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AN ARCLENGTH PROBLEM FOR m-FOLD SYMMETRIC UNIVALENT FUNCTIONS

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1. Introduction. Let S denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic and univalent in the unit disk Δ: $|z|<1$. Let S^* denote the subclass of S for which $f(z)$ is starlike, that is

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right]>0 \qquad (z\in\Delta).$$

Let C denote the subclass of S for which $f(z)$ is convex, that is

$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]>0 \quad (z\in\Delta).$$

Let K denote the subclass of S for which $f(z)$ is close-to-convex, that is

$$\mathrm{Re}\left[\frac{zf'(z)}{h(z)}\right]>0 \qquad (z\in\Delta).$$

where $h(z)$ is starlike. These classes are related by the proper inclusions $C\subset S^*\subset K\subset S$.

A function $f(z)$ analytic in Δ is said to be m-fold symmetric ($m=1, 2, \cdots$) if

$$f(e^{2\pi i/m}z)=e^{2\pi i/m}f(z).$$

In particular, every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. Let S_m denote the subclass of S for which $f(z)$ is m-fold symmetric. A simple argument shows that $f\in S_m$ is characterized by having a power series of the form

$$f(z)=z+a_{m+1}z^{m+1}+a_{2m+1}z^{2m+1}+\cdots.$$

We similarly define S_m^*, C_m and K_m.

For $f\in S$ and $0<r<1$, let

$$L_r(f)=\int_{|z|=r}|f'(z)|\,|dz|$$

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denote the arclength of the image of the circle $|z|=r$. Since $L_r(f)$ is a continuous functional and S is a normal and compact family, a solution of the extremal problem

$$\max_{f\in S} L_r(f)$$

exists and is in the class S. The problem of finding the extremal function re mains unsolved. The problem

$$\max_{f\in C_1} L_r(f)$$

has been solved by Keogh [5] who showed that

$$\max_{f\in C_1} L_r(f)=L_r\left(\frac{z}{1-z}\right)=\frac{2\pi r}{1-r^2}.$$

The problem

$$\max_{f\in S_1^*} L_r(f)$$

has been solved by Marx [7] who showed that

$$\max_{f\in S_1^*} L_r(f)=L_r(k)$$

where $k(z)$ is the Koebe function

$$k(z)=\frac{z}{(1-z)^2}=z+2z^2+3z^3+\cdots.$$

Clunie and Duren [1] have solved the extremal problem within the class K and have shown that

$$\max_{f\in K_1} L_r(f)=L_r(k).$$

Duren [2] also obtained an evaluation of $L_r(k)$ in terms of standard elliptic integrals.

In § 2 we deal with extending these results to m-fold symmetric functions ($m=1, 2, \cdots$) and solve the extremal problems

$$\text{I:} \quad \max_{f\in C_m} L_r(f),$$

$$\text{II:} \quad \max_{f\in S_m^*} L_r(f),$$

$$\text{III:} \quad \max_{f\in K_m} L_r(f),$$

We shall need the following lemmas.

LEMMA 1. (a) *$f(z)\in C_m$ if and only if $zf'(z)\in S_m^*$;*

(b) *$f(z)\in S_m^*$ if and only if $\int_0^z \frac{f(\xi)}{\xi}d\xi\in C_m$.*

LEMMA 2. *$f(z)\in S^*$ if and only if $[f(z^m)]^{1/m}\in S_m^*$.*

LEMMA 3. *$f(z)\in S_m$ if and only if $f(z)=[g(z^m)]^{1/m}$ where $g(z)\in S$.*

LEMMA 4. *$f(z)\in S_m^*$ if and only if*

$$f(z)=z\exp\int_0^{2\pi}\ln(1-z^m e^{-i\phi})^{-2/m}\,d\mu(\phi),$$

where $\mu(\phi)$ is non-decreasing on $[0, 2\pi]$ and $\mu(2\pi)-\mu(0)=1$.

