KŌDAI MATH. SEM. REP.
24 (1972), 259–269

FUNCTIONAL CENTRAL LIMIT THEOREMS FOR STRICTLY STATIONARY PROCESSES SATISFYINC THE STRONG MIXING CONDITION

BY HIROSHI OODAIRA AND KEN-ICHI YOSHIHARA

1. Summary.

The object of this paper is to prove the functional central limit theorems for strictly stationary processes satisfying the strong mixing condition under the same assumptions in Ibragimov [3]. The results generalize those of Davydov [2].

2. Main results.

Let $\{\xi_n;\ n=0, \pm1, \pm2, \cdots\}$ be a strictly stationary process with $E\xi_j=0$, satisfying the strong mixing (s. m.) condition, i.e.,

$$(1)\qquad \sup_{A\in\mathfrak{M}_{-\infty}^{a},\, B\in\mathfrak{M}_{a+s}^{\infty}} |P(AB)-P(A)P(B)|=\alpha(s)\to 0 \qquad (s\to\infty),$$

where $\mathfrak{M}_a^b$ denotes the σ-algebra generated by $\{\xi_j;\ j=a, \cdots, b\}$. Write $S_n=\xi_1+\cdots+\xi_n$ and $\sigma^2=E\xi_0^2+2\sum_{j=1}^{\infty} E\xi_0\xi_j$. Let $D=D[0,1]$ be the space of functions x on $[0,1]$ that are right-continuous and have left-hand limits, and let $\mathscr{D}$ be the σ-field of Borel sets for the Skorokhod topology (cf. [1]). When $0<\sigma<\infty$, we define random elements $X_n(t)$ of D by

$$(2)\qquad X_n(t, \omega)=\frac{1}{\sigma\sqrt{n}}S_{[nt]}(\omega), \qquad 0\leq t\leq 1;\ n=1, 2, \cdots$$

The following theorems imply that functional central limit theorems hold under the same conditions of theorems 1.6 and 1.7 in [3] which assure the validity of central limit theorems.

THEOREM 1. *If ξ_j's are bounded, i.e., $|\xi_j|<C<\infty$ with probability one and if*

$$(3)\qquad \sum_{n=1}^{\infty} \alpha(n)<\infty \qquad \textit{and} \qquad \alpha(n)\leq \frac{M}{n\log n},$$

then $\sigma^2<\infty$. If $\sigma>0$ and if X_n is defined by (2), then the distribution of X_n converges weakly to Wiener measure W on $(D, \mathscr{D})$.

Received May 20, 1971.

THEOREM 2. *If* $E|\xi_j|^{2+\delta}<\infty$ *for some* $\delta>0$ *and if*

$$\sum_{n=1}^{\infty}\{\alpha(n)\}^{\delta/2+\delta}<\infty, \tag{4}$$

then $\sigma^2<\infty$. *If* $\sigma>0$, *then the distribution of* X_n *converges weakly to Wiener measure* W *on* $(D, \mathcal{D})$.

3. Proof of theorem 1.

The first half of theorem 1 is theorem 1.6 in [3]. To prove the latter half, it suffices to show that the finite dimensional distributions of X_n converges weakly to those of W and that the sequence $\{X_n\}$ is tight. The convergence of the finite dimensional distributions is easily obtained by the method in [1] or [2].

To prove tightness, it is enough to show (cf. [1]) that for any $\varepsilon>0$ there exist a $\lambda>1$ and an integer n_0 such that

$$P\left\{\max_{i\leqq n}|S_i|\geqq 3\lambda\sigma\sqrt{n}\right\}\leqq\frac{\varepsilon}{\lambda^2}\qquad(n\geqq n_0). \tag{5}$$

For any integer n $(\geqq 2)$, put $p=[n^{1/2}\log^{-3/8}n]$ and $k=[n/p]$. Since ξ_j's are bounded, so for all sufficiently large n

$$P\{|\xi_1|+\cdots+|\xi_{2p}|\geqq\lambda\sigma\sqrt{n}\}=0. \tag{6}$$

Given $\varepsilon>0$, choose λ (>1) so that

$$P\{|S_i|>\lambda\sigma\sqrt{i}\}\leqq\frac{\varepsilon}{3\lambda^2}\qquad\text{for all } i, \tag{7}$$

which is possible because of uniform integrability of $\{S_n^2/n\}$ (cf. theorem 5.4, [1]).

