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ON TRANSVERSAL HYPERSURFACES OF AN ALMOST CONTACT MANIFOLD

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§ 0. Introduction.

Hypersurfaces of an almost contact manifold have been studied by Blair [2], [3], Eum [4], Goldberg [5], Ki [11], [12], Ludden [2], [3], Okumura [7], [13], Yamaguchi [10], Yano [3], [5], [11], [12], [13], and others.

Blair [3], Ludden [3], Okumura [13] and Yano [3], [13] found that a hypersurface of an almost contact manifold admits what they call an (f, U, V, u, v, λ)-structure.

Goldberg and one of the present authors [5] called a noninvariant hypersurface of an almost contact manifold a hypersurface such that the transform of a tangent vector of the hypersurface by the tensor φ defining the almost contact structure is never tangent to the hypersurface.

In the present paper, we study what we call transversal hypersurfaces, that is, hypersurfaces which never contain the vector field ξ defining the almost contact structure.

In § 1 we first of all show that a transversal hypersurface of an almost contact manifold admits an almost complex structure F and that a transversal hypersurface of an almost contact metric manifold admits an almost Hermitian structure (F, γ). We also derive a formula which gives the relation between the connection with respect to the Hermitian metric γ and that with respect to the induced Riemannian metric g.

In § 2, we study transversal hypersurfaces of a cosymplectic manifold. In this case the almost Hermitian structure (F, γ) is Kählerian [5].

In § 3, we study transversal hypersurfaces of a Sasakian manifold. In this case also, the almost Hermitian structure (F, γ) is Kählerian.

The last § 4 is devoted to the study of transversal hypersurfaces of a Sasakian manifold of constant curvature.

§ 1. Transversal hypersurface of an almost contact manifold.

Let $\tilde{M}$ be a $(2n+1)$-dimensional differentiable manifold covered by a system

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of coordinate neighborhoods $\{\tilde{U}; y^\kappa\}$, where, here and in the sequel, the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$ and let $\tilde{M}$ admit an almost contact structure, that is, a set $(\varphi_\lambda{}^\kappa, \xi^\kappa, \eta_\lambda)$ of a tensor field $\varphi_\lambda{}^\kappa$ of type (1, 1), a vector field ξ^κ and a 1-form η_λ satisfying

$$\varphi_\lambda{}^\mu\varphi_\mu{}^\kappa = -\delta_\lambda^\kappa + \eta_\lambda\xi^\kappa, \tag{1. 1}$$
$$\eta_\kappa\varphi_\lambda{}^\kappa = 0, \qquad \varphi_\lambda{}^\kappa\xi^\lambda = 0, \qquad \eta_\lambda\xi^\lambda = 1.$$

Consider a $2n$-dimensional differentiable manifold M covered by a system of coordinate neighborhoods $\{U; x^h\}$, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$, and assume that M is differentiably immersed in $\tilde{M}$ as a hypersurface by the immersion

$$i\colon M \longrightarrow \tilde{M},$$

which is expressed locally by

$$y^\kappa = y^\kappa(x^h).$$

We assume that for each $p \in M$, the vector field ξ^κ at $i(p)$ never belongs to the tangent hyperplane of the hypersurface $i(M)$. We call such a hypersurface $i(M)$ a *transversal hypersurface* of an almost contact manifold. In this case, we can take ξ^κ as an affine normal to the hypersurface $i(M)$.

Now the vectors $B_i{}^\kappa = \partial_i y^\kappa$ $(\partial_i = \partial/\partial x^i)$ and ξ^κ being linearly independent, the transforms $\varphi_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $\varphi_\lambda{}^\kappa$ can be expressed as

$$\varphi_\lambda{}^\kappa B_i{}^\lambda = F_i{}^h B_h{}^\kappa + \alpha_i \xi^\kappa, \tag{1. 2}$$

where $F_i{}^h$ is a tensor field of type (1, 1) and α_i a 1-form of M.

