KŌDAI MATH. SEM. REP.
21 (1969), 350—364

INVARIANT SUBMANIFOLDS OF AN ALMOST CONTACT MANIFOLD

By Kentaro Yano and Shigeru Ishihara

§ 0. Introduction.

Let V be a differentiable manifold imbedded differentiablly as a submanifold in a differentiable manifold M with an almost complex structure $\tilde{F}$ and denote by ι: $V \to M$ its imbedding. If the tangent space $T_P(\iota(V))$ of $\iota(V)$ is invariant by the linear mapping $\tilde{F}$ at each point P of $\iota(V)$, the $\iota(V)$ is called an *invariant submanifold* of an almost complex manifold M, [5].[1]

An invariant submanifold of an almost complex manifold is itself an almost complex manifold and an invariant submanifold of a complex manifold is itself a complex manifold.

It is also known that an invariant submanifold of a Kählerian manifold is itself a Kählerian manifold and is a minimal submanifold, [5], [6].

The main purpose of the present paper is to define invariant submanifolds of an almost contact manifold and to study properties of these invariant submanifolds.

In § 1, we fix our notations in the present paper and prove some formulas for submanifolds imbedded in a Riemannian manifold, and in § 2 we state some of important results in the theory of almost contact manifolds.

In § 3, we define invariant submanifolds of an almost contact manifold and study their properties.

§ 4 is devoted to the study of invariant submanifolds of a normal almost contact manifold.

In the last § 5, we study properties of invariant submanifolds in connection with the theory of fibred spaces developed by the present authors [7], [8].

The concept of invariant submanifolds in an almost contact manifold appears also in a recent paper by Okumura [1].

§ 1. Formulas for submanifolds.

As we are going to study some special kinds of submanifolds, we would like first of all to reformulate formulas for general submanifolds in a Riemannian manifold for the later use. Let V be an m-dimensional manifold imbedded differentiablly

Received April 24, 1969.

1) The numbers between brackets [ ] refer to the References at the end of the paper.

as a submanifold in an n-dimensional Riemannian manifold M, where $m<n$, and denote by ι: $V\to M$ its imbedding[2]. Denote by B: $T(V)\to T(M)$ the differential mapping of ι, i.e., $B=d\iota$, $T(V)$ and $T(M)$ being respectively the tangent bundles of V and of M. On putting $T(V,M)=BT(V)$, the set of all vectors tangent to $\iota(V)$, we see that B: $T(V)\to T(V,M)$ is an isomorphism. The set of all vectors normal to $\iota(V)$ forms a vector bundle $N(V,M)$ over $\iota(V)$, which is called the *normal bundle* of V. The vector bundle induced by ι from $N(V,M)$ is denoted by $N(V)$, which is called also the *normal bundle* of V. We denote by C: $N(V)\to N(V,M)$ the natural isomorphism.

We introduce now the following notations: $\mathcal{T}^r_s(V)$ is the space of all tensor fields of type (r,s), i.e., of contravariant degree r and covariant degree s, associated with $T(V)$. $\mathcal{T}(V)=\sum_{r,s}\mathcal{T}^r_s(V)$ is the space of all tensor fields associated with $T(V)$. $\mathcal{N}^r_s(V)$ is the space of all tensor fields of type (r,s) associated with $N(V)$. $\mathcal{N}(V)=\sum_{r,s}\mathcal{N}^r_s(V)$ is the space of all tensor fields associated with $N(V)$. Thus $\mathcal{T}^0_0(V)=\mathcal{N}^0_0(V)$ is the space of all differentiable functions defined on V. An element of $\mathcal{T}^1_0(V)$ is a vector field in V. An element of $\mathcal{N}^1_0(V)$ is a vector field normal to V.

