + u_b(-h_c{}^a B_a{}^h + l_c C^h),$$

from which

$$\nabla_c F_b{}^a = -h_{cb} U^a + h_c{}^a u_b,$$

$$\nabla_c u_b = -h_{ce} F_b{}^e - l_c u_b,$$

using the second equation of (1. 3) (i) written in the form

$$\tilde{F}_i{}^h C^i = -U^a B_a{}^h,$$

where U^a are components of the vector field U. We differentiate this covariantly along $i(M^{4n-1})$. Then we get

$$\tilde{F}_i{}^h (-h_c{}^a B_a{}^i + l_c C^i) = -(\nabla_c U^a) B_a{}^h - U^e h_{ce} C^h,$$

from which

$$\nabla_c U^a = h_c{}^e F_e{}^a + l_c U^a, \qquad h_c{}^e u_e = h_{ce} U^e.$$

Thus, we have

$$\nabla_c F_b{}^a = -h_{cb} U^a + h_c{}^a u_b, \qquad \nabla_c U^a = h_c{}^e F_e{}^a + l_c U^a,$$
(2. 4)
$$\nabla_c u_b = -h_{ce} F_b{}^e - l_c u_b, \qquad h_c{}^e u_e = h_{ce} U^e.$$

Similarly, we can prove

$$\nabla_c G_b{}^a = -h_{cb} V^a + h_c{}^a v_b, \qquad \nabla_c V^a = h_c{}^e G_e{}^a + l_c V^a,$$
(2. 5)
$$\nabla_c v_b = -h_{ce} G_b{}^e - l_c v_b, \qquad h_c{}^e v_e = h_{ec} V^e$$

and

$$\nabla_c H_b{}^a = -h_{cb} W^a + h_c{}^a w_b, \qquad \nabla_c W^a = h_c{}^e H_e{}^a + l_c W^a,$$
(2. 6)

$$\nabla_c w_b = -h_{ce} H_b{}^e - l_c w_b, \qquad h_c{}^e w_e = h_{ce} W^e,$$

where $G_b{}^a$, $H_b{}^a$, V^a, W^a, v_b, w_b are components of G, H, V, W, v, w respectively.

Now, the almost contact structure (F, U, u) is said to be normal if the tensor

$$[F, F] + du \otimes U$$

vanishes, where $[F, F]$ is the Nijenhuis tensor formed with F. We compute components of this tensor.

Using (2. 4), we have

$$\begin{aligned} &[F, F]_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) U^a \\ &= (F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a) u_b - (F_b{}^e h_e{}^a - h_b{}^e F_e{}^a - l_b U^a) u_c. \end{aligned} \tag{2. 7}$$

Similarly, computing components of the tensor

$$[G, G] + dv \otimes V,$$

we find

$$\begin{aligned} &[G, G]_{cb}{}^a + (\nabla_c v_b - \nabla_b v_c) V^a \\ &= (G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a) v_b - (G_b{}^e h_e{}^a - h_b{}^e G_e{}^a - l_b V^a) v_c. \end{aligned} \tag{2. 8}$$

We also compute components of the tensor field

$$[F, G] + du \otimes V + dv \otimes U,$$

where $[F, G]$ is the Nijenhuis tensor formed with F and G.

Using (2. 4) and (2. 5), we find

$$\begin{aligned} &[F, G]_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) V^a + (\nabla_c v_b - \nabla_b v_c) U^a \\ &= (G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a) u_b - (G_b{}^e h_e{}^a - h_b{}^e G_e{}^a - l_b V^a) u_c \\ &\quad + (F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a) v_b - (F_b{}^e h_e{}^a - h_b{}^e F_e{}^a - l_b U^a) v_c. \end{aligned} \tag{2. 9}$$

Suppose that the almost contact affine structures (F, U, u) and (G, V, v) are both normal, then we have, from (2. 7) and (2. 8),

$$(F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a) u_b - (F_b{}^e h_e{}^a - h_b{}^e F_e{}^a - l_b U^a) u_c = 0 \tag{2. 10}$$

and

$$(G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a) v_b - (G_b{}^e h_e{}^a - h_b{}^e G_e{}^a - l_b V^a) v_c = 0 \tag{2. 11}$$

respectively.