If $E_j=\{\max_{i<j}|S_i|<3\lambda\sigma\sqrt{n}\leqq|S_j|\}$, then

$$\begin{aligned}
&P\left(\max_{i\leqq n}|S_i|\geqq 3\lambda\sigma\sqrt{n}\right)\\
&\leqq P(|S_n|\geqq\lambda\sigma\sqrt{n})+P\left(\bigcup_{j=1}^{n}[E_j\cap\{|S_n-S_j|\geqq 2\lambda\sigma\sqrt{n}\}]\right)\\
&\leqq P(|S_n|\geqq\lambda\sigma\sqrt{n})\\
&\quad+\sum_{i=0}^{k-2}P\left(\bigcup_{j=1}^{p}[E_{ip+j}\cap\{|S_n-S_{ip+j}|\geqq 2\lambda\sigma\sqrt{n}\}]\right)\\
&\quad+\sum_{j=(k-1)p+1}^{n}P(|S_n-S_j|\geqq 2\lambda\sigma\sqrt{n})
\end{aligned} \tag{8}$$

$$\leqq P(|S_n|\geqq\lambda\sigma\sqrt{n})$$
$$+\sum_{i=0}^{k-2}\left\{P\left(\bigcup_{j=1}^{p}[E_{ip+j}\cap\{|S_n-S_{(i+2)p}|\geqq\lambda\sigma\sqrt{n}\}]\right)\right.$$
$$+\sum_{j=1}^{p}P(|S_{(i+2)p}-S_{ip+j}|\geqq\lambda\sigma\sqrt{n})$$
$$+\sum_{j=(k-1)p+1}^{n}P(|\xi_1|+\cdots+|\xi_{n-j}|\geqq 2\lambda\sigma\sqrt{n})$$
$$\leqq P(|S_n|\geqq\lambda\sigma\sqrt{n})$$
$$+\sum_{i=0}^{k-2}P\left(\left[\bigcup_{j=1}^{p}E_{ip+j}\right]\cap\{|S_n-S_{(i+2)p}|\geqq\lambda\sigma\sqrt{n}\}\right)$$
$$+2n\,P(|\xi_1|+\cdots+|\xi_{2p}|\geqq\lambda\sigma\sqrt{n}).$$

As $\bigcup_{j=1}^{p}E_{ip+j}\in\mathfrak{M}_{-\infty}^{(i+1)p}$ and $\{|S_n-S_{(i+2)p}|\geqq\lambda\sigma\sqrt{n}\}\in\mathfrak{M}_{(i+2)p}^{\infty}$, so we have, using (7),

$$\sum_{i=0}^{k-2}P\left(\left[\bigcup_{j=1}^{p}E_{ip+j}\right]\cap\{|S_n-S_{(i+2)p}|\geqq\lambda\sigma\sqrt{n}\}\right)$$

(9)
$$\leqq\sum_{i=0}^{k-2}P\left(\bigcup_{j=1}^{p}E_{ip+j}\right)P(|S_n-S_{(i+2)p}|\geqq\lambda\sigma\sqrt{n})+k\alpha(p)$$
$$\leqq\sum_{i=0}^{k-2}P\left(\bigcup_{j=1}^{p}E_{ip+j}\right)P(|S_{n-(i+2)p}|\geqq\lambda\sigma\sqrt{n-(i+2)p})+k\alpha(p)$$
$$\leqq\frac{\varepsilon}{3\lambda^2}+k\alpha(p).$$

Furthermore

(10)
$$\lim_{n\to\infty}k\alpha(p)\leqq M\lim_{n\to\infty}\frac{n}{[n^{1/2}\log^{-3/8}n]^2\log[n^{1/2}\log^{-3/8}n]}=0.$$

Hence, combining (6) and (7), we have (5). Thus the proof is completed.

4. Proof of theorem 2.

The first half is theorem 1.7 in [3]. We proceed as the proof of theorem 1. Since the convergence of the finite dimensional distributions is easily proved (cf. [2]), we need only to verify the tightness of $\{X_n\}$.

