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ON A SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS RELATED TO A TURNING POINT PROBLEM

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§1. Introduction.

1° In order to analyse the so called turning point problem, sometimes the given equation will be reduced to a simpler type. If the given equation, however, has a "complicated" turning point, it will be investigated in several domains separately, where the original equation behaves in a quite different manner, and each solution obtained in the corresponding domain will be *matched* with the solutions in adjacent domains by adequate methods. Iwano [2] analysed how to divide the domain where the equation is defined and how to reduce the equation in each of the divided domains. For instance, the equation with a turning point at the origin

$$\varepsilon\frac{dy}{dx}=\begin{bmatrix}0 & 1\\ x^3-\varepsilon & 0\end{bmatrix}y$$

can be changed by a transformation $y=\mathrm{diag}[1, x^{3/2}]u$ to

$$(x^{-3}\varepsilon)x^{3/2}\frac{du}{dx}=\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+(x^{-3}\varepsilon)\begin{bmatrix}0 & 0\\ -1 & -\dfrac{3}{2}x^{1/2}\end{bmatrix}\right\}u$$

in a domain $M_1|\varepsilon|^{1/3}\leqq|x|\leqq\delta_0$; by transformations $x=\varepsilon^{1/3}\xi$, $y=\mathrm{diag}[1, \varepsilon^{1/2}]v$ to

$$\varepsilon^{1/6}\frac{dv}{d\xi}=\begin{bmatrix}0 & 1\\ \xi^3-1 & 0\end{bmatrix}v$$

in a domain $M_2|\varepsilon|^{1/2}\leqq|x|\leqq M_1|\varepsilon|^{1/3}$; and by transformations $x=\varepsilon^{1/2}\eta$, $y=\mathrm{diag}[1, \varepsilon^{1/2}]w$ to

$$\frac{dw}{d\eta}=\left\{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+\varepsilon^{1/2}\begin{bmatrix}0 & 0\\ \eta^3 & 0\end{bmatrix}\right\}w$$

in a domain $|x|\leqq M_2|\varepsilon|^{1/2}$. Here δ_0 is a small constant and M_i $(i=1, 2)$ are large

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ones. The first equation would be investigated away from the turning point and the solution expansible in $x^{-3}\varepsilon$ could be obtained at least formally since its first coefficient has different characteristic roots: 1 and -1. The second equation may require the global consideration, for it must be investigated for ξ both small and large, and queerly enough three new *secondary* turning points appeared, i.e., roots of $\xi^3-1=0$. The last equation is apparently of regular perturbation type. A difference from the ordinary regular perturbation is that it must be analysed globaly because the new independent variable η varies for $|\eta|\leqq M_2$ with M_2 large and it may be infinity in some case. Notice $\xi=0$ corresponds to the original turning point $x=0$ but the roots of $\xi^3-1=0$ do not.

The differential equation of the above example does not satisfy the "one-segment condition" of its characteristic polygon (Iwano [2]), it is the case satisfying the simplest "two-segment condition" and will be investigated lateron.

Here we shall consider the case of an apparent regular pertrubation—such as the transformed last equation of the above example—and widen a central angle of the corresponding inner domain maximal in a sense in which a special type of asymptotic expansion for the solution is valid. We use a term "inner domain" to be the domain containing the original turning point.

As for the maximality of the complement of the inner domain, see, e.g., Nishimoto [5]. The widening central angles may be necessary not only for mathematical interests but also for applications, say, for boundary value problems.

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2° The equation considered is followed from an equation of the type

$$\varepsilon^{\sigma}\frac{dY}{dx}=A(x,\varepsilon)Y, \tag{1}$$

where σ is a positive integer, ε is a complex small parameter, Y is an n-dimensional column vector or an n-by-n matrix function holomorphic in x and ε for $|x|\leqq x_0$, $0<|\varepsilon|\leqq\varepsilon_0$, $|\arg\varepsilon|\leqq\varepsilon_1$, and admits an asymptotic expansion such that

$$A(x,\varepsilon)\sim\sum_{r=0}^{\infty}A_r(x)\varepsilon^r$$

as ε tends to zero.

This paper is a partial continuation of the previous one [4], which was concentrated on the formal theory and assumed that the differential equation (1) satisfies following conditions:

$$A_0(x)=\mathrm{diag}[a_1(x), a_2(x), \cdots, a_n(x)]x^k, \tag{α}$$

where $a_\nu(x)$ is holomorphic in $|x|\leqq x_0$ and $a_\nu(x)\neq a_\mu(x)$ for $\nu\neq\mu$ and for all values of x: $|x|\leqq x_0$. For $r\geqq 1$, $A_r(x)$ is of lower triangular and

$$A_r(x)=\left[\sum_{h\geqq m_{\nu\mu}^{(r)}}a_{\nu\mu h}^{(r)}x^h\right] \quad \text{and} \quad a_{\nu\mu,m_{\nu\mu}^{(r)}}^{(r)}\neq 0;$$

(β) The one-segment condition: hold the inequalities

$$k\geqq 1 \quad \text{and} \quad \frac{m_{\nu\mu}^{(r)}}{\nu+1-\mu}>k-\frac{k+1}{\sigma}\cdot\frac{r}{\nu+1-\mu} \quad \text{for} \quad \nu\geqq\mu;\ r=1,2,3,\cdots.$$

Thus the origin is *a turning point* of order $k\geqq 1$.

§ 2. The problem and notations.

3° The equation to be considered in the present paper is as follows:[1]

$$\frac{dU}{dz}=A(z,\rho)U, \tag{2}$$

where U is an n-dimensional column vector or an n-by-n matrix, ρ is a small complex parameter and $A(z,\rho)$ is an n-by-n matrix holomorphic for both z and ρ in the region

$$\mathfrak{Z}:\ \text{all } z \text{ for } |z|\geqq 0, \qquad 0<\rho\leqq\rho_0, \qquad |\arg\rho|\leqq\rho_1,$$

with ρ_0, ρ_1 small constants, and $A(z,\rho)$ asymptotically expansible such that

$$A(z,\rho)\sim\sum_{r=0}^{\infty}A_r(z)\rho^r \qquad \text{as} \quad \rho\to 0 \text{ in } \mathfrak{Z}.$$

The coefficient $A_0(z)$ possesses the form

$$A_0(z)=\mathrm{diag}[a_1,a_2,\cdots,a_n]z^k,$$

where k is a positive integer, $a_1,a_2,\cdots,a_n$ are complex constants and are characterized by

$$a_\nu\neq a_\mu \qquad \text{for} \quad \nu\neq\mu,$$

$$\arg\bar{a}_{1\mu}\leqq\arg\bar{a}_{2\mu}\leqq\cdots\leqq\arg\bar{a}_{n-1,\mu}<\arg\bar{a}_{1\mu}+2\pi,$$

in which

$$a_{\nu\mu}=a_\nu-a_\mu$$

and $\bar{a}$ designates a complex conjugate of a.

$A_r(z)$, $r=1,2,3,\cdots$, is a polynomial of degree $pr+q$, or

$$A_r(z)=z^{pr+q}\tilde{A}_r(z) \qquad (r=1,2,3,\cdots),$$

where p and q are integers such as $p\geqq 1$, $p+q\geqq 0$ and $A_r(z)$ is bounded for $|z|$ large.

1) The equation (2) is of a slightly more generalized form than one dealt in the previous paper [4] § 3, and this is followed from (1) by appropriate stretching and shearing transformations as introduced in the example of this introduction.

