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# UNIVALENCY OF ANALYTIC MAPPINGS OF A RIEMANN SURFACE INTO ITSELF

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**1.** In the present paper we shall study a Riemann surface whose every non-constant analytic mapping into itself is univalent.

Let $S$ be the class of Riemann surfaces whose every non-constant analytic mapping into itself is univalent, and let $K$ be the class of Riemann surfaces whose every non-constant analytic mapping into itself is univalent and onto. It is easy to see that $\phi \neq K \subsetneqq S \subset O_{AB} \cap H$ where $H$ is the class of Riemann surfaces whose universal covering are conformally equivalent to the unit disk. Heins [5] showed $O_G \cap H \subset S$ and $K_G \subset K$ where $K_G$ denotes the class of Riemann surfaces with a finite positive genus or with a finite number of planar boundary elements belonging to $O_G \cap H$. Kubota [8] introduced a class of Riemann surfaces and showed that the class is a subclass of $K$. In § 2 we construct an example of Riemann surface of class $O_{AB} \cap H$ on which there exists a non-univalent analytic mapping into itself. Namely we show $S \subsetneqq O_{AB} \cap H$. In § 3 we introduce a class $K_{HD}$ of Riemann surfaces and show $K_{HB} \subsetneqq K_{HD} \subset K$, where $K_{HB}$ denotes the class of Riemann surfaces introduced by Kubota. Heins [5] showed that if $W$ is of class $K_G$ and of finite genus, then the number of non-constant analytic mappings of $W$ into itself is finite. In § 4 we show the same result with respect to a Riemann surface of class $K_{HD}$.

**2.** We construct an example of a Riemann surface $W$ of class $O_{AB} \cap H$ on which there exists a non-univalent analytic mapping into itself. It will be given as a covering surface of the $z$-plane. We introduce $E$, $F$ and $D$ as follows:

$$E = \{0 < |z| < \infty\} - \bigcup_{n=-\infty}^{\infty} [4^n, 2 \cdot 4^n],$$

$$F = E - \{|z+1| \leqq 1\} - [-6, -4],$$

$$D = \{|z+5| < 2\} - [-6, -4],$$

where $[a, b] = \{z \mid a \leqq \operatorname{Re} z \leqq b, \operatorname{Im} z = 0\}$. We joint copies of $E$ and $F$ along their common slits identifying the upper edges of the slits of $E$ with the corresponding lower edges of the slits of $F$ and vice versa. The edges of the remained free slit of $F$ are identified with the opposite edges of the corresponding slit of a copy of $D$. Thereby a Riemann surface $W$ is constructed as a covering surface $(W, \pi)$ of the $z$-plane (cf. Ahlfors-Sario [1], pp. 119–120).

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Let $G$ be the covering of $\{|z+4|\leqq 4\}$ lying in the joining of $F$ and $D$. Then, by using the same arguments in Myrberg's paper [9], we see that $W-G$ is of class $O_{AB}$. Hence $W$ is of class $O_{AB}\cap H$. Let $\varphi$ be a mapping of $W$ into itself which satisfies $\pi\circ\varphi\circ\pi^{-1}(z)=4z$ and carries the points of $E$, $F$ and $D$ onto the points of $E$, $F$ and $F$ respectively. Then $\varphi$ is analytic and non-univalent.

**3.** In this section we introduce the class $K_{HD}$ of Riemann surfaces such that $K_{HD}\subset K$. We show first the following lemma.

LEMMA 1. *Let $W$ be a Riemann surface whose fundamental group is non-abelian, and let $\varphi$ be an analytic mapping of $W$ into itself whose valence function $\nu_\varphi$ is a constant $n_\varphi$ $(\leqq\infty)$ except a set of zero area. If there exists a non-constant harmonic function $u$ with finite Dirichlet integral which satisfies*

$$u\circ\varphi=cu, \tag{1}$$

*where $c$ is a real constant, then $c$ is equal to $\pm 1$ and $\varphi$ has a finite period $p$ (i.e. the $p$-th iterate $\varphi_p$ of $\varphi$ is the identity mapping $\iota$ of $W$ onto itself).*

REMARK 1. If the fundamental group of $W$ is abelian then there is an example such that $\varphi$ has no period: $W=\{r<|z|<1\}$ $(r>0)$, $\varphi(z)=e^{2\pi\theta i}\cdot z$ ($\theta$ is an irrational real number), $u=\log|z|$, $c=1$.