LEMMA 5. *Let $\mu(\phi)$ be non-decreasing on $[0, 2\pi]$ and $\mu(2\pi)-\mu(0)=1$. If $h(\phi)$ is positive and integrable with respect to $\mu(\phi)$ on $[0, 2\pi]$ then*

$$\exp\int_0^{2\pi}\ln h(\phi)\,d\mu(\phi)\leqq\int_0^{2\pi}h(\phi)\,d(\phi).$$

LEMMA 6. *If $f(z)\in K_m$ then*

$$f'(z)=e^{i\alpha}\frac{g(z)}{z}P(z),$$

where $g(z)\in S_m^*$, $\operatorname{Re}P(z)>0$ and

$$P(z)=e^{-i\alpha}+b_m z^m+b_{2m}z^{2m}+\cdots.$$

Lemma 1 is well known for the case $m=1$, Lemma 2 and Lemma 3 for $m=2$. The proofs in the general case are straightforward. Lemma 4 for the case $m=1$ may be found in [2; p. 758] and the general case is easily handled using this result and Lemma 3. Lemma 5 is in [3; Thm. 208] while Lemma 6 may be found in [6].

2. The arclength extremal problem.

THEOREM 1. *If $f\in C_m$, then $L_r(f)\leqq L_r(h_m)$ for $0<r<1$, where*

$$h_m(z)=\int_0^z\frac{[k(\xi^m)]^{1/m}}{\xi}\,d\xi\in C_m.$$

Proof. If $f\in C_m$, then by Lemma 1 (a) and Lemma 4 we have

$$zf'(z)=z\exp\int_0^{2\pi}\ln(1-z^m e^{-i\phi})^{-2/m}\,d\mu(\phi), \tag{1}$$

which yields

$$|zf'(z)|=|z|\exp\int_0^{2\pi}\ln|1-z^m e^{-i\phi}|^{-2/m}\,d\mu(\phi),$$

where $\mu(\phi)$ is non-decreasing and $\mu(2\pi)-\mu(0)=1$.

By Lemma 5 we obtain

(2) $$|zf'(z)|\leqq|z|\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2/m}\,d\mu(\phi),$$

and consequently

$$L_r(f)=\int_0^{2\pi}|zf'(z)|\,d\theta\leqq|z|\int_0^{2\pi}\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2/m}\,d\mu(\phi)\,d\theta,$$

where $z=re^{i\theta}$. Changing the order of integration and using the identity,

$$\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2/m}\,d\theta=\int_0^{2\pi}|1-z^m|^{-2/m}\,d\theta,$$

we obtain

(3) $$L_r(f)\leqq|z|\int_0^{2\pi}|1-z^m|^{-2/m}\,d\theta.$$

Since $k(z)\in S^*$, by Lemma 2 we have $[k(z^m)]^{1/m}\in S_m^*$, and by Lemma 1 (b),

$$h_m(z)\equiv\int_0^z\frac{[k(\xi^m)]^{1/m}}{\xi}\,d\xi\in C_m.$$

A simple calculation yields

$$L_r(h_m)=|z|\int_0^{2\pi}|1-z^m|^{-2/m}\,d\theta.$$

Hence (3) becomes $L_r(f)\leqq L_r(h_m)$ and we see that $h_m(z)$ is a solution of the extremal problem I.

REMARKS. (i) For $m=1$

$$h_1(z)=\frac{z}{1-z}\quad\text{and}\quad L_r(h_1)=\frac{2\pi r}{1-r^2}$$

(see Keogh [5]).

(ii) For $m=2$ (odd functions)

$$h_2(z)=\frac{1}{2}\log\frac{1+z}{1-z}$$

and

$$L_r(h_2)=\int_0^{2\pi}\frac{r}{|1-r^2e^{i2\theta}|}\,d\theta$$

which can be expressed in terms of standard elliptic integrals.