If $f_N(x)=x$ for $|x|\leqq N$ and $=0$ for $|x|>N$, then the process $\{f_N(\xi_j)\}$ satisfies the s. m. condition (1) at least with the same function $\alpha(n)$. Put $N=n^{1/2(1+\delta)}$ and $d_n^2=(1/n)E[\sum_{j=1}^{n}(f_N(\xi_j)-Ef_N(\xi_j))]^2$. It is obvious that $d_n^2\to\sigma^2$ $(n\to\infty)$. Let $\bar{f}_N(x)=x-f_N(x)$. Then

$$
\begin{aligned}
&P\Big(\max_{l\leqq n}|S_l|\geqq 2\lambda\sigma\sqrt{n}\Big)\\
&\leqq P\Big(\max_{l\leqq n}\Big|\sum_{j=1}^{l}(f_N(\xi_j)-Ef_N(\xi_j))\Big|\geqq\lambda\sigma\sqrt{n}\Big)\\
&\quad+P\Big(\max_{l\leqq n}\Big|\sum_{j=1}^{l}(\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j))\Big|\geqq\lambda\sigma\sqrt{n}\Big)\\
&\leqq P\Big(\max_{l\leqq n}\Big|\sum_{j=1}^{l}(f_N(\xi_j)-Ef_N(\xi_j))\Big|\geqq\lambda\sigma\sqrt{n}\Big)\\
&\quad+P\Big(\sum_{j=1}^{n}|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\geqq\lambda\sigma\sqrt{n}\Big).
\end{aligned}\tag{11}
$$

Since $\alpha(n)$ is monotone decreasing, it follows from (4) that $\alpha(n)=o(n^{-(2+\delta)/\delta})$. Thus if we put $p=[n^{\delta/2(1+\delta)}]$ and $k=[n/p]$, then

$$
\begin{aligned}
k\alpha(p)&\sim ko(p^{-(2+\delta)/\delta})\\
&\sim n^{(2+\delta)/2(1+\delta)}o(n^{-(2+\delta)/2(1+\delta)})\\
&\to 0 \qquad (n\to\infty).
\end{aligned}\tag{12}
$$

Now, for all sufficiently large λ

$$
\sum_{j=1}^{p}|f_N(\xi_j)-Ef_N(\xi_j)|\leqq 2pN<\frac{1}{2}\lambda\sigma\sqrt{n}\tag{13}
$$

with probability one, and so

$$
P\Big(\sum_{j=1}^{p}|f_N(\xi_j)-Ef_N(\xi_j)|\geqq\lambda\sigma\sqrt{n}\Big)=0\tag{14}
$$

for all sufficiently large λ. Thus, applying the method of the proof of theorem 1 to this case, we obtain that for any $\varepsilon>0$ there exists a λ_0 such that

$$
P\Big(\max_{l\leqq n}\Big|\sum_{j=1}^{l}(f_N(\xi_j)-Ef_N(\xi_j))\Big|\geqq\lambda\sigma\sqrt{n}\Big)\leqq\frac{\varepsilon}{2\lambda^2}\qquad(\lambda\geqq\lambda_0).\tag{15}
$$

Next, we shall estimate $P(\sum_{j=1}^{n}|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\geqq\lambda\sigma\sqrt{n})$. Using the inequality in the corollary to lemma 2.1 in [2] and Minkowsky's inequality,

$$
\begin{aligned}
&E|\bar{f}_N(\xi_0)-E\bar{f}_N(\xi_0)|\cdot|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\\
&\leqq E|\bar{f}_N(\xi_0)-E\bar{f}_N(\xi_0)|\cdot E|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\\
&\quad+12\{E|\bar{f}_N(\xi_0)-E\bar{f}_N(\xi_0)|^{2+\delta}\}^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)}
\end{aligned}\tag{16}
$$

$$\leqq 4\{E|\bar{f}_N(\xi_0)|\}^2$$
$$+48\{E|\bar{f}_N(\xi_0)|^{2+\delta}\}^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)}$$
$$\leqq (4/N^{2(1+\delta)})\{E|\bar{f}_N(\xi_0)|^{2+\delta}\}^2$$
$$+48\{E|\bar{f}_N(\xi_0)|^{2+\delta}\}^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)}$$
$$=(4/n)\gamma_N^2+48\gamma_N^{2/(2+\delta)}\{\alpha(j)\}^{\delta/(2+\delta)}$$

for $j=1, 2, \cdots, n$, where $\gamma_N=E|\bar{f}_N(\xi_0)|^{2+\delta}$. Therefore

$$\frac{1}{\sigma_n^2}E\left(\sum_{j=1}^{n}|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\right)^2$$