The above asymptotic expansion means precisely that

$$A(z, \rho)-\sum_{r=0}^{m} A_r(z)\rho^r = z^q E_{1m}(z, \rho)\cdot(z^p\rho)^{m+1} \qquad \text{for} \quad |z| \text{ large,}$$

where $E_{1m}(z, \rho)$ is bounded in $\mathfrak{Z}$, and $E_{1m}(z, \rho)z^{p(m+1)+q}$ is bounded for $|z|$ small in $\mathfrak{Z}$.

The problem is to obtain solutions of (2) as $\rho \to 0$, and the main consequence is two theorems: Theorem A in § 3 and B in § 6.

4° *Definition of admitted sectors.* We define sectors, called maximal admissible, bounded by straight lines, $\arg z = \Theta_+$ and $\arg z = \Theta_-$, passing through the origin in the z-plane.

First of all, two lines $\arg z = \hat{\Phi}_+^{(\mu)}$ and $\arg z = \hat{\Phi}_-^{(\mu)}$ are chosen such that

$$\hat{\Phi}_+^{(\mu)} < \arg \bar{a}_{1\mu} + \frac{3}{2}\pi, \qquad \arg \bar{a}_{n-1,\mu} - \frac{3}{2}\pi < \hat{\Phi}_-^{(\mu)},$$

$$\pi < \hat{\Phi}_+^{(\mu)} - \hat{\Phi}_-^{(\mu)} < 2\pi.$$

This choice is always possible, for the relation $\arg \bar{a}_{1\mu} + 3\pi/2 - (\arg \bar{a}_{n-1,\mu} - 3\pi/2) > \pi$ holds.

Notice determination of $\hat{\Phi}_\pm^{(\mu)}$ is not unique and refer 7° about the notation $\wedge$.

Further we define

$$\Phi_\pm^{(\mu)} = \frac{1}{k+1}\hat{\Phi}_\pm^{(\mu)}.$$

The sector bounded by $\Phi_\pm^{(\mu)}$ is called *admitted* for $U^{(\mu)}$, the μ-th column of the matrix solution U.

Let

$$\Theta_+^{(\mu)} = \sup \Phi_+^{(\mu)}, \qquad \Theta_-^{(\mu)} = \inf \Phi_-^{(\mu)},$$

where $\Phi_\pm^{(\mu)}$ are to satisfy all the properties above.

Let the domain $\mathfrak{S}^{(\mu)}$ be defined by the inequalities

$$\mathfrak{S}^{(\mu)}: \ \Theta_-^{(\mu)} = \frac{1}{k+1}\left(\arg \bar{a}_{n-1,\mu} - \frac{3}{2}\pi\right) < \arg z < \frac{1}{k+1}\left(\arg \bar{a}_{1\mu} + \frac{3}{2}\pi\right) = \Theta_+^{(\mu)}.$$

and let *the exterior sector* $\mathfrak{S}_e^{(\mu)}$ and *the interior* $\mathfrak{S}_i^{(\mu)}$ be defined such that $\mathfrak{S}_e^{(\mu)}$ is a subset of the sector $\mathfrak{S}^{(\mu)}$ for $|z|$ large, and $\mathfrak{S}_i^{(\mu)}$ is a complement of the exterior in $\mathfrak{S}^{(\mu)}$. The precise definition of $\mathfrak{S}_e^{(\mu)}$ and $\mathfrak{S}_i^{(\mu)}$ will be given later (as in Figure 2 in 8°).

The sector $\mathfrak{S}^{(\mu)}$ is called *maximal admissible* for $U^{(\mu)}$. We define a sector $\mathfrak{S}$ the maximal intersection of $\mathfrak{S}^{(\mu)}$ with respect to $\mu = 1, 2, 3, \cdots, n$, that is, if $\Theta_\pm$ are defined:

$$\Theta_+ = \min_{1 \leq \mu \leq n} \Theta_+^{(\mu)}, \qquad \Theta_- = \max_{1 \leq \mu \leq n} \Theta_-^{(\mu)},$$

then

$$\mathfrak{S}:\quad \Theta_{-}<\arg z<\Theta_{+} \qquad \text{or} \qquad \mathfrak{S}=\bigcap_{\mu=1}^{n}\mathfrak{S}^{(\mu)}.$$

The notations $\mathfrak{S}_e$, $\mathfrak{S}_i$ and the like are to be understood similarly to the case of $\mathfrak{S}_e^{(\mu)}$, $\mathfrak{S}_i^{(\mu)}$.

The angle of the sector $\mathfrak{S}^{(\mu)}$ is just $3\pi/(k+1)$ for $n=2$, and for the case $n=2$ and $k=1$ this result corresponds to the well-known property of the asymptotic expansion of the Bessel function.

We remark the maximal admissibility sector $\mathfrak{S}$ for $k=1$ contains possibly the whole real axis, if necessary, by rotation of the axes.

§ 3. A formal solution.

This and the following two sections are devoted to existence of formal solutions of the given equation (2).

5° One of our main purposes is the following

THEOREM A. *The differential equation* (2) *possesses the formal solution such that*

$$U(z,\rho)\sim\sum_{r=0}^{\infty}U_r(z)\rho^r \qquad \text{as} \quad \rho\to 0 \text{ in } \mathfrak{Z} \text{ and } z\in\mathfrak{S}.$$

The coefficients $U_r(z)$ are defined as follows:

$$U_0(z)=\exp B(z),$$

$$U_r(z)=U_r^*(z)\cdot\exp B(z) \qquad (r=1,2,3,\cdots),$$

where

$$B(z)=\int^{z}A_0(z)dz=z^{k+1}/(k+1)\cdot\operatorname{diag}[a_1,a_2,\cdots,a_n]=\operatorname{diag}[\beta_1(z),\beta_2(z),\cdots,\beta_n(z)].$$

In the interior sector $\mathfrak{S}_i$ $U_r^*(z)$ is bounded, and in the exterior domain $\mathfrak{S}_e$ $U_r^*=z^{m^*r}U_r^*(z)$ $(r=1,2,3,\cdots)$, $m^*=p+q+1$ and $U_r^*(z)$ is bounded in $\mathfrak{S}_e$.

The proof is long and so will be, for convenience, separated into several stages. The value μ is arbitrarily fixed in the following proof.

6° *Construction of integral equations.* Let

$$U(z,\rho)=\sum_{r=0}^{\infty}U_r(z)\rho^r$$

be a formal solution of (2). Then inserting it into (2) we obtain the following recurrence formulae:

$$(3)_0 \qquad \frac{dU_0}{dz}=A_0(z)U_0,$$

$$(3)_r \qquad \frac{dU_r}{dz}=A_0(z)U_r+\sum_{j=1}^{r}A_j(z)U_{r-j} \qquad (r=1,2,3,\cdots).$$

Since $A_0(z)$ is diagonal, we get at once the solution of $(3)_0$:

$$U_0(z)=\exp B(z).$$

The solution of the equation $(3)_r$, a non-homogeneous type of $(3)_0$, must satisfy an integral equation

$$(4) \qquad U_r(z)=\int_{\mathfrak{P}(z)} e^{B(z)-B(\zeta)}\sum_{j=1}^{r}A_j(\zeta)U_{r-j}(\zeta)d\zeta,$$

where $\mathfrak{P}(z)$ is a matrix consisting of elements $\mathfrak{P}_{\nu\mu}(z)$ $(\nu, \mu=1,2,\cdots,n)$ and each of them is respectively a path, ending z from ∞, for the (ν, μ)-element of the matrix $U_r(z)$. Here we omitted the index r of the path-matrix since the paths $\mathfrak{P}_{\nu\mu}(z)$ can be chosen independently of r as shown later.