Take a vector field $\bar{X}$ defined along $\iota(V)$, not necessarily tangent to $\iota(V)$. For any point P of $\iota(V)$, there exists in M a neighborhood Ω containing P such that there exists in Ω a vector field $\tilde{X}$ which is an extension of $\bar{X}$. Such a local vector field $\tilde{X}$ is called a local extension of $\bar{X}$ restricted to the connected component of $\Omega\cap\iota(V)$, which contains the point P. Let $\bar{X}$ and $\bar{Y}$ be two vector fields defined along $\iota(V)$ and tangent to $\iota(V)$. Taking local extensions $\tilde{X}$ of $\bar{X}$ and $\tilde{Y}$ of $\bar{Y}$ in a neighborhood Ω of M, we see that $[\tilde{X},\tilde{Y}]$ is tangent to $\iota(V)$ and its restriction $[\tilde{X},\tilde{Y}]_V$ to $\iota(V)$ is determined independently of the choice of these local extensions $\tilde{X}$ and $\tilde{Y}$. Therefore we can define $[\bar{X},\bar{Y}]$ by

$$[\bar{X},\bar{Y}]=[\tilde{X},\tilde{Y}]_V. \tag{1.1}$$

Thus we can easily see that

$$[BX,BY]=B[X,Y] \tag{1.2}$$

holds for $X, Y\in\mathcal{T}^1_0(V)$.

If we denote by $\tilde{G}$ the Riemannian metric tensor of M and put

$$g(X_1,X_2)=\tilde{G}(BX_1,BX_2),\qquad g^*(N_1,N_2)=\tilde{G}(CN_1,CN_2) \tag{1.3}$$

for $X_1, X_2\in\mathcal{T}^1_0(V)$ and $N_1, N_2\in\mathcal{N}^1_0(V)$, then g is a Riemannian metric tensor in V, which is called the *induced metric* of V, and g^* is a tensor field defining an inner product in $N(V)$. The g^* is called the *induced metric* of $N(V)$.

Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in M, i.e., the torsionless affine connection in M such that $\tilde{\nabla}\tilde{G}=0$. Suppose that a vector field $\bar{X}$

2) Manifolds, mapping, tensor fields and any geometric objects we discuss are assumed to be differentiable and of class C^∞.

tangent to $\iota(V)$ and another vector field $\bar{Y}$, tangent to $\iota(V)$ or not, are given along $\iota(V)$. We choose arbitrary local extensions $\tilde{X}$ of $\bar{X}$ and $\tilde{Y}$ of $\bar{Y}$ in a neighborhood Ω of M. Then we can prove that the restriction $(\tilde{\nabla}_{\tilde{X}}\tilde{Y})_V$ of $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ to $\iota(V)$ is independent of the choice of local extensions $\tilde{X}$ and $\tilde{Y}$. Therefore we can define $\tilde{\nabla}_{\bar{X}}\bar{Y}$ by

$$\tilde{\nabla}_{\bar{X}}\bar{Y}=(\tilde{\nabla}_{\tilde{X}}\tilde{Y})_V. \tag{1.4}$$

Thus, taking account of (1.1) and (1.4), we have the formula

$$\tilde{\nabla}_{\bar{X}}\bar{Y}-\tilde{\nabla}_{\bar{Y}}\bar{X}=[\bar{X},\bar{Y}] \tag{1.5}$$

for any vector fields $\bar{X}$ and $\bar{Y}$ defined along $\iota(V)$ and tangent to $\iota(V)$, because $\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\tilde{\nabla}_{\tilde{Y}}\tilde{X}=[\tilde{X},\tilde{Y}]$.

For any vector field $\bar{X}$ along $\iota(V)$, we denote by $\bar{X}^T$ its component tangent to $\iota(V)$ and by $\bar{X}^N$ its component normal to $\iota(V)$. Then we obtain a unique decomposition of $\bar{X}$: $\bar{X}=\bar{X}^T+\bar{X}^N$. If we put

$$B(\nabla_X Y)=(\tilde{\nabla}_{BX}BY)^T \tag{1.6}$$

for $X, Y\in\mathcal{T}_0^1(V)$, we have a unique element $\nabla_X Y$ of $\mathcal{T}_0^1(V)$ and can check easily