(17)
$$\leqq \frac{1}{\sigma^2}\left\{E|\bar{f}_N(\xi_0)-E\bar{f}_N(\xi_0)|^2\right.$$
$$\left.+2\sum_{j=1}^{n}E|\bar{f}_N(\xi_0)-E\bar{f}_N(\xi_0)|\cdot|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\right\}$$
$$\leqq \frac{1}{\sigma^2}\left\{\gamma_N/N^\delta+8\gamma_N^2+96\gamma_N^{\delta/(2+\delta)}\sum_{j=1}^{\infty}\{\alpha(j)\}^{\delta/(2+\delta)}\right\}.$$

As $\sum_{j=1}^{\infty}\{\alpha(j)\}^{\delta/(2+\delta)}<\infty$, so the last part of the above inequality tends to zero when $n\to\infty$. Consequently, for all n sufficiently large

$$P\left(\sum_{j=1}^{n}|\bar{f}_N(\xi_j)-E\bar{f}_N(\xi_j)|\geqq\lambda\sigma\sqrt{n}\right)\leqq\frac{\varepsilon}{2\lambda^2}. \tag{18}$$

From (15) and (18) we deduce that for any $\varepsilon>0$, there exist a λ and an n_0 such that

$$P\left(\max_{l\leqq n}|S_l|\geqq 2\lambda\sigma\sqrt{n}\right)\leqq\frac{\varepsilon}{\lambda^2}\qquad(n\geqq n_0), \tag{19}$$

which implies the tightness of $\{X_n\}$. Thus we have the theorem.

5. Randomly selected partial sums.

For each n, let ν_n be a positive integer-valued random variable defined on the same probability space as the ξ_n. As in [1] define Y_n by

$$Y_n(t, \omega)=\frac{1}{\sigma\sqrt{\nu_n(\omega)}}S_{[\nu_n(\omega)t]}(\omega)=X_{\nu_n(\omega)}(t, \omega),\qquad(0\leqq t\leqq 1). \tag{20}$$

THEOREM 3. *Suppose that the hypotheses of theorem* 1 (*or* 2) *are satisfied and that*

(21) $$\frac{\nu_n}{a_n} \xrightarrow{P} \theta,$$

where θ is a positive random variable and $a_n \to \infty$. If $\sigma > 0$, then the distribution of Y_n converges weakly to the distribution of W.

Proof. The proof of theorem 17.2 in [1] can be carried over to this case in exactly the same way.

6. Functions of processes satisfying the s. m. condition.

As in [3], let $\{\xi_j;\ j=0, \pm 1, \pm 2, \cdots\}$ be strictly stationary and strong mixing. Let f be a measurable mapping from the space of doubly infinite sequences $(\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots)$ of real numbers into the real line. Define random variables

(22) $$f_j = f(\cdots, \xi_{j-1}, \xi_j, \xi_{j+1}, \cdots),$$

where ξ_j occupies the 0-th place in the argument of f. It is obvious that $\{f_j\}$ is strictly stationary but need not be strong mixing. We shall obtain limit theorems for $\{f_j\}$ under the analogous assumptions in [3].

Write $S_n = f_1 + \cdots + f_n$, $\sigma^2 = Ef_0^2 + 2\sum_{j=1}^{\infty} Ef_0 f_j$. When $0 < \sigma^2 < \infty$, define Z_n by

(23) $$Z_n(t, \omega) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(\omega), \qquad 0 \leqq t \leqq 1.$$

THEOREM 4. *Let the stationary process $\{\xi_j\}$ satisfy the s. m. condition* (1), *let the random variable f be measurable with respect to $\mathfrak{M}_{-\infty}^{\infty}$, and let the process $\{f_j\}$ be obtained from $\{\xi_j\}$ as described above. Suppose that the following conditions are satisfied.*

1. $Ef = 0$ *and* $|f| < C < \infty$ *with probability one,*
2. $\sum_{k=1}^{\infty} \{E|f - E\{f|\mathfrak{M}_{-k}^{k}\}|^2\}^{1/2} < \infty,$
3. $\sum_{k=1}^{\infty} \alpha(k) < \infty$ *and* $\alpha(k) \leqq \dfrac{M}{k \log k}.$

Then $\sigma^2 < \infty$. If $\sigma > 0$ and Z_n is defined by (23), *then the distribution of Z_n converges weakly to Wiener measure W on $(D, \mathscr{D})$.*

Proof. The first part is a result in [3]. Since the convergence of finite dimensional distributions can be proved by the method used in the proof of theorem 21.1 in [1], it is enough to prove the tightness.