Let

$$V_0(z)=I \qquad \text{and} \qquad U_r(z)=V_r(z)\cdot\exp B(z) \qquad (r=1,2,3,\cdots).$$

Then from (4), we obtain

$$V_r(z)=\int_{\mathfrak{P}(z)} e^{B(z)-B(\zeta)}\sum_{j=1}^{r}A_j(\zeta)V_{r-j}(\zeta)e^{B(\zeta)-B(z)}d\zeta.$$

Let the value r be fixed and change notations:

$$V_r(z)=V(z) \qquad \text{and} \qquad \sum_{j=1}^{r}A_j(\zeta)V_{r-j}(\zeta)=M(\zeta).$$

Thus the above integral equation is written in new notations as

$$V(z)=\int_{\mathfrak{P}(z)} e^{B(z)-B(\zeta)}M(\zeta)e^{B(\zeta)-B(z)}d\zeta.$$

If the matrix $V(z)$ possesses $V_{\nu\mu}(z)$ as its (ν, μ)-element, where μ is fixed as mentioned already and ν is arbitrary, then $V_{\nu\mu}(z)$ has to satisfy

$$(5) \qquad V_{\nu\mu}(z)=\int_{\mathfrak{P}_{\nu\mu}(z)} \exp[\beta_{\nu\mu}(z)-\beta_{\nu\mu}(\zeta)]\cdot M_{\nu\mu}(\zeta)d\zeta,$$

where

$$\beta_{\nu\mu}(z)=\beta_\nu(z)-\beta_\mu(z)=a_{\nu\mu}\frac{z^{k+1}}{k+1}\,.$$

LEMMA 1. *In the sector* $\mathfrak{S}_e^{(\mu)}$ *the path* $\mathfrak{P}_{\nu\mu}(z)$ *can be so chosen that the following inequality holds*:

$$\mathrm{Re}[\beta_{\nu\mu}(z)-\beta_{\nu\mu}(\zeta)]\leqq 0$$

for all values of ν, *all points* z *in* $\mathfrak{S}_e^{(\mu)}$, *and all points* ζ *of the path* $\mathfrak{P}_{\nu\mu}(z)$.

The proof will be given in the following section.

§ 4. The paths of integration $\mathfrak{P}_{\nu\mu}(z)$.

In this section we shall construct the path $\mathfrak{P}_{\nu\mu}(z)$ with the desired property, from the point of infinity to the point z, and complete the lemma.

7° Let a symbol $\wedge$ denote a transformation:

$$\hat{z}=\frac{1}{k+1}z^{k+1} \qquad \text{or} \qquad \hat{\zeta}=\frac{1}{k+1}\zeta^{k+1}.$$

By this transformation, we get from (5)

$$\hat{Y}_{\nu\mu}(\hat{z})=\int_{\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})}\hat{\zeta}^{-k/(k+1)}\cdot\exp[a_{\nu\mu}(\hat{z}-\hat{\zeta})]\cdot\hat{M}_{\nu\mu}(\hat{\zeta})d\hat{\zeta}, \tag{6}$$

where $\hat{M}_{\nu\mu}$ consists of elements of M multiplied by factors bounded in $\hat{\mathfrak{S}}_e^{(\mu)}$, which is an image of the exterior sector $\mathfrak{S}_e^{(\mu)}$ by $\wedge$.

The inequality of Lemma 1 is equivalent to an inequality:

$$\mathrm{Re}\, a_{\nu\mu}(\hat{z}-\hat{\zeta})\leqq 0$$

for all values of ν, all points $\hat{z}$ in $\hat{\mathfrak{S}}_e^{(\mu)}$, and all points $\hat{\zeta}$ on the path $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$.

In the sequel, we consider exclusively not in the original plane but in the transformed plane, i.e., in the $\hat{\zeta}$-plane.

8° We shall define the paths $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$ as follows. For $\nu\neq\mu$

$$\hat{\mathfrak{P}}_{\nu\mu}(\hat{z}):\ \hat{\zeta}=\hat{z}+\sigma\eta_{\nu\mu} \qquad (0\leqq\sigma<\infty),$$

where the vector $\eta_{\nu\mu}$, whose magnitude is unit, satisfies properties:[2]

2) A sector in which these properties are valid is called *admitted*, and in the admitted sector defined in 4° they are fulfilled.

The path $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$ lies in the sector $\hat{\mathfrak{S}}^{(\mu)}$; and the relation

$$\operatorname{Re} a_{\nu\mu}\eta_{\nu\mu}>0$$

holds.

In order to fulfill the above properties, we have only to choose the $\eta_{\nu\mu}$ in such a way that

(i) $\arg \bar{a}_{\nu\mu}-\frac{1}{2}\pi<\arg \eta_{\nu\mu}<\arg \bar{a}_{\nu\mu}+\frac{1}{2}\pi$,

(ii) $\hat{\Phi}_{+}^{(\mu)}-\pi<\arg \eta_{\nu\mu}<\hat{\Phi}_{-}^{(\mu)}+\pi$.

Indeed, the first property is followed from (ii) and from the third inequality in the definition of $\hat{\Phi}_{\pm}^{(\mu)}$ (see **4**°), i.e., from $\hat{\Phi}_{-}^{(\mu)}<\hat{\Phi}_{+}^{(\mu)}-\pi<\arg \eta_{\nu\mu}<\hat{\Phi}_{-}^{(\mu)}+\pi<\hat{\Phi}_{+}^{(\mu)}$.

The second is just the same as (i), for the relation $|-\arg \bar{a}_{\nu\mu}+\arg \eta_{\nu\mu}| =|\arg a_{\nu\mu}\eta_{\nu\mu}|<\pi/2$ holds.

For $\nu=\mu$, the condition $\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})=0$ holds for any point $\hat{\zeta}$, and so the path $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$ is chosen as a segment combining $\hat{z}$ and an arbitrary bounded point in $\hat{\mathfrak{S}}_{e}^{(\mu)}$, say, the point $\hat{\zeta}_0$ which is an intersection of the path and the boundary of $\hat{\mathfrak{S}}_{e}^{(\mu)}$ as shown in Figure 3.

