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ON CANONICAL STRATIFICATIONS

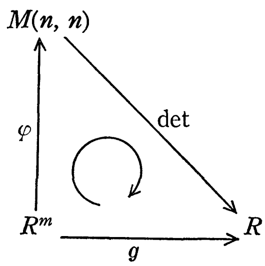
By Akira Koriyama

§ 0. Introduction.

It is well-known that every compact manifold can be imbedded into a Euclidean m-space R^m for some m. Furthermore Nash [7] proved that for a closed connected smooth manifold M, smoothly imbedded in R^m, there is a polynomial map f: $R^m \to R^q$ for some q such that M is a connected component of $f^{-1}(0)$. A polynomial map f is an ordered set $(g_1, g_2, \cdots, g_q)$ of polynomial functions.

On the other hand, by a simple calculation, we have the following

Proposition A. *Every polynomial can be expressed in a form of determinant of a certain square matrix whose entries are monomials of degree* 1 *or* 0. *More precisely, for any polynomial function* g: $R^m \to R$, *there is a positive integer* n *and an affine imbedding* φ *of* R^m *into the space* $M(n, n)$ *of all* $n \times n$ *real matrices such that the following diagram is commutative*:



(This was communicated to the author by T. Ishikawa).

Remark. For the given polynomial map $f=(g_1, \cdots, g_q)$: $R^m \to R^q$, we take the positive integer n common to all g_i.

On account of the above facts every closed connected smooth manifold can be imbedded into $M(n, n)$ for some n and is expressed as the intersection of the q affine m-spaces $\varphi_i(R^m)$ and $\det^{-1}(0)$ in $M(n, n)$. Thus it is meaningful to study the set of zeros of det: $M(n, n) \to R$, which is the same as the set of singular matrices, or the set of matrices with rank $r<n$. More generally we consider, in this paper, the set of $n \times m$ real matrices, $n \leqq m$, with rank $r<n$.

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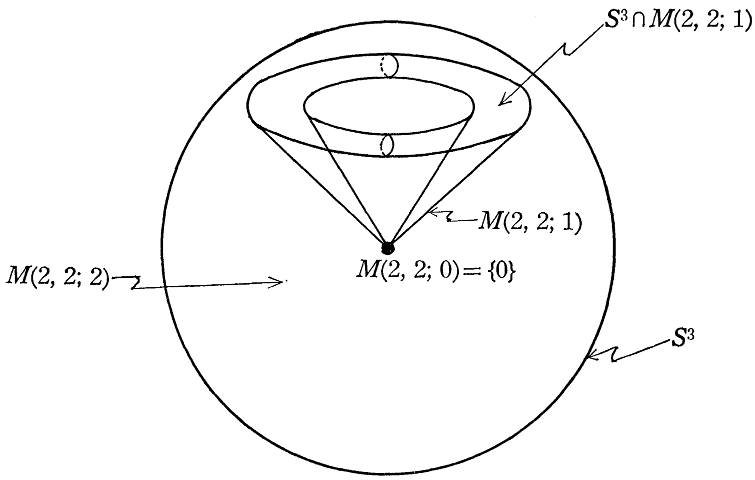
Let f: $R^m \to R^q$ be a given polynomial map. We denote by V_f the set of all zeros of f. V_f is called an affine algebraic set. The topological structure of V_f has been investigated by Lefschetz [5], Milnor [6], Nash [7], Oleĭnik [8], Thom [9], Whitney [10] and others. Whitney defined the notion of stratifications of V_f. Roughly speaking the stratification is an expression of V_f as a disjoint union of manifolds. However it is not easy to describe each component manifold, called a stratum of the stratification, and to describe geometrical relations among the strata. Let $M(n, m)$ be the set of all $n \times m$ real matrices, $n \leqq m$, and identify it with R^{nm} in a natural manner. Let V be the subset of $M(n, m)$ consisting of all matrices of rank $r < n$. Then V is an algebraic set and by making use of ranks of matrices we have a stratification of V, which Thom called the *canonical stratification.* The purpose of this paper is to study and discribe explicitly the canonical stratification of V.

A simple example of the stratification defined by rank is the following

PROPOSITION B. *Let $M(2, 2; 1)$ be the set of all 2×2 matrices of rank 1. Let D^4 be the unit closed ball with center at the origin in R^4 and S^3 be the boundary sphere. Then $S^3 \cap M(2, 2; 1)$ is a torus defined by*

$$\left\{ \sqrt{\frac{1}{2}}(\cos 2\pi t,\ \sin 2\pi t,\ \cos 2\pi t',\ \sin 2\pi t') \middle| 0 \leqq t,\ t' \leqq 1 \right\}$$

in S^3. And $D^4 \cap (\{0\} \cup M(2, 2; 1))$ is a cone over $S^3 \cap M(2, 2; 1)$ from the origin.



We are going to show how this simple stratification seen in the above picture comes into the stratifications of higher dimensional case. In this context we have the following theorems.

THEOREM A. *Let K and L be the manifolds $M(n, m; r)$ and $M(n, m; s)$, respec-*

tively, for $s>r\geqq 1$. *Then there exists a tubular neighborhood* T_K *of* K *in* $K\cup L$ *(details are explained in* §4) *and* (T_K, p, K, F) *becomes a fibre bundle. Here* F *is the cone over the manifold* $S^{(n-r)(m-r)-1}\cap M(n-r, m-r; s-r)$ *from the origin; that is* $F=\{\alpha A|A\in S^{(n-r)(m-r)-1}\cap M(n-r, m-r; s-r), 0\leqq\alpha\leqq 1\}$.

This theorem implies that the stratum L is attached to the stratum K through the fibre bundle (T_K, p, K, F) with the singular fibre F.

THEOREM B. *Let* N_A *be the normal space of* $M(n, m; 1)$ *in* R^{nm} *at the point* $A\in M(n, m; 1)$.

(i) *Then* $\cup_{r\geqq 1} M(n, m; r)=\cup_{A\in M(n,m;1)}\{N_A\cap\{\cup_{s\geqq 2} M(n, m; s)\}\}\cup M(n, m; 1)$ *as sets.* (Remark: The right hand side is not a fibre bundle, because there are points A and B in $M(n, m; 1)$ such that $N_A\cap N_B\neq\phi$.)

(ii) *On the other hand* $N_A\cap\{\cup_{s\geqq 2} M(n, m; s)\}$ *is isomorphic to* $\cup_{s'\geqq 1} M(n-1, m-1; s')$ *as stratified sets for each* $A\in M(n, m; 1)$.

Let $V_{n,r}=O(n)/I_r\times O(n-r)$ be the real Stiefel manifold of orthonormal r-frames in R^n, where $O(n)$ is the orthogonal group, and $\Lambda_{m,r}$ the manifold made from certain matrices (the precise definition of $\Lambda_{m,r}$ is given in §6). Let G be the discrete subgroup of $O(r)$ defined by $G=\{T|T=(t_{ij})\in O(r), t_{ij}=\pm\delta_{ij}\}$. Then we have the following

THEOREM C. *The manifold* $S^{nm-1}\cap M(n, m; r)$ *is homeomorphic to* $V_{n,r}\times_G \Lambda_{m,r}$ *where* $V_{n,r}\times_G \Lambda_{m,r}$ *is the orbit space of* G *under the action defined by* $T\cdot(E, B)=(ET, TB)$ *for* (E, B) *of* $V_{n,r}\times\Lambda_{m,r}$.

COROLLARY. $(S^{nm-1}\cap M(n, m; 1), p, P^{n-1}, S^{m-1})$ *is a fibre bundle over the real projective space* P^{n-1} *with fibre* S^{m-1}.

The author wishes to express his deep gratitude to Prof. H. Omori for his valuable advice, and Professors M. Adachi and T. Fukuda for their several enlightening discussions.

§1. Proof of Proposition A.

Since every polynomial is a linear combination of some monomials, it suffices to show that the product xy and the sum $x+y$ of the monomials x and y can be expressed by the determinant of matrices of the desired type.

(i) Product.

Since the determinant of any triangular matrix is a product of the diagonal elements, the matrices

$$(xy) \quad \text{and} \quad \begin{bmatrix} 1 & * \\ 0 & xy \end{bmatrix}$$

have the same determinant, where * is arbitrary. Furthermore one can add a scalar multiple of one row vector to another without changing the value of the determinant. Hence, when we put $*=-y$, for example, and multiply the 1st row by x and add it to the 2nd row, we have the matrix

$$\begin{bmatrix} 1 & -y \\ x & 0 \end{bmatrix}$$

without changing the value of the determinant.

(ii) Sum.