$$\nabla_{fX}Y=f\nabla_X Y, \qquad \nabla_X fY=f\nabla_X Y+(Xf)Y$$

for $f\in\mathcal{T}_0^0(V)$. Thus the correspondence $(X, Y)\rightarrow\nabla_X Y$ defines in V a covariant differentiation with respect to an affine connection ∇, which, as is well known, coincides with the Riemannian connection determined by the induced metric g. The affine connection ∇ thus introduced in V is called the *induced connection* of V. If we put

$$C(\nabla_X U)=(\tilde{\nabla}_{BX}CU)^N \tag{1.7}$$

for any $X\in\mathcal{T}_0^1(V)$ and $U\in\mathcal{N}_0^1(V)$, we have a unique element $\nabla_X U$ of $\mathcal{N}_0^1(V)$ and can check easily

$$\nabla_{fX}U=f\nabla_X U, \qquad \nabla_X fU=f\nabla_X U+(Xf)U \tag{1.8}$$

for any $f\in\mathcal{T}_0^0(V)$. Thus the correspondence $(X, U)\rightarrow\nabla_X U$ defines in $N(V)$ a covariant differentiation with respect to a linear connection ∇ in $N(V)$, which satisfies the condition $\nabla g^*=0$. The linear connection ∇ thus introduced in $N(V)$ is called the *induced connection* of $N(V)$.

If we put

$$(\nabla_X B)Y=(\tilde{\nabla}_{BX}BY)^N=\tilde{\nabla}_{BX}BY-B(\nabla_X Y), \tag{1.9}$$

$$(\nabla_X C)U=(\tilde{\nabla}_{BX}CU)^T=\tilde{\nabla}_{BX}CU-C(\nabla_X U) \tag{1.10}$$

for any $X, Y\in\mathcal{T}_0^1(V)$ and $U\in\mathcal{N}_0^1(V)$, then the correspondences $Y\rightarrow(\nabla_X B)Y$ and $U\rightarrow(\nabla_X C)U$ define respectively linear mappings $\nabla_X B$: $\mathcal{T}_0^1(V)\rightarrow\mathcal{N}_0^1(V)$ and $\nabla_X C$:

$\mathcal{N}_0^1(V)\to\mathcal{T}_0^1(V)$, i.e.,

(1.11) $$(\nabla_X B)(fY)=f(\nabla_X B)Y, \qquad (\nabla_X C)(fU)=f(\nabla_X C)U$$

for any $f\in\mathcal{T}_0^0(V)$, $X, Y\in\mathcal{T}_0^1(V)$ and $U\in\mathcal{N}_0^1(V)$. We have from (1.9) and (1.10)

(1.12) $$\tilde{\nabla}_{BX}BY=(\nabla_X B)Y+B(\nabla_X Y),$$

(1.13) $$\tilde{\nabla}_{BX}CU=(\nabla_X C)U+C(\nabla_X U)$$

for any $X, Y\in\mathcal{T}_0^1(V)$ and $U\in\mathcal{N}_0^1(V)$ respectively.

We extend naturally the operations of the induced connections ∇ of $T(V)$ and that of $N(V)$ respectively to $\mathcal{T}(V)$ and $\mathcal{N}(V)$ as derivations and denote the extended covariant differentiation also by the same symbol ∇. We shall now define a derivation ∇_X, $X\in\mathcal{T}_0^1(V)$, in $\mathcal{T}(V)\otimes\mathcal{N}(V)$ as follows:

$$\nabla_X(T\otimes U)=(\nabla_X T)\otimes U+T\otimes(\nabla_X U)$$

for $T\in\mathcal{T}(V)$ and $U\in\mathcal{N}(V)$. The derivation ∇_X thus defined in $\mathcal{T}(V)\otimes\mathcal{N}(V)$ is the so-called *van der Waerden-Bortolotti covariant differentiation* along V.