Applying $\varphi_\kappa{}^\mu$ again to (1. 2) and taking account of (1. 1), we find

$$F_i{}^t F_t{}^h = -\delta_i^h, \tag{1. 3}$$

$$\alpha_t F_i{}^t = \eta_i, \tag{1. 4}$$

where

$$\eta_i = \eta_\lambda B_i{}^\lambda.$$

Thus M admits an almost complex structure F and a 1-form α.

We now assume that $\tilde{M}$ admits an almost contact metric structure, that is, a set $(\varphi_\lambda{}^\kappa, \xi^\kappa, \eta_\lambda, G_{\mu\lambda})$ of $\varphi_\lambda{}^\kappa, \xi^\kappa, \eta_\lambda$ and a positive definite Riemannian metric $G_{\mu\lambda}$ satisfying, in addition to (1. 1),

$$G_{\tau\sigma}\varphi_\mu{}^\tau\varphi_\lambda{}^\sigma = G_{\mu\lambda} - \eta_\mu\eta_\lambda, \tag{1. 5}$$

and

$$\eta_\mu = G_{\mu\lambda}\xi^\lambda, \qquad G_{\mu\lambda}\xi^\mu\xi^\lambda = 1.$$

Transvecting (1. 5) with $B_j{}^\mu B_i{}^\lambda$ and using (1. 2), we find

$$G_{\tau\sigma}(F_j{}^t B_t{}^\tau+\alpha_j\xi^\tau)(F_i{}^s B_s{}^\sigma+\alpha_i\xi^\sigma)=g_{ji}-\eta_j\eta_i,$$

g_{ji} being the Riemannian metric on M induced from that of $\tilde{M}$, that is,

$$g_{ji}=G_{\mu\lambda}B_j{}^\mu B_i{}^\lambda.$$

Thus, remembering $G_{\tau\sigma}B_t{}^\tau\xi^\sigma=\eta_t$ and $F_j{}^t\eta_t=-\alpha_j$, we have

$$F_j{}^t F_i{}^s g_{ts}-\alpha_j\alpha_i=g_{ji}-\eta_j\eta_i,$$

or

$$F_j{}^t F_i{}^s(g_{ts}-\eta_t\eta_s)=g_{ji}-\eta_j\eta_i. \tag{1.6}$$

This shows that

$$\gamma_{ji}=g_{ji}-\eta_j\eta_i \tag{1.7}$$

is an almost Hermitian metric with respect to the almost complex structure F. The fact that the metric γ_{ji} is positive definite will be verified as follows.

Let X^κ be an arbitrary vector in $\tilde{M}$, then the square of the length of the vector $X^\kappa-(\eta_\alpha X^\alpha)\xi^\kappa$ is given by

$$G_{\mu\lambda}\{X^\mu-(\eta_\gamma X^\gamma)\xi^\mu\}\{X^\lambda-(\eta_\beta X^\beta)\xi^\lambda\}=(G_{\mu\lambda}-\eta_\mu\eta_\lambda)X^\mu X^\lambda,$$

which shows that $(G_{\mu\lambda}-\eta_\mu\eta_\lambda)X^\mu X^\lambda\geqq 0$ and $=0$ if and only if $X^\mu=(\eta_\gamma X^\gamma)\xi^\mu$. Thus the Riemannian metric

$$\gamma_{ji}=(G_{\mu\lambda}-\eta_\mu\eta_\lambda)B_j{}^\mu B_i{}^\lambda$$

on M induced from $G_{\mu\lambda}-\eta_\mu\eta_\lambda$ of $\tilde{M}$ is also positive definite.