The method is completely similar to that of (i). The matrices

$$(x+y) \quad \text{and} \quad \begin{bmatrix} 1 & -y \\ 0 & x+y \end{bmatrix}$$

have the same determinant. In the latter matrix we may add the 1st row to the 2nd without changing the determinant, namely the matrices

$$(x+y) \quad \text{and} \quad \begin{bmatrix} 1 & -y \\ 1 & x \end{bmatrix}$$

have the same determinant. To any polynomial, by appling the methods (i) or (ii) repeatedly, we have a matrix of the desired type. q. e. d.

EXAMPLE. Let $f(x, y)=ax^2+by^2$. Through the processes stated below, determinants are not changed.

$$(ax^2+by^2) \longrightarrow \begin{bmatrix} 1 & -ax \\ 0 & ax^2+by^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -ax \\ x & by^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -by \\ 0 & 1 & -ax \\ 0 & x & by^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -by \\ 0 & 1 & -ax \\ y & x & 0 \end{bmatrix}.$$

COROLLARY. *Every polynomial f can be expressed by the determinant of a certain $n(f)\times n(f)$ matrix.*

Proof. Obviously every polynomial f is written as uniquely $f=f_1+f_2$, where f_1 consists of monomials of degree $\geqq 2$ and f_2 consists of that of degree <2. To each polynomial f we assign the following integers:

$d(i, f_1)=$degree of the i-th term of f_1 (where we assumed all the terms of f_1 are ordered in a certain way),

$e(f_2)=$the number of terms of f_2.

And we put $n(f)=e(f_2)+\sum d(i, f_1)$.

Note that when we apply the methods (i) and (ii) stated above to f we may apply (i) and (ii) to f_1 at the same time if necessary. The proof of Corollary is easy by induction with respect to $n(f)$.

REMARK. The number $n(f)$ is not the least degree of the matrices to f.

EXAMPLE. Let C^6 be the 6-dimensional hermitian space, and $g\colon C^6 \to C$ be the polynomial function defined by

$$g(z_0, z_1, \cdots, z_5) = z_0^3 + z_1^2 + \cdots + z_5^2.$$

We put $V = g^{-1}(0)$ and $S = \{z \mid z_0\bar{z}_0 + z_1\bar{z}_1 + \cdots + z_5\bar{z}_5 = 1\}$.

It is well-known, due to Hirzebruch [4] and Brieskorn [1], that $V \cap S = \Sigma^9$ is an exotic sphere. It is not difficult to show that the polynomial g stated above is expressed by the determinant of the following matrix A.

$$A = \begin{bmatrix} 1 & & & & & & & z_5 \\ & 1 & & & \mathbf{0} & & & z_4 \\ & & 1 & & & & & z_3 \\ & & & 1 & & & & z_2 \\ & & & & 1 & & & z_1 \\ & & \mathbf{0} & & & 1 & z_0 & 0 \\ & & & & & & -1 & z_0 \\ -z_5 & -z_4 & -z_3 & -z_2 & -z_1 & -z_0 & 0 & 0 \end{bmatrix}.$$

§ 2. Expression of $M(n, m; r)$.

By $M(n, m)$ we denote the smooth manifold of all $n \times m$ real matrices and by $M(n, m; r)$ the manifold of all $n \times m$ real matrices with rank exactly r. $G_{n-r,r}$ denotes the Grassmann manifold of r-dimensional subspaces through the origin in R^n. By $GL(m)$ we denote the general linear group and we set

$$GL'(m-r) = \left\{ \left[\begin{array}{c|c} I_r & * \\ \hline 0 & X \end{array} \right] \,\middle|\, \begin{array}{l} X \in GL(m-r) \\ * \text{ is arbitrary} \end{array} \right\} \quad \text{and} \quad GL''(m-r) = \{{}^tA \mid A \in GL'(m-r)\},$$

where ${}^tA$ denotes the transposed matrix of A. $V'_{m,r} = GL(m)/GL'(m-r)$ denotes the real Stiefel manifold of r-frames in R^m. We naturally identify $V'_{m,r}$ with $M(m, r; r)$ or $M(r, m; r)$ under the expression $V'_{m,r} = GL(m)/GL'(m-r)$ or $V'_{m,r} = GL(m)/GL''(m-r)$.

THEOREM 2.1. *For each r, $n-1 \geqq r \geqq 1$, $(M(n, m; r), p, G_{n-r,r}, V'_{m,r})$ is a smooth fibre bundle over $G_{n-r,r}$ with fibre $V'_{m,r}$.*

REMARK. When we consider the singularities of a differentiable map, these singularities are expressed and classified by the rank of the Jacobian matrices of the map at the points. (see Fukuda [3] for example). In this sense, the manifolds $M(n, m; r)$ are the most fundamental one in the study of singularities.

The proof of Theorem is devided into three steps.

LEMMA 2.1. *Let $M=M(n, m; r)$ and $G=GL(n)\times GL(m)$. Then G acts on M, and M turns out to be a homogeneous space of G.*

Proof. For any (P, Q) of G and for any A of M, we define the action of (P, Q) on A by $(P, Q)\cdot A=PAQ^{-1}$. Then G acts transitively on M. Let H be the isotropy subgroup of G at the point E_0 of M, where E_0 is the matrix of the form

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right].$$

By simple calculation we find that H has the form:

$$H=\left\{\left(\left[\begin{array}{c|c} P_1 & P_2 \\ \hline 0 & P_4 \end{array}\right], \left[\begin{array}{c|c} P_1 & 0 \\ \hline Q_3 & Q_4 \end{array}\right]\right) \,\middle|\, \begin{array}{l} P_1\in GL(r),\ P_4\in GL(n-r),\ Q_4\in GL(m-r), \\ P_2 \text{ and } Q_3 \text{ are arbitrary} \end{array}\right\}.$$

Therefore the manifold $M(n, m; r)$ is diffeomorphic to G/H. q.e.d.

LEMMA 2.2. *G/H is the total space of a fibre bundle over a Grassmann manifold.*

Proof. We put

$$H_1=\left\{\left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right] \,\middle|\, \begin{array}{l} A\in GL(r),\ D\in GL(n-r) \\ B\in M(r, n-r) \end{array}\right\},$$

which is a closed subgroup of $GL(n)$. Therefore $H_1\times GL(m)$ is a closed subgroup of G, and the subgroup H in Lemma 2.1 is closed in $H_1\times GL(m)$. Since $H_1\times GL(m)$ admits a local cross-section in G, $(G/H, p, GL(n)/H_1, (H_1\times GL(m))/H)$ is a locally trivial fibre bundle. Obviously, $GL(n)/H_1\approx G_{n-r,r}$, where $X\approx Y$ means that X is diffeomorphic to Y. q.e.d.

LEMMA 2.3. *The fibre of the above fibre bundle $(G/H, p, GL(n)/H_1, (H_1\times GL(m))/H)$ is diffeomorphic to the real Stiefel manifold $V'_{m,r}$.*

Proof. For simplicity we denote by I_n the one point space $\{I_n\}$. Let $K=I_n\times GL(m)$. Then from $KH=HK=H_1\times GL(m)$, where H is the group mentioned in Lemma 2.1, it follows that KH is a closed subgroup of G. Let H_2 be the following subgroup of $GL(m)$;

$$H_2=\left\{\left[\begin{array}{c|c} I_r & 0 \\ \hline C' & D' \end{array}\right] \,\middle|\, \begin{array}{l} I_r \text{ is the } r\times r \text{ unit matrix,} \\ C'\in M(m-r, r),\ D'\in GL(m-r) \end{array}\right\}.$$

Since K is a Lie group, the map of $K/K\cap H$ to KH/H defined by $k(K\cap H)\longmapsto kH$ is a diffeomorphism. On the other hand $K\cap H=I_n\times H_2$. Hence $(H_1\times GL(m))/H \approx I_n\times GL(m)/I_n\times H_2\approx GL(m)/H_2\approx V'_{m,r}$. This completes the proof of Lemma 2.3.

Combining the above lemmas, the proof of Theorem 2. 1 is complete.

§ 3. Normal bundle of $M(n, m; r)$.

In this section we study the normal bundle of $M(n, m; r)$ in $M(n, m)$, which is naturally identified with R^{nm}. For any two points A and B of $M(n, m)$ the inner product $\langle A, B\rangle$ is expressed in the form $\langle A, B\rangle=\mathrm{trace}\ (A^{t}B)$.