The proof is in the same line as that of theorem 21.1 in [1]. Let $p = [n^{1/2} \log^{-3/8} n]$ and $k = [n/p]$. Define $U_i = E\{S_{i-2p} | \mathfrak{M}_{-\infty}^{i-p}\}$ and $V_i = E\{S_n - S_{i+2p} | \mathfrak{M}_{i+p}^{\infty}\}$. In these definitions we adopt the conventions that $S_{i-2p} = 0$ if $i < 2p$ and $S_n - S_{i+2p} = 0$ if $i + 2p > n$. If we put

$$\mu(p)=\sum_{k=p}^{\infty}\{E|f_0-E\{f_0|\mathfrak{M}_{-k}^{k}\}|^2\}^{1/2}, \tag{24}$$

then for all k and i

$$E|S_k-E\{S_k|\mathfrak{M}_{-\infty}^{k+p}\}|^2\leqq\mu^2(p), \tag{25}$$

$$E|U_i-S_i|^2\leqq 2ES_{2p}^2+2\mu^2(p) \tag{26}$$

and

$$E|V_i-(S_n-S_i)|^2\leqq 2ES_{2p}^2+2\mu^2(p). \tag{27}$$

Since $|f_0|<C$ with probability one, so

$$P\left(|f_1|+\cdots+|f_{2p}|\geqq\frac{1}{2}\lambda\sigma\sqrt{n}\right)=0 \tag{28}$$

for all sufficiently large n. Thus, for all i

$$\begin{aligned} &P(|S_i-U_i|\geqq\lambda\sigma\sqrt{n})\\ &\leqq P\left(|S_{i-2p}-E\{S_{i-2p}|\mathfrak{M}_{-\infty}^{i-p}\}|\geqq\frac{1}{2}\lambda\sigma\sqrt{n}\right)\\ &\quad+P\left(|f_1|+\cdots+|f_{2p}|\geqq\frac{1}{2}\lambda\sigma\sqrt{n}\right)\\ &\leqq\frac{4\mu^2(p)}{\lambda^2\sigma^2 n} \end{aligned} \tag{29}$$

and similarly for all i

$$P(|V_i-(S_n-S_i)|\geqq\lambda\sigma\sqrt{n})\leqq\frac{4\mu^2(p)}{\lambda^2\sigma^2 n}. \tag{30}$$

Since $\{S_n^2/n\}$ is uniformly integrable (cf. the proof of theorem 21.1, [1]), there is a $\lambda>1$ such that

$$P(|S_k|\geqq\lambda\sigma\sqrt{k})\leqq\frac{\varepsilon}{3\lambda^2} \tag{31}$$

for all k. By (29)

$$P\left(\max_{i\leqq n}|S_i|\geqq 6\lambda\sigma\sqrt{n}\right)\leqq P\left(\max_{i\leqq n}|U_i|\geqq 5\lambda\sigma\sqrt{n}\right)+\frac{4\mu^2(p)}{\lambda^2\sigma^2}. \tag{32}$$

Let $E_j=\{\max_{i<j}|U_i|<5\lambda\sigma\sqrt{n}\leqq|U_j|\}$. As $E_j\in\mathfrak{M}_{-\infty}^{j-p}$ and V_{j+2p} is measurable with respect to $\mathfrak{M}_{j+3p}^{\infty}$, so using (30) and (31),

$$P\left(\bigcup_{j=1}^{n-1}[E_j\cap\{|V_j|\geqq 2\lambda\sigma\sqrt{n}\}]\right)$$

$$\leqq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^{p}[E_{ip+j}\cap\{|V_{(i+2)p}|\geqq \lambda\sigma\sqrt{n}\}]\right)$$

$$+nP(|f_1|+\cdots+|f_{2p}|\geqq\lambda\sigma\sqrt{n})$$

$$=\sum_{i=0}^{k-2} P\left(\left[\bigcup_{j=1}^{p}E_{ip+j}\right]\cap\{|V_{(i+2)p}|\geqq \lambda\sigma\sqrt{n}\}\right) \tag{33}$$

$$\leqq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^{p}E_{ip+j}\right)P(|V_{(i+2)p}|\geqq\lambda\sigma\sqrt{n})+k\alpha(p)$$

$$\leqq \sum_{i=0}^{k-2} P\left(\bigcup_{j=1}^{p}E_{ip+j}\right)\left\{P(|S_{n-(i+2)p}|\geqq\lambda\sigma\sqrt{n-(i+2)p})+\frac{4\mu^2(p)}{\lambda^2\sigma^2 n}\right\}+k\alpha(p)$$

$$\leqq \frac{\varepsilon}{3\lambda^2}+\frac{4\mu^2(p)}{\lambda^2\sigma^2 n}+k\alpha(p).$$