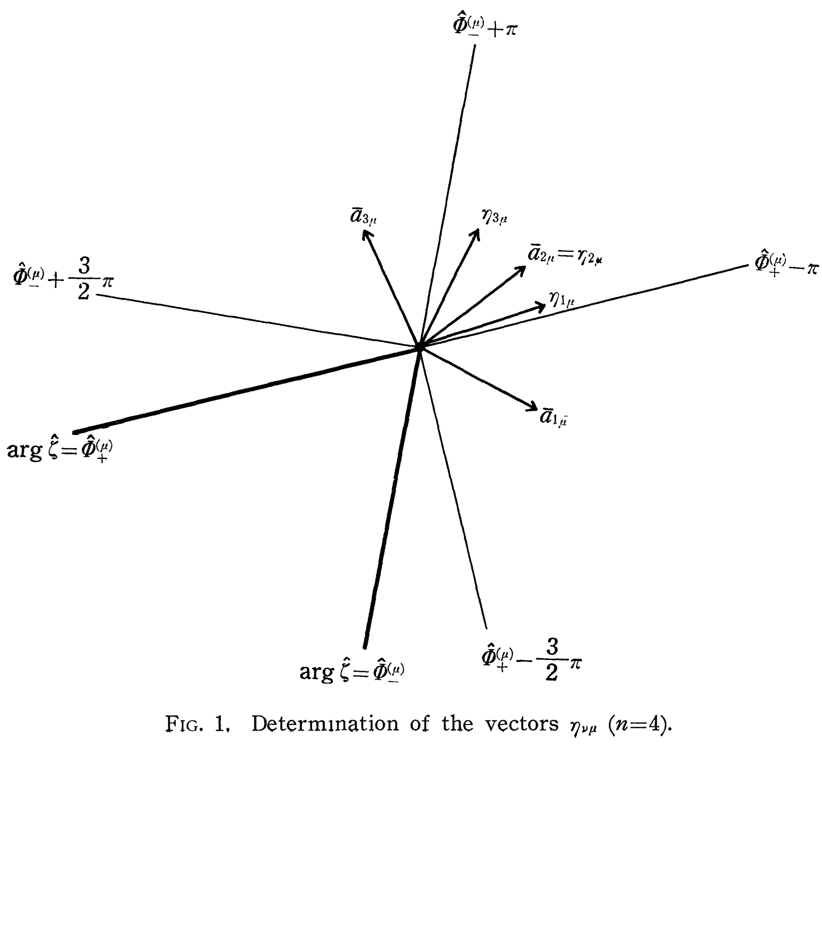

FIG. 1. Determination of the vectors $\eta_{\nu\mu}$ ($n=4$).

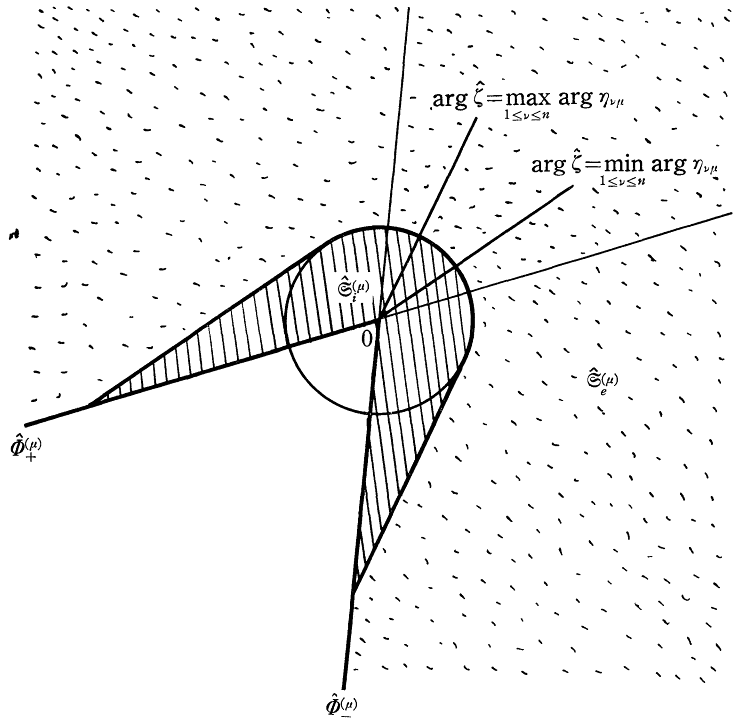

FIG. 2. The exterior and the interior sectors.

If we define the sector $\hat{\mathfrak{S}}_e^{(\mu)}$ as the region bounded by $\hat{\Phi}_\pm^{(\mu)}$ and the circle $|\hat{\zeta}|=\hat{z}_0$, $\hat{z}_0$ large, then some paths $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$, which end in certain regions, would intersect the circle. These regions are called *shadow zones*, whose definition is obvious and see Figures 2 and 3.

Thus we define the sector $\hat{\mathfrak{S}}_e^{(\mu)}$, for simplicity, as the region bounded by $\hat{\Phi}_\pm^{(\mu)}$, the circle $|\hat{\zeta}|=\hat{z}_0$ and out of shadow zones. More precisely, we must define the sector $\hat{\mathfrak{S}}_{e,\nu}^{(\mu)}$ for each value of ν and a fixed value of μ, which is bounded by $\hat{\Phi}_\pm^{(\mu)}$, the circle $|\hat{\zeta}|=\hat{z}_0$ and the shadow zone——this shadow zone is a set bounded by $\hat{\Phi}_\pm^{(\mu)}$, the circle $|\hat{\zeta}|=\hat{z}_0$ and the lines, tangent to the circle, with the same direction as the vector $\eta_{\nu\mu}$.

Therefore the exterior sector $\hat{\mathfrak{S}}_e^{(\mu)}$ is equal to the set $\bigcap_{\nu=1}^n \hat{\mathfrak{S}}_{e,\nu}^{(\mu)}$, and consequently the interior sector $\hat{\mathfrak{S}}_i^{(\mu)}$ is a subregion of the set $\hat{\mathfrak{S}}^{(\mu)}$ cut out of the set $\bigcap_{\nu=1}^n \hat{\mathfrak{S}}_{e,\nu}^{(\mu)}=\hat{\mathfrak{S}}_e^{(\mu)}$.

9° In view of the above choice of the paths and the definition of sectors, we can show the validity of the lemma. Since on every path the condition $\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})\leqq 0$ is always true, we have $\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})=-\sigma \operatorname{Re} a_{\nu\mu}\eta_{\nu\mu}\leqq 0$ for all values of ν. Thus Lemma 1 is completely proved.

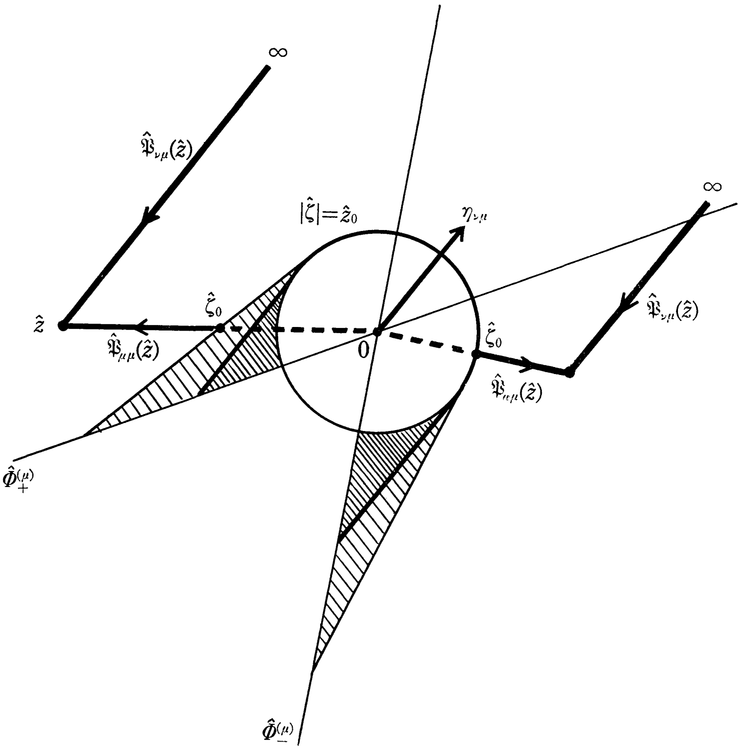

FIG. 3. Shadow zones and the paths of integration $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$.

