FIXED POINT THEOREM FOR AMENABLE SEMIGROUP
OF NONEXPANSIVE MAPPINGS

BY WATARU TAKAHASHI

1. Introduction.

Let $K$ be a subset of a Banach space $B$. A mapping $s$ of $K$ into $B$ is said to be nonexpansive if for each pair of elements $x$ and $y$ of $K$, we have \(|sx-sy| \leq |x-y|\).

Kakutani [5] and Markov [7] proved the following theorem: Let $K$ be a compact convex subset of a locally convex linear topological space $B$ and $S$ be a commuting family of linear continuous mappings of $K$ into itself. Then $S$ has a common fixed point in $K$.

Day [2] showed that this is true even if $S$ is an amenable semigroup.

On the other hand, de Marr [3] proved a fixed point theorem for a family of nonlinear mappings: Let $K$ be a nonempty compact convex subset of a Banach space $B$. If $S$ is a nonempty commutative family of nonexpansive mappings of $K$ into itself, then the family $S$ has a common fixed point in $K$.

The question naturally arises as to whether this is true if one considers an amenable semigroup of nonexpansive mappings.

In this paper, we shall show that the answer is affirmative.

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2. Amenable semigroup.

Let $S$ be an abstract semigroup and $m(S)$ be the space of all bounded real valued functions of $S$, where $m(S)$ has the supremum norm. An element $\lambda \in m(S)^*$ (the dual space of $m(S)$) is mean on $m(S)$ if $\lambda(e) = |e| = 1$ where $e$ denotes the constant 1 function on $S$. A mean $\lambda$ is left [right] invariant if $\lambda(l_s f) = \lambda(f) \ [\lambda(r_s f) = \lambda(f)]$ for all $f \in m(S)$ and $s \in S$, where the left [right] translation $l_s \ [r_s]$ of $m(S)$ by $s$ is given by $(l_s f)(s') = f(ss') \ [(r_s f)(s') = f(s's)]$. An invariant mean is a left and a right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called left amenable [right amenable]. A semigroup that has an invariant mean is called amenable.

Let $M$ be a nonempty compact Hausdorff space and $C(M)$ be the space of bounded continuous real valued functions on $M$. The norm will be the supremum norm.
norm. Let $S$ be a semigroup of continuous mappings of $M$ into $M$ and define a mapping $U_s$ for each $s$ in $S$ from $C(M)$ into $C(M)$ by attaching to each $f \in C(M)$, the function $U_s f$ on $M$ such that $(U_s f)(x) = f(sx)$ for each $x$ in $M$.

We shall prove the following Lemma by using Day’s fixed point theorem [2].

**Lemma.** Let $M$ be a nonempty compact Hausdorff space and $S$ be an amenable semigroup of continuous mappings of $M$ into $M$. Then there exists $L^* \in C(M)^*$ (the dual space of $C(M)$) such that $L^*(e) = ||L^*|| = 1$ where $e$ is the constant 1 function on $M$ and $L^*(U_s f) = L^*(f)$ for all $f \in C(M)$ and $s \in S$.

**Proof.** Let $K[C(M)] = \{L \in C(M)^* : L(e) = ||L|| = 1\}$. Since $U_s$ for each $s$ in $S$ is a linear mapping of $C(M)$ into itself such that $U_s(e) = e$ and $||U_s|| = 1$, a mapping $U_s^*$ that is given by $(U_s^* L)(f) = L(U_s f)$ for all $L \in C(M)^*$ and $f \in C(M)$ is a weak*-continuous affine mapping of $K[C(M)]$ into itself.

If $\{U_s^* : s \in S\}$ is an amenable semigroup, from Day’s fixed point theorem [2], there exists $L^* \in K[C(M)]$ such that $(U_s^* L^*)(f) = L^*(f)$ for all $f \in C(M)$.

We shall show that $\{U_s^* : s \in S\}$ is an amenable semigroup.

Since the mapping $\sigma$ of $S$ onto $\{U_s^* : s \in S\}$ that is given by $\sigma(s) = U_s^*$ for each $s$ in $S$ is a homomorphism, $\{U_s^* : s \in S\}$ is an amenable semigroup from [1]. Q.E.D.

3. Main theorem.

**Theorem.** Let $K$ be a nonempty compact convex subset of a Banach space $B$ and $S$ be an amenable semigroup of nonexpansive mappings of $K$ into $K$. Then there exists an element $z$ in $K$ such that $sz = z$ for each $s$ in $S$.

**Proof.** By using Zorn’s lemma, we can find a minimal nonempty compact convex set $X \subset K$ such that $X$ is invariant under each $s$ in $S$. If $X$ consists of a single point, then the theorem is proved.

By using Zorn’s lemma again, we can find a minimal nonempty compact set $M \subset X$ such that $M$ is invariant under each $s$ in $S$.

We will now that $M = \{sx : x \in M\}$ for each $s$ in $S$.

Since the semigroup of restrictions of all mappings $s$ in $S$ to $M$ is amenable [1], by Lemma there exists an element $L^* \in K[C(M)]$ such that $L^*(U_s f) = L^*(f)$ for all $f \in C(M)$. The Riesz theorem asserts that to the element $L^*$, there corresponds a unique probability measure $m$ on $M$ such that

$$L^*(f) = \int_M f \, dm$$

for each $f$ in $C(M)$.

Since $M$ is a compact metric space and $m$ is a probability measure on $M$, there exists a unique closed set $F \subset M$ called support of $m$ satisfying (i) $m(F) = 1$, (ii) if $D$ is any closed set such that $m(D) = 1$, then $F \subset D$. Moreover $F$ is the set of all point $x \in M$ having the property that $m(G) > 0$ for each open set $G$ containing $x$.

It is obvious that $F$ is contained in $s(M)$ for each $s$ in $S$, since each $s$ in $S$ is
a measurable transformation of $M$ into $M$ and hence $m(sM)=m(M)=1$.

Let $\chi_F$ be the characteristic function of the closed subset $F$ in $M$. Since for each $s$ in $S$

$$1=m(F)=\int_M \chi_F(x)dm$$

$$=\int_M \chi_F(sx)dm=m(s^{-1}F),$$

it is clear that $F$ is contained in $s^{-1}(F)$ for each $s$ in $S$. Therefore $F$ is invariant under each $s$ in $S$.

If $M$ contains more than one point, there exists an element $u$ in the closed convex hull of $M$ such that

$$\rho=\sup \{|u-x|: x\in M\}<\delta(M)$$

where $\delta(M)$ is the diameter of $M$.

Let us define

$$X_0=\cap \{y\in X: \|x-y\|\leq \rho\},$$

then $X_0$ is the nonempty closed convex proper subset of $X$ such that $s(X_0)\subset X_0$ for each $s$ in $S$. This is a contradiction to the minimality of $X$. Therefore $M$ contains only one point which is a common fixed point for the semigroup of nonexpansive mappings of $K$ into itself. Q.E.D.

**Corollary** (de Marr [3]). Let $K$ be a compact convex subset of a Banach space $B$ and $S$ be a family of commutative nonexpansive mappings of $K$ into itself. Then $S$ has a common fixed point in $K$.

**Proof.** Since a commutative semigroup is an amenable semigroup, Corollary is obvious from Theorem.

**Remark.** Theorem is true even if $S$ is a left amenable semigroup. We can discuss the above by using purely metric methods [6] [8].

**References**


Department of Mathematics,
Tokyo Institute of Technology.