We now assume that M is orientable and choose a unit vector field C^κ of $\tilde{M}$ normal to $i(M)$ in such a way that $2n+1$ vectors $B_i{}^\kappa$ and C^κ give the positive orientation of $\tilde{M}$. We put

$$\xi^\kappa=B_i{}^\kappa v^i+\lambda C^\kappa, \tag{1.8}$$

where $v^i=\eta^i=\eta_j g^{ji}$ and λ is a scalar field which never vanishes along $i(M)$, because ξ^κ is never tangent to $i(M)$. The length of ξ^κ being 1, we have from (1. 8)

$$g_{ji}\eta^j\eta^i=1-\lambda^2.$$

Since

$$(g_{ji}-\eta_j\eta_i)\left(g^{ih}+\frac{1}{\lambda^2}\eta^i\eta^h\right)=\delta_j^h,$$

we see that the contravariant components γ^{ih} of the metric γ are given by

$$\gamma^{ih}=g^{ih}+\frac{1}{\lambda^2}\eta^i\eta^h. \tag{1.9}$$

The transforms $\varphi_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $\varphi_\lambda{}^\kappa$ and the transform $\varphi_\lambda{}^\kappa C^\lambda$ of C^λ by $\varphi_\lambda{}^\kappa$ can be respectively expressed as

$$\varphi_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa, \tag{1.10}$$

$$\varphi_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa, \tag{1.11}$$

where $f_i{}^h$ is a tensor field of type (1, 1), u_i a 1-form of M and $u^i = u_j g^{ji}$.

From (1. 1), (1. 5), (1. 8), (1. 10) and (1. 11), we find [13], [14],

$$\begin{aligned} &f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h, \\ &u_t f_i{}^t = \lambda v_i, \qquad f_t{}^h u^t = -\lambda v^h, \\ &v_t f_i{}^t = -\lambda u_i, \qquad f_t{}^h v^t = \lambda u^h, \\ &u_t u^t = v_t v^t = 1 - \lambda^2, \qquad u_t v^t = 0, \\ &g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i, \end{aligned} \tag{1.12}$$

that is, M admits an (f, g, u, v, λ)-structure [13].

We can easily see that

$$f_{ji} = f_j{}^t g_{ti}$$

is skew-symmetric with respect to j and i.

Substituting (1. 8) into (1. 2), we find

$$\varphi_\lambda{}^\kappa B_i{}^\lambda = (F_i{}^h + \alpha_i v^h) B_h{}^\kappa + \lambda \alpha_i C^\kappa.$$

Comparing this with (1. 10), we find

$$F_i{}^h + \alpha_i v^h = f_i{}^h, \qquad u_i = \lambda \alpha_i,$$

and consequently

$$F_i{}^h = f_i{}^h - \frac{1}{\lambda} u_i v^h, \tag{1.13}$$

$$\alpha_i = \frac{1}{\lambda} u_i, \tag{1.14}$$

from which, using (1. 4) and (1. 12),

$$u_t F_i{}^t = \lambda v_i, \qquad F_t{}^h u^t = -\frac{1}{\lambda} v^h, \tag{1.15}$$

$$v_t F_i{}^t = -\frac{1}{\lambda} u_i, \qquad F_t{}^h v^t = \lambda u^h \tag{1.16}$$

We also have, using (1. 7),

(1. 17) $$F_i^t \gamma_{th} = f_{ih}.$$

We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_i the operator of covariant differentiation with respect to these Christoffel symbols.

We denote by $\Gamma_j{}^h{}_i$ the Christoffel symbols formed with γ_{ji}. Substituting (1. 7) and (1. 9) into

$$\Gamma_j{}^h{}_i = \frac{1}{2}(\partial_j \gamma_{it} + \partial_i \gamma_{jt} - \partial_t \gamma_{ji})\gamma^{th},$$

we get

$$\Gamma_j{}^h{}_i = \left\{ {h \atop j\ i} \right\} - \frac{1}{2}\{(\nabla_j v_t - \nabla_t v_j)v_i + (\nabla_i v_t - \nabla_t v_i)v_j\}g^{th}$$

$$- \frac{1}{2\lambda^2}\{(\nabla_j v_i + \nabla_i v_j) + (\nabla_j v_t - \nabla_t v_j)v^t v_i + (\nabla_i v_t - \nabla_t v_i)v^t v_j\}v^h.$$