§ 5. Some lemmas and the proof of Theorem A: completion.

In this section we shall get formal solutions of the given equation in the sectors $\mathfrak{S}_e$ and $\mathfrak{S}_i$ respectively and show the relation between them.

10° Lemma 1 yields some results in regard to the integral equation. The first is

LEMMA 2. *In the integral equation* (5), *if the function* $M_{\nu\mu}(z)z^{-c}$ $(c>0)$ *is bounded in* $\mathfrak{S}_e^{(\mu)}$ *then* $V_{\nu\mu}(z)z^{-(c+1)}$ *is bounded in* $\mathfrak{S}_e^{(\mu)}$. *In other words,* $M^{(\mu)}(z)=O(z^c)$ *in* $\mathfrak{S}_e^{(\mu)}$ *implies* $V^{(\mu)}(z)=O(z^{c+1})$ *in* $\mathfrak{S}_e^{(\mu)}$.[3)]

Proof. Let

$$M_{\nu\mu}(z)=O(z^c) \text{ in } \mathfrak{S}_e^{(\mu)}, \text{ that is}$$

$$M_{\nu\mu}(z)=O(\hat{z}^{c/(k+1)}) \text{ in } \hat{\mathfrak{S}}_e^{(\mu)}$$

$$=\hat{z}^{c/(k+1)}\tilde{M}_{\nu\mu}(\hat{z}),\ \tilde{M}_{\nu\mu}(\hat{z}) \text{ is bounded in } \hat{\mathfrak{S}}_e^{(\mu)}.$$

3) $M^{(\mu)}$ and $V^{(\mu)}$ denote the μ-th columns of matrices M and V respectively.

For $\nu \neq \mu$, we have

$$|V_{\nu\mu}(z)| \leqq c_1 \int_{\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})} |\hat{\zeta}|^{(c-k)/(k+1)} \cdot \exp[\mathrm{Re}\, a_{\nu\mu}(\hat{z}-\hat{\zeta})]|d\hat{\zeta}|$$

$$\leqq c_2 |\hat{z}|^{(c-k)/(k+1)} \int_0^\infty \left(1+\frac{\sigma}{|\hat{z}|}\right)^{(c-k)/(k+1)} \cdot \exp\left[-\min_{1\leqq\nu\leqq n} \mathrm{Re}\, a_{\nu\mu}\eta_{\nu\mu}\right] d\sigma$$

$$= O(\hat{z}^{(c-k)/(k+1)}) \text{ in } \hat{\mathfrak{S}}_e^{(\mu)}$$

$$= O(z^{c+1}) \text{ in } \mathfrak{S}_e^{(\mu)},$$

in which the constants c_1 and c_2 are independent of ν.

Along the path $\mathfrak{P}_{\mu\mu}(z)$ the following relations hold:

$$|V_{\nu\mu}(z)| = \left| \int_{\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})} \hat{\zeta}^{(c-k)/(k+1)} M_{\mu\mu}(\hat{\zeta}) d\hat{\zeta} \right|$$

$$\leqq c_3 \int_0^{|z|} (|\hat{z}|+\sigma)^{(c-k)/(k+1)} d\sigma$$

$$= O(\hat{z}^{(c-k)/(k+1)+1}) \text{ in } \hat{\mathfrak{S}}_e^{(\mu)}$$

$$= O(z^{c+1}) \text{ in } \mathfrak{S}_e^{(\mu)}.$$

Here c_3 is a constant dependent on μ only. *Q.E.D.*

LEMMA 3. *The n-dimensional vector function $V_r^{(\mu)}$, the μ-th column of the matrix V_r, is of order z^{rm^*}, $m^*=p+q+1$, as z tends to the infinity in $\mathfrak{S}_e^{(\mu)}$. In other words,*

$$V_r^{(\mu)}(z) = z^{rm^*} \tilde{V}_r^{(\mu)}(z) \qquad (r=1, 2, 3, \cdots),$$

where $\tilde{V}_r^{(\mu)}(z)$ is bounded in $\mathfrak{S}_e^{(\mu)}$.

The n-by-n matrix $V_r(z)$ is, in fact, a polynomial of the degree rm^*.

Proof. For $r=1$,

$$M(z) = \sum_{j=1}^{1} A_j(z) V_{1-j}(z) = A_1(z)$$

is, by definition, a polynomial of the degree $p+q \geqq 0$. Therefore, in view of the previous lemma, $V_1^{(\mu)}(z)$ is a polynomial of the degree $p+q+1$.

For $r \geqq 2$, we can show after a short calculation that

$$M(z) = \sum_{j=1}^{r} A_j(z) V_{r-j}(z)$$

is a polynomial of the degree $r(p+q)+r-1$. The application of the previous lemma implies $V_r^{(\mu)}(z)$ is a polynomial of the degree $r(p+q+1)$.

The value of μ is fixed and arbitrary. Then the lemma is proved. *Q.E.D.*

From the previous lemmas and the definition of V_r we obtain the μ-th column of a formal series solution of the given differential equation.

LEMMA 4. *In the exterior sector* $\mathfrak{S}_e^{(\mu)}$ *the differential equation* (2) *possesses the formal vector solution*

$$\underset{\infty}{U}^{(\mu)}(z,\rho)\sim\sum_{r=0}^{\infty}\underset{\infty}{U_r^{(\mu)}}(z)\rho^r$$

$$=\left[\sum_{r=0}^{\infty}z^{rm^*}\underset{\infty}{\tilde{U}_r^{(\mu)}}(z)\rho^r\right]\cdot[\exp B(z)]_{\mu\mu}\qquad\text{as}\quad \rho\to 0\ \textit{in}\ \mathfrak{Z}\ \textit{and}\ \mathfrak{S}_e^{(\mu)},$$

where $z^{rm^*}\underset{\infty}{\tilde{U}_r^{(\mu)}}(z)$ *is a polynomial of the degree* rm^* *and so* $\underset{\infty}{\tilde{U}_r^{(\mu)}}(z)$ *is bounded in* $\mathfrak{S}_e^{(\mu)}$, $[\exp B(z)]_{\mu\mu}$ *is equal to* $\exp(a_\mu z^{k+1}/(k+1))$ *i.e., to the* μ*-th diagonal element of the diagonal matrix* $\exp B(z)$.

Since $U^{(\mu)}$ is the μ-th column of the matrix U, Lemma 4 yields

COROLLARY TO LEMMA 4. *In the exterior sector* $\mathfrak{S}_e$ *the differential equation* (2) *possesses the formal matrix solution*

$$\underset{\infty}{U}(z,\rho)\sim\sum_{r=0}^{\infty}\underset{\infty}{U_r}(z)\rho^r$$

$$=\left[\sum_{r=0}^{\infty}z^{rm^*}\underset{\infty}{\tilde{U}_r}(z)\rho^r\right]\cdot\exp B(z)\qquad \textit{as}\quad \rho\to 0\ \textit{in}\ \mathfrak{Z}\ \textit{and}\ \mathfrak{S}_e,$$

where $\underset{\infty}{\tilde{U}_0}(z)\equiv I$ *and* $z^{rm^*}\underset{\infty}{\tilde{U}_r}(z)$ *is a polynomial of the degree* rm^* *and so* $\underset{\infty}{\tilde{U}_r}(z)$ *is bounded in* $\mathfrak{S}_e$.