Putting $c=a$ in (2. 10) and (2. 11) and summing up, we find

$$-(l_c U^c) u_b - F_b{}^e h_e{}^c u_c + l_b = 0 \tag{2. 12}$$

and

$$-(l_c V^c)v_b - G_b{}^e h_e{}^c v_c + l_b = 0 \tag{2.13}$$

respectively.

Transvecting (2.12) and (2.13) with W^b and using (1.15), (1.16), (2.4) and (2.5), we find

$$h_{cb} U^c V^b + l_b W^b = 0$$

and

$$-h_{cb} U^c V^b + l_b W^b = 0$$

respectively, from which

$$h_{cb} U^c V^b = 0, \qquad l_b W^b = 0. \tag{2.14}$$

Transvecting (2.12) with V^b and (2.13) with U^b, we have respectively

$$h_{cb} W^c U^b = l_c V^c, \qquad h_{cb} V^c W^b = -l_c U^c. \tag{2.15}$$

Transvecting (2.10) and (2.11) with $w_a W^b$, we obtain

$$h_{cb} V^c W^b = 0, \qquad h_{cb} W^c U^b = 0, \tag{2.16}$$

from which, using (2.15),

$$l_c U^c = 0, \qquad l_c V^c = 0. \tag{2.17}$$

Summing up, we have

$$\begin{aligned} &h_{cb} V^c W^b = 0, \qquad h_{cb} W^c U^b = 0, \qquad h_{cb} U^c V^b = 0, \\ &l_b U^b = 0, \qquad l_b V^b = 0, \qquad l_b W^b = 0. \end{aligned} \tag{2.18}$$

Transvecting (2.10) with U^b and (2.11) with V^b and using (2.18), we find

$$F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a = -(h_b{}^e F_e{}^a U^b) u_c \tag{2.19}$$

and

$$G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a = -(h_b{}^e G_e{}^a V^b) v_c \tag{2.20}$$

respectively.

Transvecting (2.19) and (2.20) with W^c and using (1.15), (1.16) and (2.18), we find

$$-h_e{}^a V^e - h_c{}^e F_e{}^a W^c = 0$$

and

$$h_e{}^a U^e - h_c{}^e G_e{}^a W^c = 0$$

respectively, and consequently

$$h_b{}^e F_e{}^a U^b = +h_c{}^e G_e{}^d W^c F_d{}^a = h_c{}^e H_e{}^a W^c$$

and

$$h_b{}^e G_e{}^a V^b = -h_c{}^e F_e{}^d W^c G_d{}^a = h_c{}^e H_e{}^a W^c$$

by virtue of (1. 13). Thus we can write (2. 19) and (2. 20) in the form

$$F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a = u_c P^a \tag{2. 21}$$

and

$$G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a = v_c P^a \tag{2. 22}$$

respectively, where

$$P^a = -h_b{}^e F_e{}^a U^b = -h_b{}^e G_e{}^a V^b.$$

Substituting (2. 21) and (2. 22) into (2. 9), we find

$$[F, G] + du \otimes V + dv \otimes U = 0. \tag{2. 23}$$

Conversely, suppose that two almost contact affine structures (F, U, u) and (G, V, v) satisfy (2. 23). Then we have from (2. 9)

$$\begin{aligned} &(G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a)u_b - (G_b{}^e h_e{}^a - h_b{}^e G_e{}^a - l_b V^a)u_c \\ &+ (F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a)v_b - (F_b{}^e h_e{}^a - h_b{}^e F_e{}^a - l_b U^a)v_c = 0. \end{aligned} \tag{2. 24}$$

Contracting (2. 24) with respect to a and b and using (1. 14) and (1. 15), we find

$$G_c{}^e h_e{}^a u_a + F_c{}^e h_e{}^a v_a + (l_a V^a)u_c + (l_a U^a)v_c = 0, \tag{2. 25}$$

from which, transvecting U^c, V^c and W^c respectively, we find

$$h_{cb} W^c U^b = l_a V^a, \tag{2. 26}$$

$$h_{cb} V^c W^b = -l_a U^a, \tag{2. 27}$$

$$h_{cb} U^c U^b = h_{cb} V^c V^b. \tag{2. 28}$$

Transvecting (2. 24) with U^b and using (1. 15) and (1. 16), we find

$$\begin{aligned} &G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a \\ &= -(h_e{}^a W^e + h_b{}^e G_e{}^a U^b + l_b U^b V^a)u_c - (h_b{}^e F_e{}^a U^b + l_b U^b U^a)v_c. \end{aligned} \tag{2. 29}$$