(iii) For $m=3, 4, \cdots,$

$$h_m(z)=\int_0^z\frac{[k(\xi^m)]^{1/m}}{\xi}d\xi=\int_0^z\frac{d\xi}{(1-\xi^m)^{2/m}}$$

maps Δ onto the interior of a regular convex polygon P_m of order m with perimeter of length

$$L_m=2^{1-4/m}\frac{\Gamma^2\left(\frac{1}{2}-\frac{1}{m}\right)}{\Gamma\left(1-\frac{2}{m}\right)}$$

[8; p. 196]. Since $h_m(z)$ maps $|z|\leqq r<1$ onto a convex subdomain of P_m we must have $L_r(h_m)\leqq L_m$ for all $0<r<1$. Hence

$$L_r(f)\leqq 2^{1-4/m}\frac{\Gamma^2\left(\frac{1}{2}-\frac{1}{m}\right)}{\Gamma\left(1-\frac{2}{m}\right)}$$

for $f\in C_m$, where $m=2, 3, \cdots$ and $0<r<1$.

COROLLARY. *If $f\in C_m$ and $\rho(f\,;r,\theta)$ is the radius of curvature of the image of $|z|=r(0<r<1)$ at $f(re^{i\theta})$ then*

$$\rho(f\,;r,\theta)\leqq\frac{r(1+r^m)^{2-2/m}}{1-r^{2m}}$$

and this bound is sharp.

Proof. The radius of curvature $\rho(f\,;r,\theta)$ is given by

$$\rho(f\,;r,\theta)=\frac{|zf'(z)|}{\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]} \tag{4}$$

[4; p. 359]. Using (1) we can show that

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]=\int_0^z\frac{1-r^{2m}}{|1-z^m e^{-i\phi}|^2}\,d\mu(\phi). \tag{5}$$

From (2) we have

$$\begin{aligned}|zf'(z)|&\leqq r\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2/m}\,d\mu(\phi)\\&=r\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2}\,|1-z^m e^{-i\phi}|^{2-2/m}\,d\mu(\phi)\\&\leqq r[\max_{\phi}|1-z^m e^{-\phi}|^{2-2/m}]\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2}\,d\mu(\phi),\end{aligned}$$

and consequently

$$|zf'(z)|\leqq r(1+r^m)^{2-2/m}\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2}\,d\mu(\phi). \tag{6}$$

From (4), (5) and (6) we obtain

$$\rho(f\,;\,r,\,\theta)\leqq\frac{r(1+r^m)^{2-2/m}}{1-r^{2m}}.$$

A simple calculation shows that

$$\rho(h_m\,;\,r,\,\pi/m)=\frac{r(1+r^m)^{2-2/m}}{1-r^{2m}},$$

and since $h_m\in C_m$ we see that the bound in the corollary is sharp.

THEOREM 2. *If* $f\in K_m$ *then* $L_r(f)\leqq L_r([k(z^m)]^{1/m})$ *for* $0<r<1$.

Proof. Let

$$\mathscr{P}=\{P(z)\text{ regular in }\Delta|\operatorname{Re}P(z)>0,$$
$$|P(0)|=1,\text{ and }P(z)-P(0)=b_m z^m+b_{2m}z^{2m}+\cdots\}.$$

If $f\in K_m$, then by Lemma 6 we have

$$zf'(z)=e^{i\alpha}g(z)P(z),$$

where $g(z)\in S_m^*$ and $P(z)\in\mathscr{P}$. Since $g(z)\in S_m^*$, from Lemma 4 we obtain

$$g(z)=z\exp\int_0^{2\pi}\ln(1-z^m e^{-i\phi})^{-2/m}\,d\mu(\phi),$$

where $\mu(\phi)$ is non-decreasing and $\mu(2\pi)-\mu(0)=1$. Applying Lemma 5, we obtain the inequality

$$|g(z)|\leqq|z|\int_0^{2\pi}|1-z^m e^{-i\phi}|^{-2/m}\,d\mu(\phi).$$

Thus

$$|zf'(z)|=|g(z)P(z)|\leqq|z|\int_0^{2\pi}\frac{d\mu(\phi)}{|1-z^m e^{-i\phi}|^{2/m}}|P(z)|$$

and for $z=re^{i\theta}$ $(0<r<1)$ we have

$$L_r(f)=\int_0^{2\pi}|zf'(z)|d\theta\leqq\int_0^{2\pi}\int_0^{2\pi}\frac{|z|\cdot|P(z)|}{|1-z^m e^{-i\phi}|^{2/m}}\,d\mu(\phi)\,d\theta.$$