Since the vector field $(\nabla_X B)Y$ $(X, Y\in\mathcal{T}_0^1(V))$ appearing in (1.12) belongs to $C\mathcal{N}_0^1(V)$, we have a unique element $H(X, Y)$ of $\mathcal{N}_0^1(V)$ such that

(1.14) $$(\nabla_X B)Y=CH(X, Y),$$

where H is an element of $\mathcal{T}_2^0(V)\otimes\mathcal{N}_0^1(V)$ because of (1.8) and (1.11). Since the vector field $(\nabla_X C)N$ $(X\in\mathcal{T}_0^1(V), N\in\mathcal{N}_0^1(V))$ appearing in (1.13) belongs to $B\mathcal{T}_0^1(V)$, we have a unique element $h(X, N)$ of $\mathcal{T}_0^1(V)$ such that

(1.15) $$(\nabla_X C)U=-Bh(X, U),$$

where h is an element of $\mathcal{T}_1^1(V)\otimes\mathcal{N}_1^0(V)$ because of (1.8) and (1.11). These two tensor fields H and h are the so-called *second fundamental tensors* of the submanifold V. We can thus write down (1.12) and (1.13) respectively as follows:

(1.16) $$\tilde{\nabla}_{BX}BY=B(\nabla_X Y)+CH(X, Y),$$

(1.17) $$\tilde{\nabla}_{BX}CU=C(\nabla_X U)-Bh(X, U)$$

for any $X, Y\in\mathcal{T}_0^1(V)$ and $U\in\mathcal{N}_0^1(V)$.

We have $\tilde{G}(BY, CU)=0$ for any $Y\in\mathcal{T}_0^1(V)$, $U\in\mathcal{N}_0^1(V)$, because BY and CU are perpendicular to each other. Thus, taking account of $\tilde{\nabla}\tilde{G}=0$, we obtain

$$\tilde{G}((\nabla_X B)Y, CU)+\tilde{G}(BY, (\nabla_X C)U)=0,$$

which implies together with (1.14) and (1.15)

(1.18) $$g^*(H(X, Y), U)=g(h(X, U), Y)$$

for any $X, Y\in\mathcal{T}_0^1(V)$ and $U\in\mathcal{N}_0^1(V)$. On the other hand, taking account of (1.2) and (1. 5), we have

$$\nabla_{BX}BY-\nabla_{BY}BX=[BX, BY]$$
$$=B[X, Y]$$

for any $X, Y\in\mathcal{T}_0^1(V)$. Substituting (1. 16) in the equation above, we find

$$H(X, Y)=H(Y, X) \tag{1. 19}$$

for $X, Y\in\mathcal{T}_0^1(V)$.

Let $X_1, X_2, \cdots, X_m$ be m local unit vector fields in V, which are perpendicular to each other, where $m=\dim V$. Then an element A of $\mathcal{N}_0^1(V)$ is defined by

$$mA=\sum_{j=1}^{m} H(X_j, X_j), \tag{1. 20}$$

which is called the *mean curvature vector* of the submanifold V. When the mean curvature vector A vanishes identically in V, V is called a *minimal submanifold* in M.

Taking three elements X, Y and Z of $\mathcal{T}_0^1(V)$, we have

$$\tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ=\tilde{\nabla}_{BX}\{B(\nabla_Y Z)+CH(Y, Z)\}$$
$$=B\{\nabla_X\nabla_Y Z-h(X, H(Y, Z))\}+C\{H(X, \nabla_Y Z)+\nabla_X H(Y, Z)\}$$

by virtue of (1. 16) and (1. 17). Therefore, denoting by $\tilde{K}$ and K respectively the curvature tensors of M and V, we have, by definition,

$$\tilde{K}(BX, BY)BZ=\tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ-\tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ-\tilde{\nabla}_{[BX,BY]}BZ,$$
$$K(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

and hence

$$\begin{aligned}\tilde{K}(BX, BY)BZ=BK(X, Y)Z-B\{h(X, H(Y, Z))-h(Y, H(X, Z))\}\\ +C\{(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)\}\end{aligned} \tag{1. 21}$$