11° In the region near the origin, i.e., in the interior sector $\mathfrak{S}_i^{(\mu)}$, we at once obtain solutions of (2). That is to say, we have

LEMMA 5. *In the interior sector* $\mathfrak{S}_i^{(\mu)}$ *the differential equation* (2) *possesses the formal vector solution*

$$\underset{0}{U}^{(\mu)}(z,\rho)\sim\sum_{r=0}^{\infty}\underset{0}{U_r^{(\mu)}}(z)\rho^r$$

$$=\left[\sum_{r=0}^{\infty}\underset{0}{U_r^{(\mu)}}(z)\rho^r\right]\cdot[\exp B(z)]_{\mu\mu}\qquad \textit{as}\quad \rho\to 0\ \textit{in}\ \mathfrak{Z}\ \textit{and}\ \mathfrak{S}_i^{(\mu)},$$

where $\underset{0}{U_r^{(\mu)}}(z)$ *is bounded in* $\mathfrak{S}_i^{(\mu)}$.

COROLLARY TO LEMMA 5. *In the interior sector* $\mathfrak{S}_i$, *the differential equation* (2) *possesses the formal matrix solution*

$$\underset{0}{U}(z,\rho)\sim\sum_{r=0}^{\infty}\underset{0}{U_r}(z)\rho^r$$

$$=\left[\sum_{r=0}^{\infty}\underset{0}{\tilde{U}_r}(z)\rho^r\right]\cdot\exp B(z)\qquad as\quad \rho\to 0\ in\ \mathfrak{Z}\ and\ \mathfrak{S}_i,$$

where $\underset{0}{\tilde{U}_0}(z)\equiv I$ *and* $\underset{0}{\tilde{U}_r}(z)$ *is bounded in* $\mathfrak{S}_i$.

12° The solution $\underset{\infty}{U}(z,\rho)$ is an expression of the solution $U(z,\rho)$ for $|z|$ large and the solution $\underset{0}{U}(z,\rho)$ is an expression of the same solution, i.e., of $U(z,\rho)$, for $|z|$ small. Thus the relation between the two solutions is to be obtained by calculating a constant matrix C_r $(r=1,2,3,\cdots)$ of

$$\underset{0}{U_r}(z)=e^{B(z)}C_r+\int_0^z e^{B(z)-B(\zeta)}M(\zeta)d\zeta.$$

Here the constant C_r is given by

$$C_r=\int_\infty^0 e^{-B(\zeta)}M(\zeta)d\zeta,$$

which converges by the choice of the paths of integration.

Indeed, since $\underset{\infty}{U_r}(z)$ is the solution of the integral equation (4):

$$\underset{\infty}{U_r}(z)=\int_\infty^z e^{B(z)-B(\zeta)}M(\zeta)d\zeta$$

we can reform this as follows

$$\underset{\infty}{U_r}(z)=\int_0^z e^{B(z)-B(\zeta)}M(\zeta)d\zeta+e^{B(z)}\int_\infty^0 e^{-B(\zeta)}M(\zeta)d\zeta,$$

and this must be also a solution for $|z|$ small, i.e., $\underset{0}{U_r}(z)$.

Therefore we have completed the proof of the theorem A.

§ 6. Existence of an actual solution.

13° In the sequel we shall show existence of an actual solution asymptotically expansible in the formal solution obtained so far.

THEOREM B. *In the* (z,ρ)*-domain defined by*

$$\mathfrak{D}:\quad z\in\mathfrak{S},\quad |\rho|\leqq\rho_0,\quad |\arg\rho|\leqq\rho_1,\quad |z^{m^*}\rho|\leqq c_0$$

with ρ_0, ρ_1 *and* c_0 *small constants, the formal solution in Theorem* A *is, for every integer* $m>0$, *the asymptotic representation up to order* m *of an actual solution. That is to say,*

$$U(z,\rho)-\sum_{r=0}^{m}U_r(z)\rho^r=E_{2m}(z,\rho)\cdot[z^{m^*}\rho]^{m+1}\cdot\exp B(z) \qquad \textit{for} \quad z\in\mathfrak{S}_e,$$

where $E_{2m}(z,\rho)$ *is bounded in* $\mathfrak{D}$, $z^{(m+1)m^*}\cdot E_{2m}(z,\rho)$ *is bounded in* $\mathfrak{D}$ *for* $z\in\mathfrak{S}_i$, *and* $m^*=p+q+1$.

The proof will be for convenience divided into several steps.

14° *Construction of integral equations.* Let $U_m(z,\rho)$ be the truncated series of $U(z,\rho)$, i.e., $U_m(z,\rho)=\sum_{r=0}^{m}U_r(z)\rho^r$. Then $U_m(z,\rho)$ is a solution of the differential equation

$$\frac{dU_m(z,\rho)}{dz}=A_m(z,\rho)U_m(z,\rho), \qquad A_m=\frac{dU_m}{dz}\cdot U_m^{-1}.$$

Notice U_m^{-1} really exists. Because $U_m(z,\rho)=\{I+O(z^{\kappa(z)m^*})\}\cdot\exp B(z)$, where the function $\kappa(z)$ is defined by

$$\kappa(z)=\begin{cases}0 & \text{for} \quad z\in\mathfrak{S}_i,\\ 1 & \text{for} \quad z\in\mathfrak{S}_e,\end{cases}$$

and since $\exp B(z)$ is clearly non-singular, if we take c_0 of $\mathfrak{D}$ small enough the determination of $U_m(z,\rho)$ is nearly equal to the one of $\exp B(z)$ for $|z^{\kappa(z)m^*}\rho|\leqq c_0$. Therefore $U_m(z,\rho)$ is non-singular and bounded for c_0 sufficiently small. We notice c_0 depends on m.

In order to obtain an integral equation, we reform the equation (2)

$$\frac{dU}{dz}=A(z,\rho)U=A_mU+(A-A_m)U$$

$$=A_mU+(AU_m-A_mU_m)U_m^{-1}U.$$

Namely,

$$\frac{dU}{dz}=A_mU+(AU_m-U_m')U_m^{-1}U,$$

where $'$ denotes differentiation with respect to z.

The last equation is equivalent to the following integral equation

$$U(z,\rho)=U_m(z,\rho)+U_m(z,\rho)\int_{\mathcal{P}(z)}U_m^{-1}(\zeta,\rho)[A(\zeta,\rho)U_m(\zeta,\rho)-U_m'(\zeta,\rho)]U_m^{-1}(\zeta,\rho)U(\zeta,\rho)d\zeta,$$

where the path-matrix $\mathcal{P}(z)$ is so chosen that the integral converges, and the precise choice of $\mathcal{P}(z)$ is given later.

Let

$$U_m(z,\rho)=W_m(z,\rho)\cdot\exp B(z), \qquad U(z,\rho)=W(z,\rho)\cdot\exp B(z).$$

Then remarking the relation

$$A(z,\rho)U_m(z,\rho)-U_m'(z,\rho)=E(z,\rho)[z^{\kappa(z)m^*}\rho]^{m+1}z^{-\kappa(z)}\cdot\exp B(z)$$

with $E(z,\rho)$ bounded on $\mathfrak{D}$, we can rewrite the above integral equation in regard to $U(z,\rho)$ as follows:

$$\begin{aligned}W(z,\rho)=W_m(z,\rho)+W_m(z,\rho)\int_{\mathfrak{P}(z)} & e^{B(z)-B(\zeta)}W_m^{-1}(\zeta,\rho)E(\zeta,\rho)W_m^{-1}(\zeta,\rho)\\ &\times W(\zeta,\rho)e^{B(\zeta)-B(z)}\cdot\zeta^{\kappa(\zeta)[(m+1)m^*-1]}\rho^{m+1}d\zeta,\end{aligned}\tag{7}$$

where $W_m^{-1}(z,\rho)E(z,\rho)W_m^{-1}(z,\rho)$ is bounded in $\mathfrak{D}$ in view of the definition of $W_m(z,\rho)$ and of the boundedness of $U_m(z,\rho)$ and $\exp B(z)$ for c_0 sufficiently small.