Therefore, we have the following

LEMMA. *The connection $\Gamma_j{}^h{}_i$ with respect to γ_{ji} is given by*

(1. 18) $$\Gamma_j{}^h{}_i = \left\{ {h \atop j\ i} \right\} - \frac{1}{2}\{(\nabla_j v_t - \nabla_t v_j)v_i + (\nabla_i v_t - \nabla_t v_i)v_j\}\gamma^{th} - \frac{1}{2\lambda^2}(\nabla_j v_i + \nabla_i v_j)v^h.$$

For the hypersurface $i(M)$, the equations of Gauss and Weingarten are respectively

(1. 19) $$\nabla_j B_i^\kappa = h_{ji} C^\kappa,$$

(1. 20) $$\nabla_j C^\kappa = -h_j^i B_i^\kappa,$$

where h_{ji} is the second fundamental tensor of $i(M)$ and $h_j^i = h_{jt} g^{ti}$.

Differentiating (1. 10) covariantly along $i(M)$ and taking account of (1. 19) and (1. 20), we have

(1. 21) $$(\tilde{\nabla}_\mu \varphi_\lambda{}^\kappa) B_j{}^\mu B_i{}^\lambda - h_{ji} u^h B_h{}^\kappa = (\nabla_j f_i{}^h - h_j{}^h u_i) B_h{}^\kappa + (\nabla_j u_i + h_{jt} f_i{}^t) C^\kappa,$$

where $\tilde{\nabla}_\mu$ is the operator of covariant differentiation with respect to $G_{\mu\lambda}$ of $\tilde{M}$.

Similarly, we have, from (1. 8),

(1. 22) $$(\tilde{\nabla}_\lambda \xi^\kappa) B_j{}^\lambda = (\nabla_j v^h - \lambda h_j{}^h) B_h{}^\kappa + (\nabla_j \lambda + h_{jt} v^t) C^\kappa.$$

§ 2. Transversal hypersurface of a cosymplectic manifold.

An almost contact metric manifold $\tilde{M}(\varphi_\lambda{}^\kappa, \xi^\kappa, \eta_\lambda, G_{\mu\lambda})$ is said to be cosymplectic if the 2-form $\Phi_{\mu\lambda} = \varphi_\mu{}^\alpha G_{\alpha\lambda}$ and the 1-form η_λ are both closed. It is known [1] that the cosymplectic structure is characterized by

(2. 1) $$\tilde{\nabla}_\mu \varphi_\lambda{}^\kappa = 0, \qquad \tilde{\nabla}_\mu \xi^\kappa = 0.$$

In this section, we consider a transversal hypersurface $i(M)$ of a cosymplectic manifold.

From (1. 21), (1. 22) and (2. 1), we find

$$\nabla_j f_i^{\ h} = -h_{ji}u^h + h_j^{\ h}u_i, \tag{2. 2}$$

$$\nabla_j u_i = -h_{jt}f_i^{\ t}, \tag{2. 3}$$

$$\nabla_j v_i = \lambda h_{ji}, \tag{2. 4}$$

$$\nabla_j \lambda = -h_{jt}v^t. \tag{2. 5}$$

Differentiating (1. 13) covariantly and using (2. 2)~(2. 5), we find

$$\nabla_j F_i^{\ h} = -h_{ji}u^h + \frac{1}{\lambda} h_{jt}F_i^{\ t}v^h. \tag{2. 6}$$

Using (2. 6), we compute the Nijenhuis tensor

$$[F, F]_{ji}^{\ \ h} = F_j^{\ t}\nabla_t F_i^{\ h} - F_i^{\ t}\nabla_t F_j^{\ h} - (\nabla_j F_i^{\ t} - \nabla_i F_j^{\ t})F_t^{\ h},$$

and find

$$[F, F] = 0. \tag{2. 7}$$

From (2. 2), we have

$$\nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji} = 0,$$

which shows that the exterior differential of the 2-form

$$F_i^{\ t}\gamma_{th}$$

vanishes. Thus the almost Hermitian structure (F, γ) is Kählerian, and consequently denoting by ∇_i^* the operator of covariant differentiation with respect to the Christoffel symbols $\Gamma_j{}^h{}_i$ formed with γ_{ji}, we have, from (1. 18) and (2. 4),

$$\nabla_j^* F_i^{\ h} = 0, \tag{2. 8}$$

$$\Gamma_j{}^h{}_i = \left\{ {h \atop j\ i} \right\} - \frac{1}{\lambda} h_{ji}v^h. \tag{2. 9}$$