Accordingly, from (29), (30) and (33),

$$P\left(\max_{i\leqq n}|U_i|\geqq 5\lambda\sigma\sqrt{n}\right)$$

$$\leqq P(|S_n|\geqq\lambda\sigma\sqrt{n})+P\left(\bigcup_{j=1}^{n-1}[E_j\cap\{|S_n-U_j|\geqq 4\lambda\sigma\sqrt{n}\}]\right)$$

$$\leqq P(|S_n|\geqq\lambda\sigma\sqrt{n})+\sum_{j=1}^{n-1}P(|S_n-S_j-V_j|\geqq\lambda\sigma\sqrt{n})$$

$$+P\left(\bigcup_{j=1}^{n-1}[E_j\cap\{|V_j|\geqq 2\lambda\sigma\sqrt{n}\}]\right)+\sum_{j=1}^{n-1}P(|S_j-U_j|\geqq\lambda\sigma\sqrt{n}) \tag{34}$$

$$\leqq\frac{\varepsilon}{3\lambda^2}+\frac{4\mu^2(p)}{\lambda^2\sigma^2}$$

$$+\left(\frac{\varepsilon}{3\lambda^2}+\frac{4\mu^2(p)}{\lambda^2\sigma^2 n}+k\alpha(p)\right)+\frac{4\mu^2(p)}{\lambda^2\sigma^2}$$

$$=\frac{\varepsilon}{3\lambda^2}+4\left(2+\frac{1}{n}\right)\frac{\mu^2(p)}{\lambda^2\sigma^2}+k\alpha(p)$$

and so from (32) and (34)

$$P\left(\max_{i\leqq n}|S_i|\geqq 6\lambda\sigma\sqrt{n}\right)\leqq\frac{2\varepsilon}{3\lambda^2}+4\left(3+\frac{1}{n}\right)\frac{\mu^2(p)}{\lambda^2\sigma^2}+k\alpha(p). \tag{35}$$

Since $4(3+n^{-1})\mu^2(p)/\lambda^2\sigma^2\to 0$ and $k\alpha(p)\to 0$ as $n\to\infty$, we have

$$P\left(\max_{i\leq n}|S_i|\geqq 6\lambda\sigma\sqrt{n}\right)\leqq\frac{\varepsilon}{\lambda^2} \tag{36}$$

for all sufficiently large n. This completes the proof.

Using the methods of proofs of theorems 2 and 4, we have the following

THEOREM 5. *Suppose that the following conditions are satisfied*:

1. $Ef=0$ *and* $E|f|^{2+\delta}<\infty$ *for some* $\delta>0$,

2. $\sum_{k=1}^{\infty}\{E|f-E\{f|\mathfrak{M}_{-k}^{k}\}|^2\}^{1/2}<\infty$,

3. $\sum_{k=1}^{\infty}\{\alpha(k)\}^{\delta/(2+\delta)}<\infty$.

Then $\sigma^2<\infty$. *If* $\sigma>0$, *then the distribution of* Z_n, *defined by* (23), *converges weakly to Wiener measure* W *on* $(D, \mathscr{D})$.

Proof. As before, it suffices to prove the tightness of $\{Z_n\}$. Define $g_N(f_0)=f_0$ $(|f_0|\leqq N)$, $=0$ $(|f_0|>N)$, and $\bar{g}_N(f_0)=f_0-g_N(f_0)$. Then

$$\begin{aligned}
&P\left(\max_{j\leq n}|S_j|\geqq 2\lambda\sigma\sqrt{n}\right)\\
&\leqq P\left(\max_{j\leq n}\left|\sum_{i=1}^{j}(g_N(f_i)-Eg_N(f_i))\right|\geqq\lambda\sigma\sqrt{n}\right)\\
&\quad+P\left(\max_{j\leq n}\left|\sum_{i=1}^{j}(\bar{g}_N(f_i)-E\bar{g}_N(f_i))\right|\geqq\lambda\sigma\sqrt{n}\right)\\
&\leqq P\left(\max_{j\leq n}\left|\sum_{i=1}^{j}(g_N(f_i)-Eg_N(f_i))\right|\geqq\lambda\sigma\sqrt{n}\right)\\
&\quad+P\left(\sum_{i=1}^{n}|\bar{g}_N(f_i)-E\bar{g}_N(f_i)|\geqq\lambda\sigma\sqrt{n}\right).
\end{aligned} \tag{37}$$