The (ν,μ)-component of the integral part of (7) can be written in

$$\int_{\mathfrak{P}_{\nu\mu}(z)}\exp[\beta_{\nu\mu}(z)-\beta_{\nu\mu}(\zeta)]\cdot N_{\nu\mu}[W(\zeta,\rho)]\cdot\zeta^{\kappa(\zeta)[(m+1)m^*-1]}\rho^{m+1}d\zeta,$$

and by introducing new variables, likewise in **7°**, defined by $\hat{z}=z^{k+1}/(k+1)$ or $\hat{\zeta}=\zeta^{k+1}/(k+1)$, it is further rewritten as

$$\int_{\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})}\exp[a_{\nu\mu}(\hat{z}-\hat{\zeta})]\cdot\hat{N}_{\nu\mu}[W(\hat{\zeta},\rho)]\cdot\hat{\zeta}^{\{\kappa(\zeta)[(m+1)m^*-1]-k\}/(k+1)}\rho^{m+1}d\hat{\zeta},$$

where $\hat{N}_{\nu\mu}$ is the image of $N_{\nu\mu}$ by the $\wedge$-transformation and $N_{\nu\mu}[W(z,\rho)]$ is a linear form of the μ-th column of $W(z,\rho)$ with bounded coefficients.

15° The integral equation (7) can be regarded as an operator from some space into intself whose point W is defined: $W(\hat{z},\rho)$ is a matrix function defined on $\hat{\mathfrak{D}}^{(\mu)}$, holomorphic for $z\neq\infty$ and

$$\|W(\hat{z},\rho)\|=\max_{1\leqq\nu\leqq n}\sum_{\mu=1}^{n}|W_{\nu\mu}(\hat{z},\rho)|\leqq c_4|z^{m^*}\rho|^{m+1}\leqq c_4'.$$

The domain $\hat{\mathfrak{D}}^{(\mu)}$ is an obvious notation, i.e.,

$$\hat{\mathfrak{D}}^{(\mu)}:\quad \hat{z}\in\hat{\mathfrak{S}}^{(\mu)},\quad |\hat{\rho}|\leqq\rho_0',\quad |\arg\hat{\rho}|\leqq\rho_1',\quad |\hat{z}^{m^*}\hat{\rho}|\leqq c_0'$$

with ρ_0', ρ_1' and c_0' appropriate constants.

The integral operator thus defined is written as

$$W(z,\rho)=W_m(z,\rho)\{I+\mathcal{L}[W]\},\tag{8}$$

and we shall show that this operator is of the contraction. In order to show it we shall first of all prove the following

LEMMA 6. *Denote by $\tilde{W}$ the least upper bound of W on the domain $\hat{\mathfrak{D}}^{(\mu)}$, and choose appropriately the integral path $\hat{\mathfrak{P}}_{\nu\mu}(\hat{z})$. Then the following inequality is valid:*

$$\left|\rho^{m+1}\int_{\hat{\mathscr{P}}_{\nu\mu}(\hat{z})}\exp[a_{\nu\mu}(\hat{z}-\hat{\zeta})]\cdot\hat{N}_{\nu\mu}(W)\cdot\hat{\zeta}^{\{\kappa(\hat{\zeta})[(m+1)m^*-1]-k\}/(k+1)}d\hat{\zeta}\right|$$

$$\leqq|\rho|^{m+1}c_5\widetilde{W}\hat{z}^{\{\kappa(\hat{z})[(m+1)m^*-1]-k\}/(k+1)+1}=c_5\widetilde{W}|\rho\hat{z}^{\kappa(z)m^*/(k+1)}|^{m+1}.$$

Proof. Since the value of $|\hat{N}_{\nu\mu}(W)|$ is always not greater than $c_6\widetilde{W}$, the lemma would be followed if we could show the estimate

$$|I_e|=\left|\int_{\hat{\mathscr{P}}_{\nu\mu}(\hat{z})\in\hat{\mathfrak{S}}_e^{(\mu)}}\exp[a_{\nu\mu}(\hat{z}-\hat{\zeta})]\cdot\hat{\zeta}^{\gamma}d\hat{\zeta}\right|\leqq c_7|\hat{z}|^{\gamma+1}$$

is valid for the appropriate path and γ a positive constant, and a quantity of the integral

$$I_i=\int_{\hat{\mathscr{P}}_{\nu\mu}(\hat{z})\in\hat{\mathfrak{S}}_i^{(\mu)}}\exp[a_{\nu\mu}(\hat{z}-\hat{\zeta})]\cdot\hat{\zeta}^{-k/(k+1)}d\hat{\zeta}$$

is bounded.

The validity of the above estimate will be shown in the following sections.

§ 7. Paths of integration $\hat{\mathscr{P}}_{\nu\mu}(\hat{z})$.

16° Paths of integration $\hat{\mathscr{P}}_{\nu\mu}(\hat{z})$, from the initial point $\hat{z}_{\nu\mu}$ to $\hat{z}$, are chosen as follows. The initial point $\hat{z}_{\nu\mu}$ situated on the circle $|\hat{\zeta}^{m^*}\hat{\rho}|=c_0'$ is common to all the values of $\hat{z}\in\hat{\mathfrak{D}}^{(\mu)}$ and it will be defined precisely in the following paragraph. We want to choose $\hat{\mathscr{P}}_{\nu\mu}(\hat{z})$ so that the relation $\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})\leqq 0$ holds along it.

17° In the following discussion, we shall assume $\arg\eta_{\nu\mu}=0$. This does not lose generality, for other cases could be reduced to this case by an appropriate rotation of axes.

On the circular part of the boundary of $\hat{\mathfrak{D}}^{(\mu)}$ for $|\arg\hat{\zeta}|\leqq\pi/2$ there exists for every pair ν,μ $(\nu\neq\mu)$ a point $\hat{z}_{\nu\mu}$ at which $\operatorname{Re} a_{\nu\mu}\hat{\zeta}$ assumes its maximum in $\hat{\mathfrak{D}}^{(\mu)}$ for $|\arg\hat{\zeta}|\leqq\pi/2$. In fact, since we have $|\arg a_{\nu\mu}|<\pi/2$ from the assumption $\arg\eta_{\nu\mu}=0$ and $|\arg\hat{\zeta}|\leqq\pi/2$, the value of $\arg a_{\nu\mu}\hat{\zeta}$ varies between $-\pi$ and π.

The quantity $\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})$ increases as $\hat{\zeta}$ moves from $\hat{z}_{\nu\mu}$ to a point $\hat{z}$, $|\arg\hat{z}|\leqq\pi/2$ in $\hat{\mathfrak{D}}^{(\mu)}$, along a segment (see *I*, *III* or *V* in FIG. 4).