THEOREM 3. 1. *Let $N(M)$ be the normal bundle of $M(n, m; r)$ in $M(n, m)$. Then there are maps φ: $GL(n)\times GL(m)\rightarrow M(n, m; r)$ and ψ: $GL(n)\times GL(m)\times R^{(n-r)(m-r)}\rightarrow N(M)$ satisfying the following*

(a)

$$\begin{array}{ccc} GL(n)\times GL(m)\times R^{(n-r)(m-r)} & \xrightarrow{\ \psi\ } & N(M) \\ p\Big\downarrow & & \Big\downarrow\pi \\ GL(n)\times GL(m) & \xrightarrow[\ \varphi\ ]{} & M(n, m; r) \end{array}$$

diagram (1)

The commutativity holds in the diagram.

(b) *ψ is fibre preserving and is a linear isomorphism at each fibre, where $GL(n)\times GL(m)\times R^{(n-r)(m-r)}$ is regarded as a trivial vector bundle with the natural projection p.*

(c) *More precisely, we have the following bundle isomorphism*

$$\begin{array}{ccc} \{GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}\}/\Delta & \xrightarrow{\ \psi_0\ } & N(M) \\ \pi'\Big\downarrow & & \Big\downarrow\pi \\ (GL(n)\times GL(m)/K_1\times K_2)/\Delta & \xrightarrow[\ \varphi_0\ ]{} & M(n, m; r), \end{array}$$

diagram (2)

where $K_1\times K_2$ and Δ are the following:

$$K_1=\left\{\left[\begin{array}{c|c} I_r & A_2 \\ \hline 0 & A_4 \end{array}\right] \,\middle|\, \begin{array}{l} A_4\in GL(n-r), \\ A_2 \textit{ is arbitrary} \end{array}\right\},$$

$$K_2=\left\{\left[\begin{array}{c|c} I_r & B_2 \\ \hline 0 & B_4 \end{array}\right] \,\middle|\, \begin{array}{l} B_4\in GL(m-r), \\ B_2 \textit{ is arbitrary} \end{array}\right\} \quad \textit{and}$$

$$\Delta=\{(G_1, G_2)\mid G_1=G_2\in GL(r)\}.$$

The proof of this theorem consists of several lemmas mentioned below. First of all, one has to imbed the trivial bundle $GL(n)\times GL(m)\times R^{(n-r)(m-r)}$ into $GL(n)\times GL(m)\times R^{nm}$.

LEMMA 3.1. *For any point (A, B) of $GL(n)\times GL(m)$ we define the liner space $E^{\perp}_{(A,B)}$ as follows:*

$$E^{\perp}_{(A,B)}=\left\{ {}^{t}A^{-1}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right]B^{-1} \;\middle|\; \begin{array}{r} \\ r\{ \\ n-r\{ \end{array}\overset{\begin{array}{cc} r & m-r \end{array}}{\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right]};\quad * \text{ is arbitrary}\right\}.$$

We define the map ξ

$$\begin{array}{ccc} GL(n)\times GL(m)\times R^{(n-r)(m-r)} & \xrightarrow{\ \xi\ } & GL(n)\times GL(m)\times R^{nm} \\ \downarrow & & \downarrow \\ GL(n)\times GL(m) & \xrightarrow[\text{id.}]{} & GL(n)\times GL(m) \end{array}$$

diagram (3)

by $\xi(A, B, Z)=(A, B, {}^{t}A^{-1}\tilde{Z}B^{-1})$, where

$$Z\in M(n-r, m-r) \quad \textit{and} \quad \tilde{Z}=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Z \end{array}\right]\in M(n, m).$$

Then ξ is an imbedding of $GL(n)\times GL(m)\times R^{(n-r)(m-r)}$ into $GL(n)\times GL(m)\times R^{nm}$.

Proof. It is trivial from the definition of ξ.

LEMMA 3.2. *Let*

$$E_n=\left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right]\in M(n, r; r) \quad \textit{and} \quad E_m=\left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right]\in M(m, r; r).$$

We define the map φ: $GL(n)\times GL(m)\to M(n, m; r)$ by $\varphi(A, B)=(AE_n)^{t}(BE_m)$ mentioned in (a) *of Theorem* 3.1. *Then $E^{\perp}_{(A,B)}$ is the normal space to $M(n, m; r)$ in R^{nm} at $\varphi(A, B)$.*

Proof. The map φ is smooth and onto. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be tangent vectors to $GL(n)$ at A and to $GL(m)$ at B, respectively. Since $GL(n)$ is an open submanifold of $M(n, n)$, we may regard $M(n, n)$ as the tangent space of $GL(n)$ at each point. $M(m, m)$ can be thought of as the tangent space of $GL(m)$. Moreover, A and B are linear isomorphisms of $M(n, n)$ and $M(m, m)$ respectively. $\boldsymbol{u}$ and $\boldsymbol{v}$ may be written as the form $\boldsymbol{u}=AX$ and $\boldsymbol{v}=BY$, where $X\in M(n, n)$ and $Y\in M(m, m)$. Let $T_{(A,B)}(GL(n)\times GL(m))$ and $T_{\varphi(A,B)}M$ be the tangent spaces to $GL(n)\times GL(m)$ at (A, B) and to $M=M(n, m; r)$ at $\varphi(A, B)$, respectively. And let $d\varphi_{(A,B)}$: $T_{(A,B)}(GL(n)\times GL(m))\to T_{\varphi(A,B)}M$ be the linear map induced by φ. For any tangent vector $(\boldsymbol{u}, \boldsymbol{v})=(AX, BY)$ we see that

$$d\varphi_{(A,B)}(\boldsymbol{u}, \boldsymbol{v})=(\boldsymbol{u}E_n)^{t}(BE_m)+(AE_n)^{t}(\boldsymbol{v}E_m)=A\left[\begin{array}{c|c} * & * \\ \hline \underbrace{*}_{r} & \underbrace{0}_{m-r} \end{array}\right]^{t}B \quad \}n-r \;.$$

Hence for each fixed pair (A, B), the tangent space to $M(n, m; r)$ at $\varphi(A, B)$ has the following form:

$$E_{(A,B)}=\left\{A\left[\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right]^{t}B \,\middle|\, * \text{ are arbitrary matrices}\right\}.$$

And $E_{(A,B)}^{\perp}$ defined in Lemma 3.1 becomes the orthogonal complement of $E_{(A,B)}$. In fact, for any element

$$A\left[\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right]^{t}B$$

of $E_{(A,B)}$ and for any element

$${}^{t}A^{-1}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right]B^{-1}$$

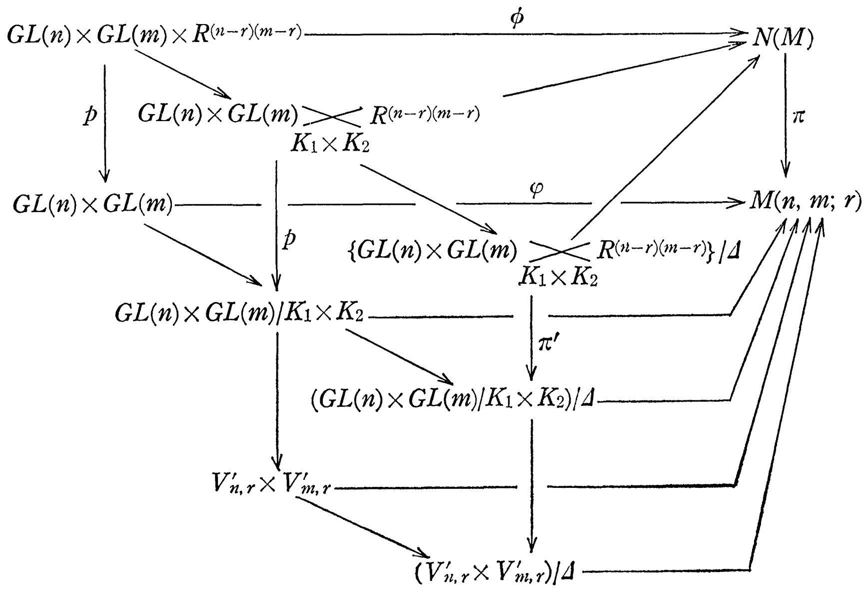
of $E_{(A,B)}^{\perp}$, we see that

$$\text{trace}\left\{A\left[\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right]^{t}B^{t}B^{-1}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right]A^{-1}\right\}=\text{trace}\left\{AA^{-1}\left[\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right]\right\}=0. \qquad \text{q.e.d.}$$

LEMMA 3.3. (*Proof of* (a) *of Theorem* 3.1).