We denote by $R_{kji}{}^h$ the curvature tensor of the connection $\Gamma_j{}^h{}_i$, that is, we put

$$R_{kji}{}^h = \partial_k \Gamma_j{}^h{}_i - \partial_j \Gamma_k{}^h{}_i + \Gamma_k{}^h{}_t \Gamma_j{}^t{}_i - \Gamma_j{}^h{}_t \Gamma_k{}^t{}_i. \tag{2. 10}$$

In general, if

$$\Gamma_j{}^h{}_i = \left\{ {h \atop j\ i} \right\} + T_{ji}{}^h, \tag{2. 11}$$

$T_{ji}{}^h$ being a tensor field of type (1, 2), we have

(2. 12) $$R_{kji}{}^{h}=K_{kji}{}^{h}+\nabla_{k}T_{ji}{}^{h}-\nabla_{j}T_{ki}{}^{h}+T_{kt}{}^{h}T_{ji}{}^{t}-T_{jt}{}^{h}T_{ki}{}^{t},$$

where $K_{kji}{}^{h}$ is the curvature tensor of the Levi-Civita connection $\{{}_{j}{}^{h}{}_{i}\}$.

Differentiating

(2. 13) $$T_{ji}{}^{h}=-\frac{1}{\lambda}h_{ji}v^{h}$$

covariantly and using (2. 4) and (2. 5), we find

(2. 14) $$\nabla_{k}T_{ji}{}^{h}=-\frac{1}{\lambda^{2}}h_{kt}v^{t}h_{ji}v^{h}-\frac{1}{\lambda}(\nabla_{k}h_{ji})v^{h}-h_{ji}h_{k}{}^{h}.$$

Substituting (2. 13) and (2. 14) into (2. 12), we obtain

(2. 15) $$R_{kji}{}^{h}=K_{kji}{}^{h}-(h_{k}{}^{h}h_{ji}-h_{j}{}^{h}h_{ki})-\frac{1}{\lambda}(\nabla_{k}h_{ji}-\nabla_{j}h_{ki})v^{h}.$$

On the other hand, differentiating (2. 4) covariantly and using (2. 5), we find

$$\nabla_{k}\nabla_{j}v_{i}=-h_{kt}v^{t}h_{ji}+\lambda\nabla_{k}h_{ji},$$

from which, using Ricci identity,

$$\nabla_{k}\nabla_{j}v_{i}-\nabla_{j}\nabla_{k}v_{i}=-K_{kji}{}^{h}v_{h},$$

we have

$$K_{kji}{}^{t}v_{t}=(h_{k}{}^{t}h_{ji}-h_{j}{}^{t}h_{ki})v_{t}-\lambda(\nabla_{k}h_{ji}-\nabla_{j}h_{ki}),$$

or

(2. 16) $$\nabla_{k}h_{ji}-\nabla_{j}h_{ki}=-\frac{1}{\lambda}(K_{kji}{}^{t}-h_{k}{}^{t}h_{ji}+h_{j}{}^{t}h_{ki})v_{t}.$$