Let $p=[n^{\delta/2(1+\delta)}]$, $k=[n/p]$ and $N=n^{1/2(1+\delta)}$. Define $\bar{\xi}_j^{(s)}=E\{\bar{g}_N(f_j)-E\bar{g}_N(f_j)|\mathfrak{M}_{-s+j}^{s+j}\}$ and $\bar{\eta}_j^{(s)}=\bar{g}_N(f_j)-E\bar{g}_N(f_j)-\bar{\xi}_j^{(s)}$. Then, for $j>2s$, $\bar{\xi}_j^{(s)}$ is measurable with respect to $\mathfrak{M}_{j-s}^{j+s}$, and $\bar{\xi}_0^{(s)}$ is measurable with respect to $\mathfrak{M}_{-s}^{s}$. Hence

$$\begin{aligned}
&E|\bar{\xi}_0^{(s)}|\cdot|\bar{\xi}_j^{(s)}|\\
&\leqq\{E|\bar{\xi}_0^{(s)}|\}^2+8\{E|\bar{\xi}_0^{(s)}|^{2+\delta}\}^{2/(2+\delta)}\{\alpha(j-2s)\}^{\delta/(2+\delta)}\\
&\leqq\frac{4}{N^{2(1+\delta)}}\{E|\bar{g}_N(f_0)|^{2+\delta}\}^2\\
&\quad+32\{E|\bar{g}_N(f_0)|^{(2+\delta)}\}^{2/(2+\delta)}\cdot\{\alpha(j-2s)\}^{\delta/(2+\delta)}.
\end{aligned} \tag{38}$$

Moreover

$$\begin{aligned}E|\bar{\xi}_i^{(s)}|\cdot|\bar{\eta}_j^{(s)}| &\leqq \{E|\bar{\eta}_j^{(s)}|^{(2+\delta)/(1+\delta)}\}^{(1+\delta)/(2+\delta)}\cdot\{E|\bar{\xi}_i^{(s)}|^{2+\delta}\}^{1/(2+\delta)} \\ &\leqq \{E|\bar{\xi}_0^{(s)}|^{2+\delta}\}^{1/(2+\delta)}\{E|\bar{\eta}_0^{(s)}|^2\}^{1/2}\end{aligned} \tag{39}$$

and

$$\begin{aligned}&E|\bar{\eta}_i^{(s)}|\cdot|\eta_j^{(s)}| \\ &\leqq \{E|\bar{\eta}_0^{(s)}|^{(2+\delta)/(1+\delta)}\}^{(1+\delta)/(2+\delta)}\{E|\bar{\eta}_0^{(s)}|^{2+\delta}\}^{1/(2+\delta)} \\ &\leqq \{E|\bar{\eta}_0^{(s)}|^{2+\delta}\}^{1/(2+\delta)}\{E|\bar{\eta}_0^{(s)}|^2\}^{1/2}.\end{aligned} \tag{40}$$