Proof. We define the map ψ: $GL(n)\times GL(m)\times R^{(n-r)(m-r)}\to N(M)$ by $\psi(A, B, Z)=(\varphi\times 1_R)\circ\xi(A, B, Z)=((AE_n)^{t}(BE_m), {}^{t}A^{-1}\tilde{Z}B^{-1})$, where 1_R is the identity map of R^{nm}. From the definition of the maps, the commutativity holds. q.e.d.

Next we will consider the construction of the normal bundle $N(M)$ from the trivial bundle $(GL(n)\times GL(m)\times R^{(n-r)(m-r)}, p, GL(n)\times GL(m))$. We consider the following commutative diagram.



Definitions of maps in the diagram are given in the lemmas mentioned below.

We recall the following well-known

THEOREM (cf. H. Cartan [2]). *Let X be a principal G-bundle and Y any G-space. Then $(X\times_G Y, p, X/G, Y)$ is a fibre bundle with fibre Y, where we write $X\times_G Y$ for $(X\times Y)/G$ and p is the projection induced by the canonical projection π_1: $X\times Y\to X$.*

Applying this theorem we have

LEMMA 3. 4. *Let $K_1\times K_2$ be the closed subgroup of $GL(n)\times GL(m)$ defined in* (c) *of Theorem* 3. 1. *Then $K_1\times K_2$ acts on $GL(n)\times GL(m)\times R^{(n-r)(m-r)}$, and its orbit space*

$$GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}$$

becomes a total space of a fibre bundle.

Proof. For any (P, Q) of $GL(n)\times GL(m)$ we define the action of (k_1, k_2) of $K_1\times K_2$ by $(k_1, k_2)\circ(P, Q)=(Pk_1^{-1}, Qk_2^{-1})$. Then $(GL(n)\times GL(m), q, GL(n)\times GL(m)/K_1\times K_2, K_1\times K_2)$ becomes a principal fibre bundle, where q: $GL(n)\times GL(m)\to GL(n)\times GL(m)/K_1\times K_2$ is the canonical projection. To each element Z of $M(n-r, m-r)$ we assign

$$\tilde{Z}=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Z \end{array}\right]$$

of $M(n, m)$ and by this correspondence we identify $R^{(n-r)(m-r)}=M(n-r, m-r)$ with the subset $R^{(n-r)(m-r)}\times 0$ of $R^{nm}=M(n, m)$. For any point Z of $R^{(n-r)(m-r)}$ we define the action of (k_1, k_2) by $(k_1, k_2)\cdot Z={}^tk_1^{-1}\tilde{Z}k_2^{-1}$. Then $K_1\times K_2$ acts on $R^{(n-r)(m-r)}$. Applying the theorem stated above,

$$(GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}, p, GL(n)\times GL(m)/K_1\times K_2, R^{(n-r)(m-r)})$$

becomes a fibre bundle. The projection p is defined by $p([P, Q, Z])=q(P, Q)$, where $[P, Q, Z]$ is the coset containing (P, Q, Z). q. e. d.

LEMMA 3. 5. *Let Δ be the closed subgroup of $GL(r)\times GL(r)$ defined by $\Delta=\{(G_1, G_2)\mid G_1=G_2\in GL(r)\}$. Then Δ acts on*

$$GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}$$

as a topological transformation group.

Proof. For any two elements (G_1', G_2') and (G_1, G_2) of Δ we define the product , $\cdot$, in Δ by $(G_1', G_2')\cdot(G_1, G_2)=(G_1'G_1, G_2'G_2)$. We identify each element (G_1, G_2) of Δ with the element (G_1, G_2) of $GL(n)\times GL(m)$, where

$$\tilde{G}_1=\left[\begin{array}{c|c} G_1 & 0 \\ \hline 0 & I_{n-r} \end{array}\right]\in GL(n) \quad\text{and}\quad \tilde{G}_2=\left[\begin{array}{c|c} G_2 & 0 \\ \hline 0 & I_{m-r} \end{array}\right]\in GL(m).$$

By the above identification we define the action of Δ on

$$GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}$$

by $(G_1, G_2)\cdot[P, Q, \tilde{Z}]=[P\tilde{G}_1^{-1}, Q{}^t\tilde{G}_2, {}^t\tilde{G}_1^{-1}\tilde{Z}\tilde{G}_2^{-1}]$. We will show that the definition stated above does not depend on the choice of representatives and this action is well-defined. For this, it suffices to show that

$$(Pk_1^{-1}\tilde{G}_1^{-1}, Qk_2^{-1t}\tilde{G}_2, {}^t\tilde{G}_1^{-1t}k_1^{-1}\tilde{Z}k_2^{-1}\tilde{G}_2^{-1})\equiv(P\tilde{G}_1^{-1}, Q{}^t\tilde{G}_2, {}^t\tilde{G}_1^{-1}\tilde{Z}\tilde{G}_2^{-1}) \pmod{K_1\times K_2}$$

holds for any element $(Pk_1^{-1}, Qk_2^{-1}, {}^tk_1^{-1}\tilde{Z}k_2^{-1})$ of $[P, Q, \tilde{Z}]$. We assume that

$$k_1=\left[\begin{array}{c|c} I & K_2 \\ \hline 0 & K_4 \end{array}\right] \quad\text{and}\quad \tilde{G}_1=\left[\begin{array}{c|c} G_1 & 0 \\ \hline 0 & I \end{array}\right].$$

Then

$$k_1^{-1}=\left[\begin{array}{c|c} I & -K_2K_4^{-1} \\ \hline 0 & K_4^{-1} \end{array}\right] \quad\text{and}\quad \tilde{G}_1^{-1}=\left[\begin{array}{c|c} G_1^{-1} & 0 \\ \hline 0 & I \end{array}\right].$$

We put

$$k_1'^{-1}=\left[\begin{array}{c|c} I & -G_1K_2K_4^{-1} \\ \hline 0 & K_4^{-1} \end{array}\right].$$

Then

$$\left[\begin{array}{c|c} I & -K_2K_4^{-1} \\ \hline 0 & K_4^{-1} \end{array}\right]\left[\begin{array}{c|c} G_1^{-1} & 0 \\ \hline 0 & I \end{array}\right]=\left[\begin{array}{c|c} G_1^{-1} & -K_2K_4^{-1} \\ \hline 0 & K_4^{-1} \end{array}\right]=\left[\begin{array}{c|c} G_1^{-1} & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} I & -G_1K_2K_4^{-1} \\ \hline 0 & K_4^{-1} \end{array}\right].$$

Hence $k_1^{-1}\tilde{G}_1^{-1}=\tilde{G}_1^{-1}k_1'^{-1}$, and $Pk_1^{-1}\tilde{G}_1^{-1}=P\tilde{G}_1^{-1}k_1'^{-1}\equiv P\tilde{G}_1^{-1}$ (mod K_1). In the same way, for

$$k_2=\left[\begin{array}{c|c} I & K_2' \\ \hline 0 & K_4' \end{array}\right],$$

we put

$$k_2'^{-1}=\left[\begin{array}{c|c} I & -{}^t G_2^{-1}K_2'K_4'^{-1} \\ \hline 0 & K_4'^{-1} \end{array}\right].$$

Then $k_2^{-1\,t}\tilde{G}_2={}^t\tilde{G}_2k_2'^{-1}$ and $Qk_2^{-1\,t}\tilde{G}_2=Q{}^t\tilde{G}_2k_2'^{-1}\equiv Q{}^t\tilde{G}_2$ (mod K_2). Therefore the definition does not depend on representatives. Since $(G_1', G_2')\cdot((G_1, G_2)\cdot[P, Q, Z])=(G_1'G_1, G_2'G_2)\cdot[P, Q, Z]$, the action of Δ is well-defined. q. e. d.

By

$$(GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\Delta$$

we denote the orbit space of Δ; that is

$$(GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\Delta=\{\{[P\tilde{G}_1^{-1}, Q{}^t\tilde{G}_2, {}^t\tilde{G}_1^{-1}\tilde{Z}\tilde{G}_2^{-1}] \mid (G_1, G_2)\in\Delta\}\}.$$

LEMMA 3. 6. *The action of Δ on the fibre $R^{(n-r)(m-r)}$ is trivial. Hence*

$$(GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\Delta=\{\{[P\tilde{G}_1^{-1}, Q{}^t\tilde{G}_2, \tilde{Z}] \mid (G_1, G_2)\in\Delta\}\}.$$

Moreover commutativity holds in the following diagram

$$\begin{array}{ccc} GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)} & \longrightarrow & (GL(n)\times GL(m) \underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\Delta \\ \downarrow & & \downarrow \\ GL(n)\times GL(m)/K_1\times K_2 & \longrightarrow & (GL(n)\times GL(m)/K_1\times K_2)/\Delta. \end{array}$$

diagram (4)

Proof. For all (G_1, G_2) of Δ,

$$ {}^t\tilde{G}_1^{-1}\tilde{Z}\tilde{G}_2^{-1}=\left[\begin{array}{c|c} {}^tG_1^{-1} & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Z \end{array}\right]\left[\begin{array}{c|c} G_2^{-1} & 0 \\ \hline 0 & I \end{array}\right]=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Z \end{array}\right]=\tilde{Z}. $$

Hence the action of $\varDelta$ is trivial.