Substituting (2. 16) into (2. 15), we find

$$R_{kji}{}^{h}=(K_{kji}{}^{t}-h_{k}{}^{t}h_{ji}+h_{j}{}^{t}h_{ki})\left(\delta_{t}^{h}+\frac{1}{\lambda^{2}}v_{t}v^{h}\right)$$

that is,

$$R_{kji}{}^{h}=(K_{kjit}-h_{kt}h_{ji}+h_{jt}h_{ki})\gamma^{th}.$$

Thus we have

(2. 17) $$R_{kjih}=K_{kjih}-(h_{kh}h_{ji}-h_{jh}h_{ki}),$$

where

$$R_{kjih}=R_{kji}{}^{t}\gamma_{th}.$$

Applying the equations of Gauss

$$K_{kjih}=\tilde{K}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda B_h{}^\kappa+(h_{kh}h_{ji}-h_{jh}h_{ki})$$

of $i(M)$ in $\tilde{M}$, where $\tilde{K}_{\nu\mu\lambda\kappa}$ is the curvature tensor of the connection $\{_\mu{}^\kappa{}_\lambda\}$ of $\tilde{M}$, we obtain, from (2. 17),

$$R_{kjih}=\tilde{K}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda B_h{}^\kappa. \tag{2. 18}$$

In a cosymplectic manifold of constant φ-holomorphic sectional curvature, the curvature tensor $\tilde{K}_{\nu\mu\lambda\kappa}$ has the form [6]

$$\begin{aligned}\tilde{K}_{\nu\mu\lambda\kappa}=\frac{k}{4}\{&(G_{\nu\kappa}-\eta_\nu\eta_\kappa)(G_{\mu\lambda}-\eta_\mu\eta_\lambda)\\ &-(G_{\mu\kappa}-\eta_\mu\eta_\kappa)(G_{\nu\lambda}-\eta_\nu\eta_\lambda)+\varphi_{\nu\kappa}\varphi_{\mu\lambda}-\varphi_{\mu\kappa}\varphi_{\nu\lambda}-2\varphi_{\nu\mu}\varphi_{\lambda\kappa}\},\end{aligned} \tag{2. 19}$$

k being a constant.

Substituting (2. 19) into (2. 18), we have

$$R_{kjih}=\frac{k}{4}(\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}), \tag{2. 20}$$

which shows that the Kählerian manifold with complex structure F and the Hermitian metric γ is of constant holomorphic sectional curvature. Thus we have

PROPOSITION 2. 1. *If $\tilde{M}$ is a cosymplectic manifold of constant φ-holomorphic sectional curvature, then the transversal Kählerian hypersurface $i(M)$ is of constant holomorphic sectional curvature.*

§ 3. Transversal hypersurface of a Sasakian manifold.

In this section, we consider a hypersurface $i(M)$ of a Sasakian manifold $\tilde{M}$.

In a Sasakian manifold $\tilde{M}$, we have

$$\begin{aligned}&\tilde{\nabla}_\mu\varphi_\lambda{}^\kappa=-G_{\mu\lambda}\xi^\kappa+\delta_\mu^\kappa\eta_\lambda,\\ &\tilde{\nabla}_\mu\xi^\kappa=\varphi_\mu{}^\kappa.\end{aligned} \tag{3. 1}$$

Substituting (3. 1) into (1. 21) and (1. 22) and using (1. 8), (1. 10) and (1. 11), we find [13]

$$\nabla_j f_i{}^h=-h_{ji}u^h+h_j{}^h u_i-g_{ji}v^h+\delta_j^h v_i, \tag{3. 2}$$

$$\nabla_j u_i=-h_{jt}f_i{}^t-\lambda g_{ji}, \tag{3. 3}$$

$$\nabla_j v_i=f_{ji}+\lambda h_{ji}, \tag{3. 4}$$

$$\nabla_j\lambda=u_j-h_{jt}v^t. \tag{3. 5}$$

Differentiating (1. 13) covariantly and using (3. 2)~(3. 5), we find

$$\nabla_j F_i{}^h = -h_{ji}u^h - \frac{1}{\lambda} F_j{}^h u_i + \frac{1}{\lambda} h_{jt} F_i{}^t v^h + \delta_j^h v_i. \tag{3. 6}$$