for any $X, Y, Z\in\mathcal{T}_0^1(V)$ by virtue of the identity

$$\nabla_X H(Y, Z)=(\nabla_X H)(Y, Z)+H(\nabla_X Y, Z)+H(Y, \nabla_X Z).$$

Denoting by K^* the curvature tensor of the induced connection of $N(V)$, we have by a similar device

$$\begin{aligned}\tilde{K}(BX, BY)CU=CK^*(X, Y)U-C\{H(X, h(Y, U))-H(Y, h(X, U))\}\\ -B\{(\nabla_X h)(Y, U)-(\nabla_Y h)(X, U)\}\end{aligned} \tag{1. 22}$$

for any $X, Y\in\mathcal{T}_0^1(V)$ and any $U\in\mathcal{N}_0^1(V)$. The equations (1. 21) and (1. 22) are the so-called *structure equations* of the submanifold V.

§ 2. Almost contact structures.

We shall now recall definitions and some properties of almost contact structures for the later use. We consider in an odd-dimensional differentiable manifold M an *almost contact structure*, that is, a structure (F, ξ, η), F, ξ and η being a tensor field of type (1, 1), a vector field and a 1-form respectively, such that

$$F^2=-I+\eta\otimes\xi, \qquad F\xi=0, \qquad \eta(F(X))=0, \qquad \eta(\xi)=1 \tag{2. 1}$$

for any $X\in\mathcal{T}_0^1(M)$, where I denotes the identity tensor of type (1, 1). The Nijenhuis tensor N of F is, by definition, a tensor field of type (1, 2) given by

$$N(X, Y)=[FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y] \tag{2. 2}$$

for any $X, Y\in\mathcal{T}_0^1(M)$. The almost contact structure is said to be *normal* when it satisfies the condition

$$S=0, \tag{2. 3}$$

S being a tensor field of type (1, 1) defined by

$$S=N+d\eta\otimes\xi. \tag{2. 4}$$

The condition (2. 3) implies

$$\mathcal{L}F=0, \tag{2. 5}$$

$$\mathcal{L}\eta=0, \qquad \text{or, equivalently } d\eta(\xi, X)=0 \tag{2. 6}$$

for any $X\in\mathcal{T}_0^1(M)$, where $\mathcal{L}$ denotes the Lie derivation with respect to ξ (Cf. [2], [3], [4]).

Suppose that a Riemannian metric G is given in M and satisfies the condition

$$G(X, Y)=G(FX, FY)+\eta(X)\eta(Y), \tag{2. 7}$$

$$\eta(X)=G(X, \xi) \tag{2. 8}$$

for any $X, Y\in\mathcal{T}_0^1(M)$. Then the structure (F, G, ξ, η) is called an *almost contact metric structure*. A tensor field Φ of type (0, 2) defined by

$$\Phi(X, Y)=G(FX, Y) \tag{2. 9}$$

for any $X, Y\in\mathcal{T}_0^1(M)$ is skewsymmetric, i.e.,

$$\Phi(X, Y)+\Phi(Y, X)=0.$$

When the condition

(2. 10) $$\Phi=d\eta$$

is satisfied, (F, G, ξ, η) is called a *contact metric structure*. When the tensor field S defined by (2. 3) vanishes identically, the contact metric structure is said to be *normal*. If the contact metric structure (F, G, ξ, η) is normal, we have

(2. 11) $$\mathcal{L}G=0,$$

(2. 12) $$(\nabla_X F)Y=\eta(Y)X-G(X, Y)\xi,$$

or, equivalently

(2. 13) $$(\nabla_X \Phi)(Y, Z)=G(X, Z)\eta(Y)-G(X, Y)\eta(Z)$$

for any $X, Y, Z \in \mathcal{T}_0^1(M)$, where ∇ denotes the Riemannian connection determined be G and $\mathcal{L}$ the Lie derivation with respect to ξ, [2], [3], [4].

§ 3. Invariant submanifold in an almost contact manifold.