Using (38), (39) and (40), we obtain

$$\begin{aligned}&\sum_{j=1}^{n} E|\bar{g}_N(f_0)-E\bar{g}_N(f_0)|\cdot|\bar{g}_N(f_j)-E\bar{g}_N(f_j)| \\ &=\sum_{j=1}^{n} E|\bar{\xi}_0^{([j/3])}+\bar{\eta}_0^{([j/3])}|\cdot|\bar{\xi}_j^{([j/3])}+\bar{\eta}_j^{([j/3])}| \\ &\leqq \sum_{j=1}^{n}\{E|\bar{\xi}_0^{([j/3])}|\cdot|\bar{\xi}_j^{([j/3])}|+E|\bar{\xi}_0^{([j/3])}|\cdot|\bar{\eta}_j^{([j/3])}| \\ &\quad +E|\bar{\eta}_0^{([j/3])}|\cdot|\bar{\xi}_j^{([j/3])}|+E|\bar{\eta}_0^{([j/3])}|\cdot|\bar{\eta}_j^{([j/3])}|\} \\ &\leqq \sum_{j=1}^{n}\left[\frac{4}{N^{2(1+\delta)}}\{E|\bar{g}_N(f_0)|^{2+\delta}\}^2\right. \\ &\quad +32\{E|\bar{g}_N(f_0)|^{2+\delta}\}^{2/(2+\delta)}\left\{\alpha\left(\left[\frac{j}{3}\right]\right)\right\}^{\delta/(2+\delta)} \\ &\quad +2\{E|\bar{\xi}_0^{([j/3])}|^{2+\delta}\}^{1/(2+\delta)}\cdot\{E|\bar{\eta}_0^{([j/3])}|^2\}^{1/2} \\ &\quad \left.+\{E|\bar{\eta}_0^{([j/3])}|^{2+\delta}\}^{1/(2+\delta)}\cdot\{E|\bar{\eta}_0^{([j/3])}|^2\}^{1/2}\right] \\ &\leqq 4\{E|\bar{g}_N(f_0)|^{2+\delta}\}^2 \\ &\quad +32\{E|\bar{g}_N(f_0)|^{2+\delta}\}^{2/(2+\delta)}\sum_{j=1}^{\infty}\left\{\alpha\left(\left[\frac{j}{3}\right]\right)\right\}^{\delta/(2+\delta)} \\ &\quad +8\{E|\bar{g}_N(f_0)|^{2+\delta}\}^{1/(2+\delta)}\sum_{j=1}^{\infty}\{E|\bar{\eta}_0^{([j/3])}|^2\}^{1/2} \\ &\leqq 4\gamma_N^2+C_1\gamma_N^{2/(2+\delta)}+C_2\gamma_N^{1/(2+\delta)},\end{aligned} \tag{41}$$

where $\gamma_N=E|\bar{g}_N(f_0)|^{2+\delta}\to 0$ $(n\to\infty)$. Hence we have

$$
\begin{aligned}
&P\left(\sum_{j=1}^{n}|\bar{g}_N(f_j)-E\bar{g}_N(f_j)|\geqq\lambda\sigma\sqrt{n}\right)\\
&\leqq\frac{1}{\lambda^2\sigma^2 n}E\left\{\sum_{j=1}^{n}|\bar{g}_N(f_j)-E\bar{g}_N(f_j)|\right\}^2\\
&\leqq\frac{1}{\lambda^2\sigma^2}\left\{E|\bar{g}_N(f_0)-E\bar{g}_N(f_0)|^2\right.\\
&\qquad\left.+2\sum_{j=1}^{n}E|\bar{g}_N(f_0)-E\bar{g}_N(f_0)|\cdot|\bar{g}_N(f_j)-E\bar{g}_N(f_j)|\right\}\\
&\leqq\frac{1}{\lambda^2\sigma^2}\left\{\frac{\gamma_N}{N^\delta}+8\gamma_N^2+2C_1\gamma_N^{2/(2+\delta)}+2C_2\gamma_N^{1/(2+\delta)}\right\}\\
&\leqq\frac{\varepsilon}{2\lambda^2}
\end{aligned}
\tag{42}
$$

for all sufficiently large n. By the same argument as in the proof of theorem 4, we have

$$
P\left(\max_{j\leqq n}\left|\sum_{i=1}^{j}(g_N(f_i)-Eg_N(f_i))\right|\geqq\lambda\sigma\sqrt{n}\right)\leqq\frac{\varepsilon}{2\lambda^2}
\tag{43}
$$

for all sufficiently large n. Thus it follows from (37), (42) and (43) that

$$
P\left(\max_{j\leqq n}|S_j|\geqq 2\lambda\sigma\sqrt{n}\right)\leqq\frac{\varepsilon}{2\lambda^2},
\tag{44}
$$

which implies the tightness of $\{Z_n\}$. The proof is completed.

References

[1] BILLINGSLEY, P., Convergence of probability measures. John Wiley, New York (1968).

[2] DAVYDOV, YU. A., Convergence of distributions generated by stationary stochastic processes. Theory of Prob. & Appl. **13** (1968), 691–696.

[3] IBRAGIMOV, I. A., Some limit theorems for stationary processes, Theory of Prob. & Appl. **7** (1962), 349–382.

DEPARTMENT OF APPLIED MATHEMATICS,
YOKOHAMA NATIONAL UNIVERSITY, AND
DEPARTMENT OF MATHEMATICS,
YOKOHAMA NATIONAL UNIVERSITY.