Using (3. 6), we compute the Nijenhuis tensor $[F, F]$ formed with F and find

$$[F, F]=0. \tag{3. 7}$$

From (3. 2), we have

$$\nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji} = 0,$$

which shows that the exterior differential of the 2-form $F_i{}^t \gamma_{th}$ vanishes. Thus the almost Hermitian structure (F, γ) is Kählerian and we have

$$\nabla_j^* F_i{}^h = 0. \tag{3. 8}$$

Substituting (3. 4) into (1. 18), we find

$$\Gamma_j{}^h{}_i = \left\{ {h \atop j\ i} \right\} - F_j{}^h v_i - F_i{}^h v_j - \frac{1}{\lambda} h_{ji} v^h. \tag{3. 9}$$

Thus we put

$$T_{ji}{}^h = -F_j{}^h v_i - F_i{}^h v_j - \frac{1}{\lambda} h_{ji} v^h, \tag{3. 10}$$

and compute

$$\begin{aligned} &\nabla_k T_{ji}{}^h - \nabla_j T_{ki}{}^h + T_{kt}{}^h T_{ji}{}^t - T_{jt}{}^h T_{ki}{}^t \\ &= F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj} F_i{}^h - (\delta_k^h v_j - \delta_j^h v_k) v_i \\ &\quad - (h_k{}^h h_{ji} - h_j{}^h h_{ki}) - \frac{1}{\lambda} (\nabla_k h_{ji} - \nabla_j h_{ki}) v^h. \end{aligned} \tag{3. 11}$$

On the other hand, differentiating (3. 4) covariantly and taking account of (3. 2) and (3. 5), we find

$$\nabla_k \nabla_j v_i = -h_{kj} u_i + h_{ki} u_j - g_{kj} v_i + g_{ki} v_j + (u_k - h_{kt} v^t) h_{ji} + \lambda \nabla_k h_{ji},$$

from which, using Ricci identity,

$$K_{kji}{}^t v_t = v_k g_{ji} - v_j g_{ki} + (h_{kt} h_{ji} - h_{jt} h_{ki}) v^t - \lambda (\nabla_k h_{ji} - \nabla_j h_{ki}),$$

or

$$\nabla_k h_{ji} - \nabla_j h_{ki} = \frac{1}{\lambda} \{ v_k g_{ji} - v_j g_{ki} - (K_{kji}{}^t - h_k{}^t h_{ji} + h_j{}^t h_{ki}) v_t \}. \tag{3. 12}$$

Substituting (3. 12) into (3. 11), we find

$$\begin{aligned}&\nabla_k T_{ji}{}^h-\nabla_j T_{ki}{}^h+T_{kt}{}^h T_{ji}{}^t-T_{jt}{}^h T_{ki}{}^t\\&=-(\delta_k^h v_j-\delta_j^h v_k)v_i+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h\\&-\frac{1}{\lambda^2}(v_k g_{ji}-v_j g_{ki})v^h+\frac{1}{\lambda^2}(K_{kji}{}^t-h_k{}^t h_{ji}+h_j{}^t h_{ki})v_t v^h\\&-h_k{}^h h_{ji}+h_j{}^h h_{ki}.\end{aligned}$$

Therefore, from (2. 12), we have

$$\begin{aligned}R_{kji}{}^h=&(K_{kjit}-h_{kt}h_{ji}+h_{jt}h_{ki})\left(g^{th}+\frac{1}{\lambda^2}v^t v^h\right)\\&-(\delta_k^h v_j-\delta_j^h v_k)v_i-\frac{1}{\lambda^2}(v_k g_{ji}-v_j g_{ki})v^h\\&+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h.\end{aligned}\tag{3. 13}$$

Transvecting (3. 13) with $\gamma_{hl}=g_{hl}-v_h v_l$ and taking account of (1. 9) and (1. 17), we have

$$\begin{aligned}R_{kjil}=&K_{kjil}-h_{kl}h_{ji}+h_{jl}h_{ki}\\&-\{(g_{kl}-v_k v_l)v_j-(g_{jl}-v_j v_l)v_k\}v_i\\&-\frac{1}{\lambda^2}(v_k g_{ji}-v_j g_{ki})v^h(g_{hl}-v_h v_l)\\&+F_{kl}F_{ji}-F_{jl}F_{ki}-2F_{kj}F_{il},\end{aligned}$$

or

$$\begin{aligned}R_{kjih}=&K_{kjih}-(h_{kh}h_{ji}-h_{jh}h_{ki})\\&-(g_{kh}v_j-g_{jh}v_k)v_i-(v_k g_{ji}-v_j g_{ki})v_h\\&+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}.\end{aligned}\tag{3. 14}$$