REMARK 2. If $\varphi$ does not satisfy the condition on the valence function, then it is easy to construct an example such that $\varphi$ is not univalent.

REMARK 3. If $u$ is a harmonic function with infinite Dirichlet integral, then there is an example such that the valence function is a constant $n(\geqq 2)$ except one point: $W=\{0<|z|<1\}-\{r^{n^{-k}}\cdot e^{l\cdot n^{-k}2\pi i}\mid 0\leqq k<\infty,\ 0\leqq l\leqq n^k-1\}$ $(0<r<1)$, $\varphi(z)=z^n$, $u=\log|z|$, $c=n$.

REMARK 4. If $u(\fallingdotseq\text{const})$ is a bounded harmonic function with finite Dirichlet integral, then we are able to replace the condition on $\varphi$ in lemma 1 by a weaker condition that $W$ is covered by the image $\varphi(W)$ of $\varphi$ except a set of zero area. In fact, we may assume without loss of generality that $\sup_W u$ is positive. For the 2nd iterate $\varphi_2$ of $\varphi$ we have

$$\sup_{\varphi_2(W)} u=\sup_W(u\circ\varphi_2)=\sup_W(c^2u)=c^2\sup_W u,$$

$$\sup_{\varphi_2(W)} u\leqq\sup_W u.$$

Hence $c^2\leqq 1$. Therefore we have

$$D_{\varphi(W)}(u)=D_W(u\circ\varphi)=D_W(cu)=c^2D_W(u)\leqq D_W(u),$$

where

$$D_{\varphi(W)}(u)=\int_W \nu_\varphi du\cdot du^*.$$

On the other hand, by the above condition we have

$$D_{\varphi(W)}(u) \geqq D_W(u).$$

Hence the valence function $\nu_\varphi$ is equal to 1 except a set of zero area.

*Proof of lemma* 1. We use the following result due to Heins [5]:

Let $W$ denote a non-compact Riemann surface whose fundamental group is non-abelian, and let $\varphi$ denote an analytic mapping of $W$ into itself. If $\varphi$ neither

i) possesses a fixed point $\zeta$, nor

ii) has a finite period $p$, then

iii) for every given compact subsets $K_1$, $K_2$ of $W$ there exists a natural number $N$ such that $\varphi_N(K_1) \subset W - K_2$.

We show first $n_\varphi = c^2 < \infty$. This follows from the following formulae.

$$D_W(u \circ \varphi) = D_{\varphi(W)}(u) = n_\varphi D_W(u),$$

$$D_W(cu) = c^2 \cdot D_W(u).$$

We show next that iii) leads to a contradiction. Let $\{W_n\}_{n=1}^\infty$ be a canonical exhaustion of $W$. Since $D_W(u)$ is finite, for any given positive number $\varepsilon$ there is a natural number $n$ such that $D_{W-\overline{W}_n}(u) < \varepsilon$. Setting $K_1 = K_2 = \overline{W}_n$, we find a natural number $N = N(n)$ such that $\varphi_N(\overline{W}_n) \subset W - \overline{W}_n$. Hence we have

$$D_{W_n}(u \circ \varphi_N) = D_{\varphi_N(W_n)}(u) \leqq n_\varphi^N \cdot D_{W-\overline{W}_n}(u).$$

By formula (1) we have

$$D_{W_n}(c^N u) = c^{2N} D_{W_n}(u) = n_\varphi^N D_{W_n}(u),$$

and hence

$$D_W(u) = D_{W_n}(u) + D_{W-\overline{W}_n}(u)$$

$$\leqq 2 D_{W-\overline{W}_n}(u) < 2\varepsilon.$$

Therefore $u$ must reduce to a constant. This is a contradiction.