For any $[P, Q]=\{(Pk_1^{-1}, Qk_2^{-1})|(k_1, k_2)\in K_1\times K_2\}$ of $GL(n)\times GL(m)/K_1\times K_2$ we define the action of (G_1, G_2) of $\varDelta$ by $(G_1, G_2)\cdot[P, Q]=[P\tilde{G}_1^{-1}, Q{}^t\tilde{G}_2]$; and denote its orbit space by $(GL(n)\times GL(m)/K_1\times K_2)/\varDelta$; that is

$$ (GL(n)\times GL(m)/K_1\times K_2)/\varDelta=\{\{[P\tilde{G}_1^{-1}, Q{}^t\tilde{G}_2]|(G_1, G_2)\in\varDelta\}\}. $$

Let π', ψ_3 and φ_3 be the following canonical projections:

$$ \pi'\colon\ (GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\varDelta\longrightarrow(GL(n)\times GL(m)/K_1\times K_2)/\varDelta, $$

$$ \psi_3\colon\ GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}\longrightarrow(GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\varDelta $$

and

$$ \varphi_3\colon\ GL(n)\times GL(m)/K_1\times K_2\longrightarrow(GL(n)\times GL(m)/K_1\times K_2)/\varDelta. $$

Then from the definitions of these maps we see that the commutativity holds in the diagram (4). q. e. d.

LEMMA 3. 7. *$GL(n)\times GL(m)/K_1\times K_2$ is diffeomorphic to $V'_{n,r}\times V'_{m,r}$.*

Proof. First of all we identify $V'_{n,r}\times V'_{m,r}$ with $M(r, n; r)\times M(r, m; r)$. We define the action of (P, Q) of $GL(n)\times GL(m)$ on (X, Y) of $V'_{n,r}\times V'_{m,r}$ by $(P, Q)\cdot(X, Y)=(X{}^tP, Y{}^tQ)$. Then $GL(n)\times GL(m)$ acts on $V'_{n,r}\times V'_{m,r}$ transitively. Let $E_n=(I_r|0)\in M(r, n; r)$ and $E_m=(I_r|0)\in M(r, m; r)$. Then the isotropy subgroup of $GL(n)\times GL(m)$ at $E_n\times E_m$ is $K_1\times K_2$. Hence there is a diffeomorphism α_1: $GL(n)\times GL(m)/K_1\times K_2\to V'_{n,r}\times V'_{m,r}$. Explicitly, α_1 has the following form. Let

$$ P=\left[\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array}\right],\quad Q=\left[\begin{array}{c|c} Q_1 & Q_2 \\ \hline Q_3 & Q_4 \end{array}\right],\quad \bar{P}=\left[\begin{array}{c} P_1 \\ \hline P_3 \end{array}\right]\quad\text{and}\quad \bar{Q}=\left[\begin{array}{c} Q_1 \\ \hline Q_3 \end{array}\right]. $$

We naturally identify $M(r, n; r)\times M(r, m; r)$ with $M(n, r; r)\times M(m, r; r)$. Then $\alpha_1([P, Q])=(\bar{P}, \bar{Q})$, where $[P, Q]$ is the coset containing (P, Q). q. e. d.

LEMMA 3. 8. *$V'_{n,r}\times V'_{m,r}/\varDelta$ is diffeomorphic to $M(n, m; r)$.*

Proof. For any $(\bar{P}, \bar{Q})$ of $V'_{n,r}\times V'_{m,r}=M(n, r; r)\times M(m, r; r)$ we define the action of (G_1, G_2) of $\varDelta$ by $(G_1, G_2)\cdot(\bar{P}, \bar{Q})=(\bar{P}G_1^{-1}, \bar{Q}{}^tG_2)$, and denote its orbit space by $V'_{n,r}\times V'_{m,r}/\varDelta$.

Next, for any $[\bar{P}, \bar{Q}]=\{(\bar{P}G_1^{-1}, \bar{Q}{}^tG_2)|(G_1, G_2)\in\varDelta\}$ of $V'_{n,r}\times V'_{m,r}/\varDelta$, we define the action of (A, B) of $GL(n)\times GL(m)$ on $[\bar{P}, \bar{Q}]$ by $(A, B)\cdot[\bar{P}, \bar{Q}]=[A\bar{P}, {}^tB^{-1}\bar{Q}]$ $=\{(A\bar{P}G_1^{-1}, {}^tB^{-1}\bar{Q}{}^tG_2)|(G_1, G_2)\in\varDelta\}$. Since every element of $V'_{n,r}=M(n, r; r)$ (resp. $V'_{m,r}=M(m, r; r)$) has maximal rank, it can be transformed into the canonical form

$$E_n=\left[\frac{I_r}{0}\right] \quad \left(\text{resp. } E_m=\left[\frac{I_r}{0}\right]\right)$$

by multiplying some element of $GL(n)$ (resp. $GL(m)$). Hence the action is transitive. Let $\mathcal{H}$ be the isotropy subgroup of $GL(n)\times GL(m)$ at $[E_n, E_m]$ $=\{(E_nG_1^{-1}, E_m{}^tG_2)|(G_1, G_2)\in\Delta\}$ of $V'_{n,r}\times V'_{m,r}/\Delta$. Then for each (A, B) of $\mathcal{H}$, $(A, B)\cdot[E_n, E_m]=[AE_n, {}^tB^{-1}E_m]=[E_n, E_m]$. Since

$$AE_n=\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]\left[\frac{I_r}{0}\right]=\left[\frac{A_1}{A_3}\right] \quad \text{and} \quad {}^tB^{-1}E_m=\left[\begin{array}{c|c} {}^tB'_1 & {}^tB'_3 \\ \hline {}^tB'_2 & {}^tB'_4 \end{array}\right]\left[\frac{I_r}{0}\right]=\left[\frac{{}^tB'_1}{{}^tB'_2}\right],$$

where

$$\left[\begin{array}{c|c} B'_1 & B'_2 \\ \hline B'_3 & B'_4 \end{array}\right]=\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]^{-1},$$

the necessary and sufficient condition for $(AE_n, {}^tB^{-1}E_m)$ to be contained in $[E_n, E_m]$ is that the following three conditions be satisfied:

(i) $A_3=0$, (ii) ${}^tB'_2=0$ and (iii) $A_1^{-1}=B'_1$.

Obviously $B'_1B_1=I_r$ and $B'_1B_2=0$. Hence $B'_1=B_1^{-1}$ and $B_2=0$. Moreover, by (iii), $A_1=B_1$. Hence the group $\mathcal{H}$ has the following form:

$$\mathcal{H}=\left\{\left(\left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_4 \end{array}\right], \left[\begin{array}{c|c} A_1 & 0 \\ \hline B_3 & B_4 \end{array}\right]\right) \middle| \begin{array}{l} A_1\in GL(r);\ A_4\in GL(n-r);\ B_4\in GL(m-r) \\ A_2 \text{ and } B_3 \text{ are arbitrary} \end{array}\right\}.$$

Therefore $\mathcal{H}$ agrees with the group H stated in Lemma 3. 1, and

$$V'_{n,r}\times V'_{m,r}/\Delta\approx\mathcal{H}\backslash GL(n)\times GL(m)\approx H\backslash GL(n)\times GL(m)$$
$$\approx GL(n)\times GL(m)/H\approx M(n, m; r).$$

Explicitly, the diffeomorphism Φ: $V'_{n,r}\times V'_{m,r}/\Delta\to M(n, m; r)$ is given by $\Phi([\bar{P}, \bar{Q}])$ $=\bar{P}{}^t\bar{Q}$. q. e. d.

LEMMA 3. 9. *The group Δ acts on $GL(n)\times GL(m)/K_1\times K_2$ and the orbit space $(GL(n)\times GL(m)/K_1\times K_2)/\Delta$ has a structure of smooth manifolds. Moreover there exists a diffeomorphism α_2: $(GL(n)\times GL(m)/K_1\times K_2)/\Delta\to V'_{n,r}\times V'_{m,r}/\Delta$.*

REMARK. First, for the proof of Lemma 3. 9, we recall the following well-known theorem of G-spaces. (cf. H. Cartan [2]).