On the other hand, we have

$$\begin{aligned}&-(g_{kh}v_j-g_{jh}v_k)v_i-(v_k g_{ji}-v_j g_{ki})v_h\\&=-(g_{kh}g_{ji}-g_{jh}g_{ki})+(\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}).\end{aligned}\tag{3. 15}$$

Substituting (3. 15) into (3. 14), we find

$$\begin{aligned}&R_{kjih}-(\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih})\\&=(K_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki})-(g_{kh}g_{ji}-g_{jh}g_{ki}).\end{aligned}\tag{3. 16}$$

Applying the equations of Gauss of $i(M)$ in $\tilde{M}$, we obtain, from (3. 16),

$$\begin{aligned}(3.17)\qquad & R_{kjih}-(\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}) \\ & =(\tilde{K}_{\nu\mu\lambda\kappa}-G_{\nu\kappa}G_{\mu\lambda}+G_{\mu\kappa}G_{\nu\lambda})B_k^{\nu}B_j^{\mu}B_i^{\lambda}B_h^{\kappa}.\end{aligned}$$

In a Sasakian manifold of constant C-holomorphic sectional curvature, the curvature tensor has the form [9]

$$\begin{aligned}\tilde{K}_{\nu\mu\lambda\kappa}= & \frac{k+3}{4}(G_{\nu\kappa}G_{\mu\lambda}-G_{\mu\kappa}G_{\nu\lambda}) \\ (3.18)\qquad & -\frac{k-1}{4}\{\eta_{\nu}(G_{\mu\lambda}\eta_{\kappa}-G_{\mu\kappa}\eta_{\lambda})-\eta_{\mu}(G_{\nu\lambda}\eta_{\kappa}-G_{\nu\kappa}\eta_{\lambda}) \\ & -\varphi_{\nu\kappa}\varphi_{\mu\lambda}+\varphi_{\mu\kappa}\varphi_{\nu\lambda}+2\varphi_{\nu\mu}\varphi_{\lambda\kappa}\}.\end{aligned}$$

Substituting (3. 18) into (3. 17), we obtain

$$(3.19)\qquad R_{kjih}=\frac{k+3}{4}(\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}),$$

which shows that the Kählerian manifold with complex structure F and the Hermitian metric γ is of constant holomorphic sectional curvature. Thus we have

PROPOSITION 3. 1. *If $\tilde{M}$ is a Sasakian manifold of constant C-holomorphic sectional curvature, then the transversal Kählerian hypersurface $i(M)$ is of constant holomorphic sectional curvature.*

§4. Transversal hypersurface of a Sasakian manifold of constant curvature.

In this section we consider a hypersurface $i(M)$ of a Sasakian manifold of constant curvature.

When $\tilde{M}$ is of constant curvature 1, the curvature tensor has the form

$$(4.1)\qquad \tilde{K}_{\nu\mu\lambda\kappa}=G_{\nu\kappa}G_{\mu\lambda}-G_{\mu\kappa}G_{\nu\lambda},$$

and consequently, we have

$$(4.2)\qquad \tilde{K}_{\nu\mu\lambda\kappa}B_k^{\nu}B_j^{\mu}B_i^{\lambda}B_h^{\kappa}=g_{kh}g_{ji}-g_{jh}g_{ki}.$$

Substituting (4. 1) into (3. 17), we find

$$(4.3)\qquad R_{kjih}=\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}.$$

Thus, we have

PROPOSITION 4. 1. *If $\tilde{M}$ is a Sasakian manifold of constant curvature* 1, *then a transversal hypersurface $i(M)$ is of constant holomorphic sectional curvature which is equal to* 4.

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