Finally we show that i) implies ii). Let $(\{|z|<1\}, \pi)$ be the universal covering surface of $W$ such that $\pi$ is analytic and satisfies $\pi(0) = \zeta$. We consider $\pi^{-1}$ in the neighborhood of $\zeta$ satisfying $\pi^{-1}(\zeta) = 0$ and set $f = \pi^{-1} \circ \varphi \circ \pi$ around 0. We continue analytically the function element of $f$ onto $\{|z|<1\}$. Then $f$ satisfies $f(0) = 0$, $|f(z)| < 1$ and $\varphi_k \circ \pi = \pi \circ f_k$ $(k = 1, 2, \cdots)$. Setting $v = u \circ \pi$, we have $v \circ f = cv$. Let $h$ be an analytic function on $\{|z|<1\}$ having $v$ as its real part and set $g = h - h(0)$. Then we have

$$g \circ f = cg \tag{2}$$

and $g(0) = 0$. If $f$ and $g$ have the expansions around the origin

$$f(z)=az^j+a_1z^{j+1}+\cdots, \qquad a\neq 0, \quad j\geqq 1,$$

$$g(z)=bz^k+b_1z^{k+1}+\cdots, \qquad b\neq 0, \quad k\geqq 1,$$

then from (2) we have $j=1$, $a^k=c$ and $|a^k|=|c|=\sqrt{n_\varphi}\geqq 1$. Using Schwarz's lemma we have $a^{2k}=c^2=1$, $f(z)=az$ and $\varphi_{2k}\circ\pi=\pi\circ f_{2k}=\pi$. Hence we have $\varphi_{2k}=\iota$. Therefore $\varphi$ has a finite period $p$. It follows immediately that $u=u\circ\varphi_p=c^pu$, and hence we have $c=\pm 1$.

We consider next a problem whether there exists a harmonic function $u(\neq\text{const})$ satisfying (1) for a given analytic mapping $\varphi$ of $W$ into itself. This is an eigenvalue problem in the following sense. For every harmonic function $u$ on $W$ the composition $u\circ\varphi$ is also harmonic on $W$. We denote by $H(W)$ the class of harmonic functions on $W$ and set $\varphi^*(u)=u\circ\varphi$. Then $\varphi^*$ is a linear operator of $H(W)$ into itself and (1) is represented using $\varphi^*$ as follows:

$$(1') \qquad \varphi^*(u)=cu$$

where $c$ is an eigenvalue of $\varphi^*$ and $u$ is its eigenelement. From this point of view we consider an eigenvalue problem of the restriction $\varphi^*|X$ of $\varphi^*$ to $X$, where $X$ is a linear subspace of $H(W)$ such that $\varphi^*(X)\subset X$. If $X$ is a finite dimensional lattice-ordered linear space (vector lattice) with respect to the natural order, then $X$ has a base consisting of $X$-minimal functions (cf. Constantinescu-Cornea [3]). From this fact we obtain a matricial representation of $\varphi^*|X$.

LEMMA 2. *Let $\varphi$ be an analytic mapping of a Riemann surface $W$ into itself such that $W$ is covered by $\varphi(W)$ except a set of zero area, and let $X\subset H(W)$ be a finite dimensional lattice-ordered linear space satisfying $\varphi^*(X)\subset X$. Choose a base $u_1, u_2, \cdots, u_n$ of $X$ consisting of $X$-minimal functions and set*

$$\begin{pmatrix}\varphi^*(u_1)\\ \varphi^*(u_2)\\ \vdots\\ \varphi^*(u_n)\end{pmatrix}=\Phi\begin{pmatrix}u_1\\ u_2\\ \vdots\\ u_n\end{pmatrix}$$

*where $\Phi$ is a square matrix of degree $n$. Then $\Phi$ is regular and equal to $(c_i\cdot\delta_{\sigma(i)j})$, where $c_i$ $(i=1, 2, \cdots, n)$ are positive constants, $\delta_{ij}$ is Kronecker's symbol and $\sigma$ is a permutation of degree $n$. Consequently, if we denote by $s$ the order of $\sigma$, then $\Phi^s$ is a diagonal matrix and all its diagonal elements are positive.*