Let G be a Lie group, X and Y be G-spaces. A map f: $X\to Y$ is *equivariant* if $f(gx)=gf(x)$ for all $(g, x)\in G\times X$. An equivariant homeomorphism of X onto Y is called an *equivalence* of X with Y. If X and Y are G-spaces and f: $X\to Y$ is equivariant, then there is a unique map f: $X/G\to Y/G$ such that $f\circ\pi_X=\pi_Y\circ f$, where π_X and π_Y are canonical projections. This map g is called the *map induced by f*.

THEOREM. *Let X and Y be G-spaces. If f: $X\to Y$ is an equivalence of X with Y then the induced map f is a homeomorphism of X/G onto Y/G.*

Proof of Lemma 3.9.

We recall that the action of $\varDelta$ on $V'_{n,r}\times V'_{m,r}=M(n, r; r)\times M(m, r; r)$ was defined by $(G_1, G_2)\cdot(\bar{P}, \bar{Q})=(\bar{P}G_1^{-1}, \bar{Q}^tG_2)$ for (G_1, G_2) of $\varDelta$, and on $GL(n)\times GL(m)/K_1\times K_2$ by $(G_1, G_2)\cdot[P, Q]=[P\tilde{G}_1^{-1}, Q^t\tilde{G}_2]$. The map α_1: $GL(n)\times GL(m)/K_1\times K_2\to V'_{n,r}\times V'_{m,r}$ was defined by $\alpha_1([P, Q])=(\bar{P}, \bar{Q})$. Hence, by simple calculation we see that $\alpha_1[(G_1, G_2)\cdot(\bar{P}, \bar{Q})]=(G_1, G_2)\cdot\alpha_1([\bar{P}, \bar{Q}])$. Thus α_1 is an equivalence. By the above theorem of G-spaces, there is a homeomorphism α_2: $(GL(n)\times GL(m)/K_1\times K_2)/\varDelta\to V'_{n,r}\times V'_{m,r}/\varDelta$ induced by α_1. Explicitly, α_2 is defined by

$$\alpha_2\{[P\tilde{G}_1^{-1}, Q^t\tilde{G}_2]\,|\,(G_1, G_2)\in\varDelta\}=\{(\bar{P}G_1^{-1}, \bar{Q}^tG_2)\,|\,(G_1, G_2)\in\varDelta\}.$$

By Lemmas 3.7 and 3.8, $(GL(n)\times GL(m)/K_1\times K_2)/\varDelta$ has a structure of smooth manifold and α_2 becomes a diffeomorphism under this structure. q.e.d.

LEMMA 3.10. (*Proof of* (b) *and* (c) *of Theorem* 3.1.) *Let ψ_0 be the map of*

$$(GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)})/\varDelta$$

to $N(M)$ defined by $\psi_0\{[P\tilde{G}_1^{-1}, Q^t\tilde{G}_2, \tilde{Z}]\,|\,(G_1, G_2)\in\varDelta\}=(PE_n{}^tE_m{}^tQ, {}^tP^{-1}\tilde{Z}Q^{-1})$ and φ_0 be the map of $(GL(n)\times GL(m)/K_1\times K_2)/\varDelta$ to $M(n, m; r)$ defined by $\varphi_0=\Phi\circ\alpha_2$. Then (ψ_0, φ_0) is a bundle isomorphism.

Proof. From the definition of ψ, defined in Lemma 3.3, Theorem (b) is obvious. Since Φ and α_2 are diffeomorphisms, so does φ_0. By Lemma 3.9, ψ_0 is well-defined. Obviously ψ_0 is one-to-one and onto. Since π and π' are locally trivial and φ_0 is a diffeomorphism, ψ_0 turns out to be a diffeomorphism. q.e.d.

§ 4. Attaching of $M(n, m; s)$ to $M(n, m; r)$; *Proof of Theorem* A.

For any manifold Y the cone CY over Y is defined to be the following quotient space:

$CY=Y\times I/Y\times 1$, where I denotes the closed unit interval $[0, 1]$.

Let $\xi=(E, \pi, B, F)$ be any fibre bundle. For ξ we define the fibre bundle $CYL(\xi)$ as follows:

Its total space is the mapping cylinder $M(\pi)$ of the projection π, the base space is B and the projection p is the natural projection p: $M(\pi)\to B$ of the mapping cylinder. That is, $CYL(\xi)=(M(\pi), p, B, CY)$.

LEMMA 4.1. *The structural group $K_1\times K_2$ of the bundle*

$$(GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)}, p, GL(n)\times GL(m)/K_1\times K_2)$$

can be reduced to $O(n-r)\times O(m-r)$.

Proof. Since $GL(n-r)=O(n-r)\times R^{(1/2)(n-r)(n-r+1)}$, Lemma 4.1 is clear.

For any fixed pair (r, s), $1\leqq r<s\leqq n$, we define a subset $D^{r,s}_{(A,B)}$ of $E^{\perp}_{(A,B)}$ as follows:

$$D^{r,s}_{(A,B)}=\left\{{}^tA^{-1}\left[\begin{array}{c|c}0&0\\\hline 0&Z\end{array}\right]B^{-1}\ \middle|\ \begin{array}{l}Z\in M(n-r,\ m-r;\ s-r),\\ \text{trace}\,(Z\,{}^tZ)=(1/2)\ \text{trace}\,(AE_n{}^tE_m{}^tB)^t(AE_n{}^tE_m{}^tB)\end{array}\right\}.$$

Let $\rho(A, B)$ be the function $\{(1/2)\ \text{trace}\,(AE_n{}^tE_m{}^tB)^t(AE_n{}^tE_m{}^tB)\}^{1/2}$ of (A, B). Then we see that $D^{r,s}_{(A,B)}=S^{(n-r)(m-r)-1}(\rho(A, B))\cap M(n-r, m-r; s-r)$, where $S^{(n-r)(m-r)-1}(\rho(A, B))$ is the sphere in $R^{(n-r)(m-r)}$ of radius $\rho(A, B)$ about the origin. Since for any $s\neq 0$ of R and A of $M(n-r, m-r; s-r)$ sA is contained again in $M(n-r, m-r; s-r)$, $D^{r,s}_{(A,B)}$ is diffeomorphic to $S^{(n-r)(m-r)-1}\cap M(n-r, m-r; s-r)$ for each (A, B) of $GL(n)\times GL(m)$. We identify $T_1\in O(n-r)$ with

$$\tilde{T}_1=\left[\begin{array}{c|c}I_r&0\\\hline 0&T_1\end{array}\right]\in O(n)\quad\text{and}\quad T_2\in O(m-r)\ \text{with}\ \tilde{T}_2=\left[\begin{array}{c|c}I_r&0\\\hline 0&T_2\end{array}\right]\in O(m).$$

Since

$${}^t\left[\begin{array}{c|c}I_r&0\\\hline 0&T_1\end{array}\right]^{-1}\left[\begin{array}{c|c}0&0\\\hline 0&Z\end{array}\right]\left[\begin{array}{c|c}I_r&0\\\hline 0&T_2\end{array}\right]^{-1}=\left[\begin{array}{c|c}0&0\\\hline 0&T_1ZT_2^{-1}\end{array}\right],$$

$D^{r,s}_{(A,B)}$ is preserved invariantly under the action of $O(n-r)\times O(m-r)$. Let $\boldsymbol{D}$ be the fibre bundle made from

$$(GL(n)\times GL(m)\underset{K_1\times K_2}{\times} R^{(n-r)(m-r)},\ p,\ GL(n)\times GL(m)/K_1\times K_2)$$

by replacing the fibre $R^{(n-r)(m-r)}$ with the fibre in the form of $D^{r,s}_{(A,B)}$. q.e.d.

LEMMA 4.2. (*Proof of Theorem* A.) *The image of the bundle* $CYL(\boldsymbol{D})$ *by the map* (ψ_1, φ_1) *is a tubular neighborhood of* $M(n, m; r)$ *in* $M(n, m; r)\cup M(n, m; s)$.

Proof. It is easy to see that

$${}^tA^{-1}\left[\begin{array}{c|c}0&0\\\hline 0&Z\end{array}\right]B^{-1}$$

is contained in $E^{\perp}_{(A,B)}\cap(M(n, m; r)\cup M(n, m; s))$ if and only if Z is in $\{0\}\cup M(n-r, m-r; s-r)$. This completes the proof of Lemma 4.2. Obviously Lemma 4.2 implies Theorem A. q.e.d.

§ 5. Proof of Theorem B.