Let V be an m-dimensional differentiable manifold imbedded as a submanifold in a $(2n+1)$-dimensional differentiable manifold M with an almost contact structure $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$ and denote by ι: $V \rightarrow M$ its imbedding. We assume that the tangent space $T_P(\iota(V))$ or the submanifold $\iota(V)$ is invariant by the linear mapping $\tilde{F}$ at each point P of $\iota(V)$ and call V an *invariant* submanifold M. We note here that the formula (1. 2) holds in the present case. Since V is invariant, following the notations introduced in § 1, we have

(3. 1) $$\tilde{F}BX=BFX$$

for any $X \in \mathcal{T}_0^1(V)$, where F is an element of $\mathcal{T}_1^1(V)$. Denoting by $\tilde{N}$ and N respectively the Nijenhuis tensors of $\tilde{F}$ and F, we have

(3. 2) $$\tilde{N}(BX, BY)=BN(X, Y)$$

for $X, Y \in \mathcal{T}_0^1(V)$. In fact, we have from the definition (2. 2) of the Nijenhuis tensor

$$\begin{aligned}\tilde{N}(BX, BY)&=[\tilde{F}BX, \tilde{F}BY]-\tilde{F}[\tilde{F}BX, BY]-\tilde{F}[BX, \tilde{F}BY]+\tilde{F}^2[BX, BY]\\&=[BFX, BFY]-\tilde{F}[BFX, BY]-\tilde{F}[BX, BFY]+\tilde{F}^2[BX, BY]\\&=B\{[FX, FY]-F[FX, Y]-F[X, FY[+F^2[X, Y]\}\\&=BN(X, Y)\end{aligned}$$

by virtue of (1. 2) and (3. 1), which proves the formula (3. 2).

We see easily that there occur only following two cases, i.e., Case I and Case II for any invariant submanifold V in an almost contact manifold M:

Case I: The vector field $\tilde{\xi}$ is never tangent to $\iota(V)$, where V is necessarily even-dimensional.

Case II: The vector field $\tilde{\xi}$ is always tangent to $\iota(V)$, where V is necessarily odd-dimensional.

We first consider the Case I.

Case I. The vector field $\tilde{\xi}$ is never tangent to the invariant submanifold $\iota(V)$, that is, $\tilde{\xi}$ is independent of any vector field of the form BX, $X\in\mathcal{T}_0^1(V)$. Applying $\tilde{F}$ to (3. 1), we have by virtue of (2. 1)

$$
\begin{aligned}
BF^2 &= \tilde{F}^2 BX \\
&= (-I+\tilde{\eta}\otimes\tilde{\xi})BX \qquad (3.3)\\
&= -BX+\tilde{\eta}(BX)\tilde{\xi}
\end{aligned}
$$

for any $X\in\mathcal{T}_0^1(V)$, which implies

$$F^2=-I, \qquad \tilde{\eta}(BX)=0 \tag{3. 4}$$

for any $X\in\mathcal{T}_0^1(V)$. The first equation of (3. 4) shows that the tensor field F appearing in (3. 1) is an almost complex structure, which is called the *induced almost complex structure* of the invariant submanifold V. Taking account of the second equation of (3. 4), we have by virtue of (1. 2)

$$d\tilde{\eta}(BX, BY)=(BX)\tilde{\eta}(BY)-(BY)\tilde{\eta}(BX)-\tilde{\eta}(B[X, Y]) \tag{3. 5}$$

for any $X, Y\in\mathcal{T}_0^1(V)$, which implies

$$d\tilde{\eta}(BX, BY)=0 \tag{3. 6}$$

for any $X, Y\in\mathcal{T}_0^1(V)$. Thus, denoting by $\tilde{S}$ the tensor field defined by (2. 4) in terms of $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$, we find

$$\tilde{S}(BX, BY)=\tilde{N}(BX, BY)$$

and hence by virtue of (3. 2)

$$\tilde{S}(BX, BY)=BN(X, Y) \tag{3. 7}$$

for any $X, Y\in\mathcal{T}_0^1(V)$. Thus we have

PROPOSITION 3. 1. *An invariant submanifold V imbedded in an almost contact manifold M in such a way that the vector field $\tilde{\xi}$ is never tangent to $\iota(V)$ is an almost complex manifold with the induced almost complex structure F. If the almost contact structure of M is normal, the invariant submanifold V is a complex manifold.*