*Proof*. The regularity of $\Phi$ follows from the fact that $\varphi^*$ is injective and $X$ is of finite dimension. If we set

$$\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{pmatrix}=\Phi^{-1}\begin{pmatrix}u_1\\ u_2\\ \vdots\\ u_n\end{pmatrix},$$

then we have

$$\begin{pmatrix} \varphi^*(v_1) \\ \varphi^*(v_2) \\ \vdots \\ \varphi^*(v_n) \end{pmatrix} = \Phi^{-1} \begin{pmatrix} \varphi^*(u_1) \\ \varphi^*(u_2) \\ \vdots \\ \varphi^*(u_n) \end{pmatrix} = \Phi^{-1}\Phi \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

and hence $v_i \circ \varphi = u_i$ $(i=1, 2, \cdots, n)$. Since $W$ is covered by $\varphi(W)$ except a set of zero area, the functions $v_i$ are positive. For any $v \in X$ such that $v>0$, $v \leqq v_i$ it follows that $v \circ \varphi \in X$, $v \circ \varphi > 0$ and $v \circ \varphi \leqq v_i \circ \varphi = u_i$. Hence $v \circ \varphi = cu_i = c(v_i \circ \varphi) = (cv_i) \circ \varphi$. This implies that $v_i$ are also $X$-minimal functions. Hence there exists a permutation $\tau$ of degree $n$ satisfying $v_i = k_i u_{\tau(i)}$ $(i=1, 2, \cdots, n)$ with positive constants $k_i$. Setting $\sigma = \tau^{-1}$ and $c_i = 1/k_{\tau^{-1}(i)}$, we have the desired result.

From lemma 1 and 2 we have the following lemma.

LEMMA 3. *Let $W$ be a Riemann surface whose fundamental group is non-abelian, and let $\varphi$ be a non-constant analytic mapping of $W$ into itself whose valence function is finite and constant except a set of zero area. If there exists a lattice-ordered linear space $X \subset H(W)$ which satisfies* (i) *$\varphi^*(X) \subset X$ and that* (ii) *$X \cap HD(W)$ is of finite dimension and contains at least one non-constant function, then $\varphi$ has a finite period.*

*Proof*. Since $HD = HD(W)$ is a lattice-ordered linear space, $X \cap HD$ is a finite dimensional lattice-ordered linear space. By the condition on the valence function we have $\varphi^*(HD) \subset HD$ and hence $\varphi^*(X \cap HD) \subset X \cap HD$. We apply now lemma 2 to $X \cap HD$. Then there exists a natural number $s$ such that every $X \cap HD$-minimal function is an eigenelement of $\varphi_s^* | X \cap HD$. We apply further lemma 1 to $X \cap HD$-minimal functions. Then the matrix $\Phi^s$ is equal to the unit one and $\varphi$ has a finite period.

We introduce now the class $K_{HD}$.

DEFINITION. We denote by $K_{HD}$ the class of Riemann surfaces $W$ which satisfy the following conditions:

i) Every non-constant analytic mapping of $W$ into itself is a Dirichlet mapping and of type $Bl$, i.e. the valence function is finite and constant except a set of capacity zero.

ii) Let $M_Y$ be the linear space generated by all $Y$ $(Y = HP, HB, HD)$-minimal functions. The space $M_Y \cap HD$ is of finite dimension and contains at least one non-constant function.

The class $K_{HD}$ is not empty. In fact, the class $O_{HB}^n - O_{HD}$ is a subclass of $K_{HD}$. If $W$ is of class $O_{HB}^n - O_{HD}$, then we have $M_{HB} = HB \supset HD$. This implies that the condition ii) is fulfilled for $Y = HB$. Since $W$ is of class $O_{HB}^n$, each non-constant analytic mapping $\varphi$ of $W$ into itself is of type $Bl$ and statisfies $\varphi^*(HD) \subset HD$. Using the same argument in the proof of lemma 3 and remark 4 $\varphi$ is univalent, and hence the condition i) is satisfied.