If a point $\hat{z}$ lies in $\hat{\mathfrak{D}}^{(\mu)}$ for $|\arg\hat{z}|\geqq\pi/2$, we choose as the path $\hat{\mathscr{P}}_{\nu\mu}(\hat{z})$ a segment, parallel to the line $\arg\hat{\zeta}=\arg\eta_{\nu\mu}$, from the point $\hat{z}$ to the point intersecting a line defined by $\arg\hat{\zeta}=\pm\pi/2$, and a segment from this intersection to the point $\hat{z}_{\nu\mu}$ (see *II* or *IV* in FIG. 4).

Along the path $\hat{\mathscr{P}}_{\nu\mu}(\hat{z})$ from $\hat{z}_{\nu\mu}$ to $\hat{z}$, the quantity $\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})$ is always negative, thus by the mean value theorem there exists a positive constant ω, independent of ν,μ and ρ, such that

$$\operatorname{Re} a_{\nu\mu}(\hat{z}-\hat{\zeta})\leqq-\omega|\hat{z}-\hat{\zeta}|$$

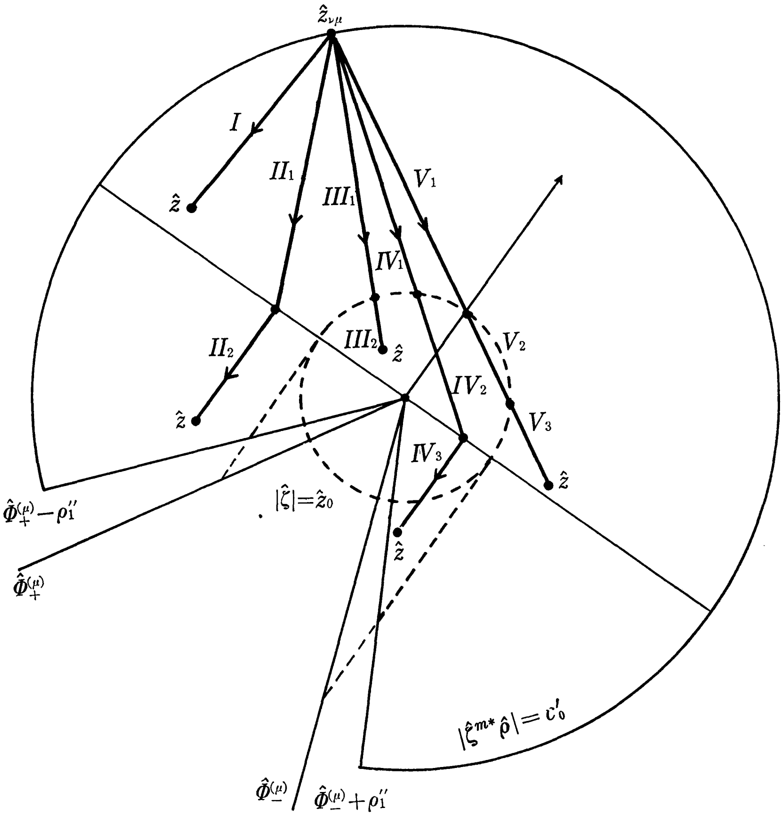

FIG. 4. Paths of integration $\hat{\mathcal{P}}_{\nu\mu}(\hat{z})$. Paths $\hat{\mathcal{P}}_{\mu\mu}(\hat{z})$ are segments from the origin to $\hat{z}$.[4)]

if $\hat{\zeta}$ is on $\hat{\mathcal{P}}_{\nu\mu}(\hat{z})$.

For $\nu=\mu$, the paths $\hat{\mathcal{P}}_{\mu\mu}(\hat{z})$ may be taken as the segments from the origin to $\hat{z}$. We take as paths $\mathcal{P}_{\nu\mu}(z)$ the antecedents of $\hat{\mathcal{P}}_{\nu\mu}(\hat{z})$ in the ζ-plane.

§ 8. Proof of Theorem B: completion.

18° We shall complete Lemma 6.

First, we consider the case when the whole path $\hat{\mathcal{P}}_{\nu\mu}(\hat{z})$ lies in $\mathfrak{S}_e^{(\mu)}$.[5)] Then we have

$$|I_e|\leqq\int e^{-\omega|\hat{z}-\hat{\zeta}|}\cdot|\hat{\zeta}|^r|d\hat{\zeta}|.$$

4) The quantity ρ_1'' vanishes if the parameter ρ is real.

5) Angles of $\mathfrak{S}_e^{(\mu)}$ and $\mathfrak{S}_i^{(\mu)}$ is less than ones in the formal theory if ρ is complex.

For each of the parts $I, II_1 \cup II_2, III_1, IV_1$ or $V_1 \cup V_3$ in FIG. 4,

$$|I_e| \leqq \hat{z}^r \int_0^\infty e^{-\omega\sigma}\left(1+\frac{\sigma}{|\hat{z}|}\right)^r d\sigma \leqq c_8 \hat{z}^r.$$

For each of the other parts $\kappa(\hat{z})=0$ or $\kappa(\hat{\zeta})=0$ and so along the path

$$|I_i| \leqq \int e^{-\omega|\hat{z}-\hat{\zeta}|} \cdot |\hat{\zeta}|^{-k/(k+1)} |d\hat{\zeta}|.$$

The quantity of the last integral is bounded for $\hat{\zeta} \in \hat{\mathfrak{S}}_i^{(\mu)}$.

Finally, if $\nu=\mu$, we find

$$\left| \int_{\hat{\mathfrak{P}}_{\mu\mu}(\hat{z})} \hat{\zeta} d\hat{\zeta} \right| \leqq \int_0^{\hat{z}} |\hat{\zeta}| d\hat{\zeta} \leqq c_9 |\hat{\zeta}|^{r+1}$$

for $\hat{\zeta} \in \hat{\mathfrak{S}}_e^{(\mu)}$, and

$$\int \hat{\zeta}^{-k/(k+1)} d\hat{\zeta} \text{ is bounded}$$

if $\hat{\zeta}$ lies in $\hat{\mathfrak{S}}_i^{(\mu)}$. Thus Lemma 6 is completed.

19° The integral operator (8) is the contraction one , that is to say, the inequality

$$\| W_m(z, \rho) \mathcal{L}[W] \| \leqq c\widetilde{W}, \qquad 0<c<1$$

will be shown.

As already shown in **14°**, the function $W_m(z, \rho)$ is bounded in $\mathfrak{D}$, and the elements of $\mathcal{L}[W]$ satisfy the estimate in Lemma 6. Therefore we have

$$\| W_m(z, \rho) \mathcal{L}[W] \| \leqq c_{10} |\rho z^{\kappa(z) m^*}|^{m+1} \widetilde{W}.$$

Then if c_0 of $|\rho z^{m^*}| \leqq c_0$ is taken sufficiently small the inequality

$$c_{10} |\rho z^{\kappa(z) m^*}|^{m+1} < 1$$

is true. The constant c_0 of the domain $\mathfrak{D}$ can be, from the outset, assumed so small that the contraction property is satisfied and that the non-singularity of the matrix $U_m(z, \rho)$ is guaranteed (cf. **14°**).

From (8) and its contraction property, we obtain

$$\| W - W_m \| \leqq c_{11} |\rho z^{\kappa(z) m^*}|^{m+1},$$

which is clearly equivalent to the asymptotic property $U(z, \rho)$ in Theorem B.

Summing up the above statements we have the following

LEMMA 7. *For each fixed integer m, there exists a unique actual solution* $U(z, \rho) = \overset{m}{U}(z, \rho)$ *of the differential equation* (2) *asymptotically expansible in the formal solution.*

Thus Theorem B has been completely proved.

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