Case II. The vector field $\tilde{\xi}$ is always tangent to the invariant submanifold $\iota(V)$, that is, $\tilde{\xi}$ is expressible as

$$\tilde{\xi}=B\xi, \tag{3. 8}$$

where ξ is a unique element of $\mathcal{T}_0^1(V)$. If we put

$$\eta(X)=\tilde{\eta}(BX) \tag{3. 9}$$

for any $X\in\mathcal{T}_0^1(V)$, then η is a 1-form in V. Thus by virtue of (3. 1), (3. 8) and (3. 9) we have, from (2. 1),

$$F^2=-I+\eta\otimes\xi \tag{3. 10}$$

and, from the conditions $\tilde{F}\tilde{\xi}=0$, $\tilde{\eta}(\tilde{F}(BX))=0$ and $\tilde{\eta}(\tilde{\xi})=1$, respectively

$$F\xi=0, \qquad \eta(FX)=0, \qquad \eta(\xi)=1 \tag{3. 11}$$

for any $X\in\mathcal{T}_0^1(V)$. Therefore (F,ξ,η) forms an almost contact structure in V, which is called the *induced almost contact structure* of V. Taking account of (3. 5) and (3. 9), we obtain along V

$$d\tilde{\eta}(BX, BY)=d\eta(X, Y) \tag{3. 12}$$

for $X, Y\in\mathcal{T}_0^1(V)$. If we denote by $\tilde{S}$ and S the tensor fields defined by (2. 3) in terms of $(\tilde{F},\tilde{\xi},\tilde{\eta})$ and (F,ξ,η) respectively, then we have by virtue of (3. 2), (3. 8) and (3. 12)

$$\tilde{S}(BX, BY)=BS(X, Y) \tag{3. 13}$$

for $X, Y\in\mathcal{T}_0^1(V)$. Thus we have

PROPOSITION 3. 2. *An invariant submanifold V imbedded in an almost contact manifold M in such a way that the vector field $\tilde{\xi}$ is always tangent to $\iota(V)$ is an almost contact manifold with the induced almost contact structure (F,ξ,η). If the almost contact structure of M is normal, the induced almost contact structure of V is so also.*

§ 4. Invariant submanifolds in a normal contact metric manifold.

Let V be an invariant submanifold of a normal contact metric manifold M and denote by $(\tilde{F},\tilde{G},\tilde{\xi},\tilde{\eta})$ the normal contact metric structure of M. We denote by g the induced metric of V in the sense of § 1.

Case I. We assume that the vector field $\tilde{\xi}$ is never tangent to the invariant submanifold $\iota(V)$. Following the notations introduced in § 1 and § 2, we have by virtue of (1. 3), (2. 7), (3. 1) and (3. 4)

$$g(X, Y)=g(FX, FY) \tag{4. 1}$$

for any $X, Y\in\mathcal{T}_0^1(V)$. Thus, taking account of (3. 4) and (4. 1), we see that the *induced structure* (F,g) of V is almost Hermitian. Denoting by $\tilde{\nabla}$ the Riemannian connection determined by $\tilde{G}$, we have by virtue of (1. 16)

$$\tilde{\nabla}_{BX}(BFY)=B(\nabla_X FY)+CH(X, FY)$$
$$=B\{(\nabla_X F)Y+F\nabla_X Y\}+CH(X, FY)$$

for $X, Y\in\mathcal{T}_0^1(V)$, which implies together with (2. 12), (3. 1) and (3. 4)