Transvecting (2. 29) with v_a and taking account of (2. 27), we find

$$G_c{}^e h_e{}^a v_a - l_c = h_{ba} W^b U^a v_c, \tag{2. 30}$$

from which, transvecting with V^c

$$-l_c V^c = h_{ba} W^b U^a.$$

Comparing (2. 26) with this, we find

$$h_{cb} W^c U^b = 0, \qquad l_c V^c = 0. \tag{2. 31}$$

Transvecting again (2. 30) with W^c, we find

$$l_c W^c = h_{cb} U^c V^b. \tag{2. 32}$$

Now, transvecting (2. 24) with V^b and using (2. 31), we find

$$F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a = -h_b{}^e V^b G_e{}^a u_c + (h_e{}^a W^e - h_b{}^e V^b F_e{}^a) v_c. \tag{2. 33}$$

Transvecting (2. 33) with u_a and using (2. 31), we find

$$F_c{}^e h_e{}^a u_a - l_c = -h_{ba} V^b W^a u_c \tag{2. 34}$$

from which, transvecting with U^c,

$$l_c U^c = h_{cb} V^c W^b,$$

and consequently, from (2. 27) and this equation, we have

$$h_{cb} V^c W^b = 0, \qquad l_c U^c = 0. \tag{2. 35}$$

Thus we have, from (2. 34),

$$l_c = F_c{}^e h_e{}^a u_a, \tag{2. 36}$$

from which, transvecting W^c,

$$l_c W^c = -h_{cb} U^c V^b.$$

Thus (2. 32) and this give

$$h_{cb} U^c V^b = 0, \qquad l_c W^c = 0. \tag{2. 37}$$

Summing up, we have

$$\begin{aligned} &h_{cb} V^c W^b = 0, \qquad h_{cb} W^c U^b = 0, \qquad h_{cb} U^c V^b = 0, \\ &l_c U^c = 0, \qquad l_c V^c = 0, \qquad l_c W^c = 0. \end{aligned} \tag{2. 38}$$

On the other hand, transvecting (2. 29) with W^c and taking account of (1. 14), (1. 15) and (2. 38),

$$h_e{}^a U^e - h_c{}^e W^c G_e{}^a = 0,$$

from which, transvecting $G_a{}^b$,

$$h_e{}^a U^e G_a{}^b - h_c{}^e W^c(-\delta_e^b + v_e V^b) = 0,$$

or

$$h_e{}^d U^e G_d{}^a + h_c{}^a W^c = 0. \tag{2.39}$$

Thus, (2. 29) becomes

$$G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a = -h_b{}^e F_e{}^a U^b v_c,$$

that is,

$$G_c{}^e h_e{}^a - h_c{}^e G_e{}^a - l_c V^a = \beta^a v_c, \tag{2.40}$$

β^a being a certain vector field.

In the same way, from (2. 33) we can deduce

$$F_c{}^e h_e{}^a - h_c{}^e F_e{}^a - l_c U^a = \alpha^a u_c, \tag{2.41}$$

α^a being a certain vector field.

Substituting (2. 41) into (2. 7), we find

$$[F, F] + du \otimes U = 0$$

and substituting (2. 40) into (2. 8), we find

$$[G, G] + dv \otimes V = 0,$$

that is, the almost contact affine structures (F, U, u) and (G, V, v) are both normal. Thus, we have proved

THEOREM 2. 1. *On a hypersurface of an almost quaternion manifold, the condition*

$$[F, F] + du \otimes U = 0 \qquad \text{and} \qquad [G, G] + dv \otimes V = 0$$

and the condition

$$[F, G] + du \otimes V + dv \otimes U = 0$$

are equivalent.

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