Proof of (i). Since $\{0\}\cup M(n, m; 1)$ is closed in R^{nm}, for any X of $M(n, m; r)$

($r>1$), there is a matrix A of $\{0\}\cup M(n, m; 1)$ such that $\sqrt{\langle X, A\rangle}$ gives the distance between X and $\{0\}\cup M(n, m; 1)$. It is easy to show that $A\neq 0$. Hence A is in $M(n, m; 1)$. And X is in N_A.

Proof of (ii). It is the same as that of Lemma 3. 2. Since rank $A=1$, there are matrices $P\in GL(n)$ and $Q\in GL(m)$ such that $A=(PE_n)^t(QE_m)$, where

$$E_n=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\in M(n, 1) \quad \text{and} \quad E_m=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\in M(m, 1).$$

Let $E=E_n{}^tE_m$. Then

$$N_E=\left\{\left[\begin{array}{c|c}0&\cdots 0\\\hline\vdots&\\0&X\end{array}\right]\middle|\, X\in M(n-1, m-1)\right\}.$$

Hence

$$N_A=\left\{{}^tP^{-1}\left[\begin{array}{c|c}0&\cdots 0\\\hline\vdots&\\0&X\end{array}\right]Q^{-1}\,\middle|\, X\in M(n-1, m-1)\right\}.$$

Since

$$\left[\begin{array}{c|c}0&\cdots 0\\\hline\vdots&\\0&X\end{array}\right]\longmapsto {}^tP^{-1}\left[\begin{array}{c|c}0&\cdots 0\\\hline\vdots&\\0&X\end{array}\right]Q^{-1}$$

is a rank-preserving linear map of N_E to N_A,

$${}^tP^{-1}\left[\begin{array}{c|c}0&\cdots 0\\\hline\vdots&\\0&X\end{array}\right]Q^{-1}\in N_A$$

is in $\cup_{s\geqq 2} M(n, m; s)$ if and only if X is in $\cup_{r\geqq 1} M(n-1, m-1; r)$. q. e. d.

§ 6. Proof of Theorem C.

Let $V_{n,r}=O(n)/I_r\times O(n-r)$ be the real Stiefel manifold of orthonormal r-frames in R^n. $V_{n,r}$ is canonically identified with the set $\{A|A\in M(n, r; r), {}^tAA=I_r\}$. We naturally identify $M(n, r)$ with R^{nr}. Then the inner product $\langle A, A\rangle$ is expressed by $\langle A, A\rangle=\text{trace}\,({}^tAA)=\text{trace}\,(A{}^tA)$.

We call a matrix A of $M(r, m)$ an upper triangular matrix if A has the following form:

$$\begin{bmatrix}a_{11}&a_{12}&\cdots\cdots&a_{1m}\\&a_{22}&\cdots\cdots&a_{2m}\\&&\ddots&\vdots\\0&&a_{rr}\cdots\cdots&a_{rm}\end{bmatrix}.$$

We consider the following sets:

$$\mathcal{T}_{r,m}=\{A|A\in M(r,m;r),\ A \text{ is upper triangular}\},$$

$$\Lambda_{m,r}=\{A|A\in\mathcal{T}_{r,m},\ \langle A,A\rangle=1\}.$$

Obviously $\Lambda_{m,r}$ is a submanifold of $M(r, m; r)$ of dimension $rm-r(r-1)/2-1$. We put $K(n, m; r)=S^{nm-1}\cap M(n, m; r)$, where S^{nm-1} is the unit sphere in R^{nm} centered at the origin.

LEMMA 6.1. *Any element of $K(n, m; r)$ can be expressed in a product of some elements of $V_{n,r}$ and $\Lambda_{m,r}$.*

Proof. For any $P=(p_{ij})$ of $K(n, m; r)$, let $\mathfrak{A}_1$ be the first non-zero column vector and $\mathfrak{A}_2$ be the first column vector that is linearly independent of $\mathfrak{A}_1$. Let $\mathfrak{A}_3$ be the first column vector that is linearly independent of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. By induction we have r linearly independent column vectors $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_r$. By these vectors P is represented in the following form:

$$P=(\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_r)\begin{bmatrix} c_{11} & c_{12} & \cdots\cdots\cdots & c_{1m} \\ & c_{22} & \cdots\cdots\cdots & c_{2m} \\ 0 & \ddots & & \vdots \\ & & c_{rr}\ \cdots\cdots & c_{rm} \end{bmatrix}.$$

Let $\{e_1, e_2, \cdots, e_r\}$ be the orthonormal r-frame obtained from $\{\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_r\}$. Then P is represented in the following form:

$$P=(e_1e_2\cdots e_r)\begin{bmatrix} b_{11} & b_{12} & \cdots\cdots\cdots & b_{1m} \\ & b_{22} & \cdots\cdots\cdots & b_{2m} \\ 0 & \ddots & & \vdots \\ & & b_{rr}\ \cdots\cdots & b_{rm} \end{bmatrix}.$$

We set

$$E=(e_1e_2\cdots e_r) \quad \text{and} \quad B=\begin{bmatrix} b_{11} & \cdots\cdots\cdots & b_{1m} \\ 0 \ \ddots & & \vdots \\ & b_{rr}\ \cdots\cdots & b_{rm} \end{bmatrix}.$$

Then $E\in V_{n,r}$ and $B\in M(r, m; r)$. Moreover, since $P\in K(n, m; r)$, $\langle EB, EB\rangle=\text{trace}\,({}^tB{}^tEEB)=\text{trace}\,({}^tBB)=1$.

Hence $B\in\Lambda_{m,r}$. q. e. d.

We define a map $\tilde{\varphi}$: $V_{n,r}\times\Lambda_{m,r}\to K(n, m; r)$ by $\tilde{\varphi}(E, B)=EB$. Since, at each point, $\tilde{\varphi}$ is a polynomial with respect to the local coordinates, $\tilde{\varphi}$ is continuous. Obviously $\tilde{\varphi}$ is onto. Next we assume $P=EB=E'B'$ for E, E' of $V_{n,r}$ and B, B' of $\Lambda_{m,r}$. Let $EB=(e_{ij}b_{jk})$ and $E'B'=(e'_{ij}b'_{jk})$. Then $e_{ij}b_{jk}=e'_{ij}b'_{jk}$ holds for all i, j and k.

Assume $b_{jk}\neq 0$. Then

$$e_{ij}=\frac{b'_{jk}}{b_{jk}}e'_{ij} \qquad (i=1, \cdots, n).$$

Since

$$\langle \boldsymbol{e}_j, \boldsymbol{e}_j\rangle=1 \qquad \text{for} \quad \boldsymbol{e}_j=\begin{bmatrix} e_{1j} \\ \vdots \\ e_{nj} \end{bmatrix},$$

$$1=\sum_{i=1}^{n} e_{ij}^2=\sum_{i=1}^{n}\left(\frac{b'_{jk}}{b_{jk}}\right)^2 e'^2_{ij}=\left(\frac{b'_{jk}}{b_{jk}}\right)^2 \sum_{i=1}^{n} e'^2_{ij}=\left(\frac{b'_{jk}}{b_{jk}}\right)^2.$$

Hence $b'_{jk}=\pm b_{jk}$ and

$e'_{ij}=\pm e_{ij}$ ($i=1, \cdots, n$; double signs in the same order). ......(*)

Also, since $e_{ij}b_{jk}=e'_{ij}b'_{jk}$ ($k=1, \cdots, m$), $b'_{jk}=\pm b_{jk}$ ($k=1, \cdots, m$) (double signs in the same order as (*)). If $b_{jk}=0$, then $b'_{jk}=0$ because

$$b'_{jk}=\frac{e_{ij}}{e'_{ij}}b_{jk} \text{ for some } e'_{ij}\neq 0.$$

Since $(b_{j1}, \cdots, b_{jm})\neq 0$, there is $b_{jh}\neq 0$ so that $b'_{jh}=\pm b_{jh}$. Since $e_{ij}b_{jh}=e'_{ij}b'_{jh}$, $e'_{ij}=\pm e_{ij}$ ($i=1, \cdots, n$) (double signs in the same order as (*)). Let G be a discrete subgroup of $O(r)$ defined by

$$G=\{T=(t_{ij})\,|\,t_{ij}=\pm\delta_{ij},\ T\in O(r)\}.$$

We define an action of $T\in G$ on $(E, B)\in V_{n,r}\times\Lambda_{m,r}$ by $T\cdot(E, B)=(ET, TB)$. Then G acts freely on $V_{n,r}\times\Lambda_{m,r}$. And for a given $P\in K(n, m; r)$ the expression of P in $P=EB$ is unique up to the action of G. Since G is a finite group, its action is totally discontinuous and $V_{n,r}\to V_{n,r}/G$ becomes a principal G-bundle. And $V_{n,r}\times_G\Lambda_{m,r}\to V_{n,r}/G$ turns out to be an associated fibre bundle with fibre $\Lambda_{m,r}$. Here $V_{n,r}\times_G\Lambda_{m,r}$ denotes the orbit space $(V_{n,r}\times\Lambda_{m,r})/G$. Hence for the given $EB\in K(n, m; r)$ we have $\tilde{\varphi}^{-1}(EB)=\{(ET, TB)\,|\,T\in G\}$. Therefore the map φ: $V_{n,r}\times_G\Lambda_{m,r}\to K(n, m; r)$ defined by $\varphi([E, B])=\tilde{\varphi}(E, B)=EB$ is a continuous map, one-to-one and onto.