On interchanging the order of integration and making the change of variable $\theta=\psi+\phi/m$, we obtain

$$L_r(f)\leqq\int_0^{2\pi}\frac{r|P(re^{i(\psi+\phi/m)})|}{|1-r^m e^{im\psi}|^{2/m}}\,d\psi\,d\mu(\phi).$$

If we let $\zeta=re^{i\psi}$, then we obtain

$$L_r(f)\leqq\int_0^{2\pi}\int_0^{2\pi}\frac{r|P(e^{i\phi/m}\zeta)|}{|1-\zeta^m|^{2/m}}\,d\psi\,d\mu(\phi).$$

$$\leqq \max_{0\leqq\phi\leqq2\pi}\int_0^{2\pi}\frac{r|P(e^{i\phi/m}\zeta)|}{|1-\zeta^m|^{2/m}}d\phi.$$

Since $P(\zeta)\in\mathcal{P}$, we have $P(e^{i\phi/m}\zeta)\in\mathcal{P}$ for $0\leqq\phi\leqq2\pi$, and consequently

$$L_r(f)\leqq\max_{P\in\mathcal{P}}\int_0^{2\pi}\frac{r|P(\zeta)|}{|1-\zeta^m|^{2/m}}d\phi. \tag{7}$$

Now $P(\zeta)$ has a Herglotz representation given by

$$P(\zeta)=\cos\alpha\int_0^{2\pi}\frac{1+\zeta^m e^{-it}}{1-\zeta^m e^{-it}}d\nu(t)-i\sin\alpha$$

$$=e^{-i\alpha}\int_0^{2\pi}\frac{1+\zeta^m e^{-i(t-2\alpha)}}{1-\zeta^m e^{-it}}d\nu(t),$$

where $P(0)=e^{-i\alpha}$, $\nu(t)$ is non-decreasing and $\nu(2\pi)-\nu(0)=1$. Thus

$$|P(\zeta)|\leqq\int_0^{2\pi}\frac{|1+\zeta^m e^{-i(t-2\alpha)}|}{|1-\zeta^m e^{-it}|}d\nu(t),$$

and (7) can be replaced by

$$L_r(f)\leqq\max_\nu\int_0^{2\pi}\frac{r}{|1-\zeta^m|^{2/m}}\int_0^{2\pi}\frac{|1+\zeta^m e^{-i(t-2\alpha)}|}{|1-\zeta^m e^{-it}|}d\nu(t)\,d\phi$$

$$=\max_\nu\int_0^{2\pi}\int_0^{2\pi}\frac{r}{|1-\zeta^m|^{2/m}}\frac{|1+\zeta^m e^{-i(t-2\alpha)}|}{|1-\zeta^m e^{-it}|}d\phi\,d\nu(t)$$

$$\leqq\max_{0\leqq t\leqq2\pi}\int_0^{2\pi}\frac{r}{|1-\zeta^m|^{2/m}}\frac{|1+\zeta^m e^{-i(t-2\alpha)}|}{|1-\zeta^m e^{-it}|}d\phi.$$

If we let $\phi=\theta+t/m$, we obtain

$$L_r(f)\leqq r\max_{0\leqq t\leqq2\pi}\int_0^{2\pi}\frac{1}{|1-r^m e^{i(m\theta+t)}|^{2/m}}\frac{|1+r^m e^{i(m\theta+2\alpha)}|}{|1-r^m e^{im\theta}|}d\theta$$

$$\equiv r\max_{0\leqq t\leqq2\pi}I(\alpha,t).$$

It now follows from a result on rearrangements of functions given in [1; p. 182] that

$$I(\alpha,t)\leqq I(0,0).$$

Therefore

$$L_r(f)\leqq\int_0^{2\pi}\frac{r|1+r^m e^{im\theta}|}{|1-r^m e^{im\theta}|^{2/m+1}}d\theta=L_r([k(z^m)]^{1/m}). \tag{8}$$

Since $[k(z^m)]^{1/m}\in S_m^*$ by Lemm 2, and $S_m^*\subset K_m$, we see from (8) that the function $[k(z^m)]^{1/m}$ is a solution of the extremal problem III. This also indicates that it is a solution of problem II.

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