The class $K_{HB}$ which is introduced by Kubota [8] is a proper subclass of $K_{HD}$. If $W$ is of class $K_{HB}$, then the condition i) is fulfilled (cf. Kubota [8]). In the following we use the notation in [8]. Let $B_i$ be a set of positive measure. Then the harmonic measure $\omega_i=\lim_{\nu\to\infty}\omega_i^{(\nu)}$ of $B_i$ is non-constant and its Dirichlet integral is finite since by the definition of $K_{HB}$ there exists another set $B_j$ of positive measure. We assume that $B_i$ consists of $HB$-indivisible sets and a set of measure zero. Then $\omega_i$ belongs to $M_{HB}$, and hence the condition ii) is satisfied. To see $K_{HB}\neq K_{HD}$, we consider a Riemann surface $W$ which is of class $O_{HB}^n-O_{HD}$ and has one ideal boundary component (cf. Constantinescu-Cornea [2], pp. 230–231). Then from the above argument $W$ is of class $K_{HD}$, but by the definition of $K_{HB}$ $W$ is not of class $K_{HB}$.

THEOREM 1. *The class $K_{HD}$ is a subclass of $K$.*

*Proof*. Suppose that $W$ is of class $K_{HD}$. Then $M_Y$ is a lattice-ordered linear space and satisfies $\varphi^*(M_Y)\subset M_Y$ for every non-constant analytic mapping $\varphi$ of $W$ into itself (cf. Constantinescu-Cornea [3], pp. 123–124). Applying lemma 3, we have that $W$ is of class $K$.

**4.** In this section we show the following theorem.

THEOREM 2. *If $W$ is of class $K_{HD}$, then the number of non-constant analytic mappings of $W$ into itself is finite.*

*Proof*. Let $\{\varphi^{(k)}\}_{k=1}^{\infty}$ be a sequence of non-constant analytic mappings of $W$ into itself. From theorem 1 we know that each $\varphi^{(k)}$ is univalent and onto. We apply lemma 2 to $M_Y\cap HD$ and denote by $\sigma_k$ the permutation of $\varphi^{(k)*}$. Then there exists a permutation $\sigma_0$ and a subsequence $\{\varphi^{(k_l)}\}$ of $\{\varphi^{(k)}\}$ such that $\sigma_{k_l}=\sigma_0$ $(l=1,2,\cdots)$. For the sake of simplicity we write $\{\varphi^{(k)}\}$ for $\{\varphi^{(k_l)}\}$. From lemma 1 all the matrices $\Psi^{(k)}$ of $\psi^{(k)}=\varphi^{(k)}\circ\varphi_{-1}^{(1)}$, where $\varphi_{-1}^{(1)}$ is the inverse mapping of $\varphi^{(1)}$, are equal to the unit one. Hence there exists at least one non-constant harmonic function $u$ with finite Dirichlet integral such that $u\circ\psi^{(k)}=u$ $(k=1,2,\cdots)$. If $\{\psi^{(k)}\}_{k=1}^{\infty}$ is a sequence of mutually distinct mappings, then for every two compact sets $K_1$, $K_2$ there exists a natural number $N$ such that $\psi^{(N)}(K_1)\subset W-K_2$ (cf. Heins [4], Komatu-Mori [6] and Kubota [7]). Let $\{W_n\}_{n=1}^{\infty}$ be a canonical exhaustion of $W$. Since $D_W(u)$ is finite, for any given positive number there exists a natural number $n$ such that $D_{W-\overline{W}_n}(u)<\varepsilon$. Setting $K_1=K_2=\overline{W}_n$, we find a natural number $N=N(n)$ such that $\psi^{(N)}(\overline{W}_n)\subset W-\overline{W}_n$. Hence we have

$$D_{W_n}(u)=D_{W_n}(u\circ\psi^{(N)})=D_{\psi^{(N)}(W_n)}(u)\leqq D_{W-\overline{W}_n}(u),$$

and

$$\begin{aligned} D_W(u)&=D_{W_n}(u)+D_{W-\overline{W}_n}(u)\\ &\leqq 2D_{W-\overline{W}_n}(u)<2\varepsilon. \end{aligned}$$

Therefore $u$ must reduce to a constant. This is a contradiction.

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