LEMMA 6. 2. *The map* φ: $V_{n,r}\times_G\Lambda_{m,r}\to K(n, m; r)$ *is a homeomorphism.*

Proof. Since $\tilde{\varphi}$ is an onto map, for any $P\in K(n, m; r)$ there is $(E, B)\in V_{n,r}\times\Lambda_{m,r}$ such that $\tilde{\varphi}(E, B)=P$. To each $P\in K(n, m; r)$ we assign the class $[E, B]\in V_{n,r}\times_G\Lambda_{m,r}$ which containing (E, B). Then this correspondence ψ becomes a map ψ: $K(n, m; r)\to V_{n,r}\times_G\Lambda_{m,r}$. At each point the component functions of ψ have the form of rational functions with respect to the local coordinates. Hence ψ is a continuous function. Since $\psi\circ\varphi=$id. and $\varphi\circ\psi=$id., φ is a homeomorphism. q. e. d.

Let D^{nm} be the unit closed ball $\{x\,|\,x\in R^{nm}, \sum_i (x_i)^2\leqq 1\}$. If A is in $M(n, m; r)$, sA is also in $M(n, m; r)$ for any $s\neq 0$. Hence we have the following

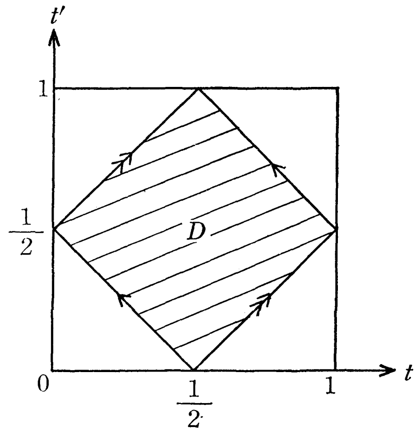
COROLLARY. *$D^{nm} \cap (\{0\} \cup M(n, m; r))$ is equal to the cone* Cone $(S^{nm-1} \cap M(n, m; r))$ *over $S^{nm-1} \cap M(n, m; r)$. Moreover the pair $(D^{nm}, D^{nm} \cap (\{0\} \cup M(n, m; r)))$ is equal to the pair* (Cone (S^{nm-1}), Cone $(S^{nm-1} \cap M(n, m; r))$).

§ 7. Proof of Proposition B.

By the corollary to Theorem C, $S^3 \cap M(2, 2; 1)$ is diffeomorphic to a 2-dimensional torus. Hence it suffices to parametrize $S^3 \cap M(2, 2; 1)$ by $I \times I$, where $I = [0, 1]$. Let D' be the following closed subset of $I \times I$ defined by four inequalities:

$$D' = \left\{(t, t') \mid (t, t') \in I \times I;\ \frac{1}{2} \leqq t + t' \leqq \frac{3}{2};\ -\frac{1}{2} \leqq t' - t \leqq \frac{1}{2}\right\}.$$

We identify the boundary of D' by the following relations: $(t, t+1/2) \sim (1/2+t, t)$ and $(t, -t+1/2) \sim (1/2+t, -t+1)$ for $0 \leqq t \leqq 1/2$.



Let D be the quotient space $D'/\sim$. Obviously D is diffeomorphic to a 2-dimensional torus. We define a map h of D to $S^3 \cap M(2, 2; 1)$ as follows:

$$h(t, t') = (\cos 2\pi t \cos 2\pi t',\ \cos 2\pi t \sin 2\pi t',\ \sin 2\pi t \cos 2\pi t',\ \sin 2\pi t \sin 2\pi t').$$

This definition is compatible with the relation "$\sim$" and h is well-defined. Clearly $h(D) \subset S^3 \cap M(2, 2; 1)$. By the two lemmas stated below, we see that h is a diffeomorphism.

LEMMA 7.1. *h: $D \to S^3 \cap M(2, 2; 1)$ is an onto map.*

Proof. Lemma 7.1 is equivalent to the following condition. For any 4-tuple (a_1, a_2, a_3, a_4) of real numbers satisfing the following relations:

$$(1)\qquad \begin{aligned} &a_1a_4-a_2a_3=0,\\ &a_1^2+a_2^2+a_3^2+a_4^2=1, \end{aligned}$$

there is a solution (t_0, t_0'), in D, of the following system of equations with two unknowns t and t';

$$(2)\qquad \begin{aligned} &\cos 2\pi t\cos 2\pi t'=a_1,\\ &\cos 2\pi t\sin 2\pi t'=a_2,\\ &\sin 2\pi t\cos 2\pi t'=a_3,\\ &\sin 2\pi t\sin 2\pi t'=a_4. \end{aligned}$$

Substituting $\alpha=2\pi t$ and $\beta=2\pi t'$ into (2) we have the following equations:

$$(\mathrm{A})\qquad \begin{aligned} &\cos(\alpha+\beta)=a_1-a_4,\\ &\sin(\alpha+\beta)=a_2+a_3; \end{aligned}$$

$$(\mathrm{B})\qquad \begin{aligned} &\sin(\alpha-\beta)=a_3-a_2,\\ &\cos(\alpha-\beta)=a_1+a_4. \end{aligned}$$

Making use of (1) we see that (A) and (B) have solutions. If we solve the above system under the conditions $\pi\leqq\alpha+\beta<3\pi$ and $-\pi\leqq\alpha-\beta<\pi$, we obtain its unique solution. By $\alpha_0=2\pi t_0$ and $\beta_0=2\pi t_0'$ we denote the unique solution. Clearly t_0 and t_0' satisfy the following inequalities: $1/2\leqq t_0+t_0'<3/2$ and $-1/2\leqq t_0-t_0'<1/2$. Hence this solution (t_0, t_0') is in D. Therefore h is an onto map and also it is proved in the above arguments that h is a one-to-one map. q. e. d.

LEMMA 7. 2. *h is a diffeomorphism.*

Proof. We regard the map h as a map of D into R^4 and consider the Jacobian matrix Jh of h with respect to the coordinates (t, t') and (x_1, x_2, x_3, x_4). It is sufficient to show that h has maximal rank on D. Jh has the following form:

$$Jh=\begin{bmatrix} -\sin 2\pi t\cos 2\pi t' & -\sin 2\pi t\sin 2\pi t' & \cos 2\pi t\cos 2\pi t' & \cos 2\pi t\sin 2\pi t' \\ -\cos 2\pi t\sin 2\pi t' & \cos 2\pi t\cos 2\pi t' & -\sin 2\pi t\sin 2\pi t' & \sin 2\pi t\cos 2\pi t' \end{bmatrix}.$$

It is easy to see that Jh has maximal rank on D. q. e. d.

Let Φ be the map of $I\times I$ to $S^3\cap M(2, 2; 1)$ defined by

$$\Phi(t, t')=\Big(-\frac{1}{2}(\cos 2\pi t+\cos 2\pi t'),\ -\frac{1}{2}(\sin 2\pi t+\sin 2\pi t'),\ -\frac{1}{2}(\sin 2\pi t-\sin 2\pi t'),\ \frac{1}{2}(\cos 2\pi t-\cos 2\pi t')\Big).$$

By Lemmas 7.1 and 7.2 Φ is a parametrization of $S^3 \cap M(2, 2; 1)$ with two parameters t and t'. Let A be the following matrix

$$A=\sqrt{\frac{1}{2}}\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Let

$$CT^2=\left\{\sqrt{\frac{1}{2}}(\cos 2\pi t,\ \sin 2\pi t,\ \cos 2\pi t',\ \sin 2\pi t') \mid 0 \leqq t,\ t' \leqq 1\right\}.$$

Then it is easy to show that A is in $SO(4)$ and A maps CT^2 on $S^3 \cap M(2, 2; 1)$ by the right operation. q. e. d.

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Tokyo Metropolitan University.