$$\begin{aligned}
&B\{(\nabla_X F)Y+F\nabla_X Y\}+CH(X, FY)\\
&=\tilde{\nabla}_{BX}(BFY)\\
&=\tilde{\nabla}_{BX}(\tilde{F}BY)\\
&=(\tilde{\nabla}_{BX}\tilde{F})BY+\tilde{F}\{B\nabla_X Y+CH(X, Y)\}\\
&=BF\nabla_X Y+\tilde{F}CH(X, Y)-\tilde{G}(BX, BY)\tilde{\xi}
\end{aligned}$$

for any $X, Y\in\mathcal{T}_0^1(V)$. Since the last two terms in the last expression are normal to $\iota(V)$, we have

$$\nabla_X F=0 \tag{4. 2}$$

and

$$CH(X, FY)=\tilde{F}CH(X, Y)-g(X, Y)\tilde{\xi} \tag{4. 3}$$

for any $X, Y\in\mathcal{T}_0^1(V)$. We have from (4. 3)

$$H(X, FY)=H(Y, FX) \tag{4. 4}$$

because of the fact that the right hand side of (4. 3) is symmetric with respect to X and Y. The equation (4. 2) shows that *V is a Kählerian manifold with the induced structure (F, g)*. We take m local unit vector fields $X_1, X_2, \cdots, X_m$ in V, which are mutually orthognal and satisfy the conditions

$$FX_k=X_{k+r}, \qquad FX_{k+r}=-X_k \qquad (k=1, 2, \cdots, r), \tag{4. 5}$$

where $m=2r$. Then the mean curvature vector A defined by (1. 20) is given by

$$mA=\sum_{j=1}^{m} H(X_j, X_j)=\sum_{k=1}^{r} H(X_k, X_k)+\sum_{k=1}^{r} H(X_{k+r}, X_{k+r})$$
$$=-\sum_{k=1}^{r} H(X_k, FX_{k+r})+\sum_{k=l}^{r} H(X_{k+r}, FX_k)=0$$

because of (4. 4) and (4. 5). That is to say, *V is a minimal submanifold in M.* Summing up, we have

PROPOSITION 4. 1. *An invariant submanifold V imbedded in a normal contact metric manifold M in such a way that the vector field $\tilde{\xi}$ is never tangent to $\iota(V)$ is a Kählerian manifold with the induced structure (F, g) and is a minimal submanifold in M.*

Case II. We assume that the vector field $\tilde{\xi}$ is always tangent to the invariant submanifold $\iota(V)$. Following the notations used in the above, we have by virtue of (1. 3), (2. 7), (3. 8) and (3. 9)

$$g(X, Y)=g(FX, FY)+\eta(X)\eta(Y),\qquad \eta(X)=g(X, \xi) \tag{4. 6}$$

for $X, Y\in\mathcal{T}_0^1(V)$. If we put

$$\phi(X, Y)=g(FX, Y)=\tilde{G}(\tilde{F}BX, BY)$$

for any $X, Y\in\mathcal{T}_0^1(V)$, then we have

$$\phi(X, Y)=d\tilde{\eta}(BX, BY)=d\eta(X, Y)$$

by virtue of (2. 8), (2. 9), (3. 9) and (3. 12). That is to say, we have

$$\phi=d\eta. \tag{4. 7}$$

Thus, taking account of Proposition 3. 2, (4. 6) and (4. 7), we see that *the induced structure* (F, g, ξ, η) *of* V *is a normal contact metric structure.* Taking account of (1. 16), we have

$$\tilde{\nabla}_{BX}BFY=B\{(\nabla_X F)Y+F\nabla_X Y\}+CH(X, FY)$$

for any $X, Y\in\mathcal{T}_0^1(V)$, which implies together with (2. 12), (3. 1), (3. 8) and (3. 9)

$$\begin{aligned} B\{(\nabla_X F)Y+F\nabla_X Y\}+CH(X, FY)&=\tilde{\nabla}_{BX}(BFY)\\ =\tilde{\nabla}_{BX}(\tilde{F}BY)&=(\tilde{\nabla}_{BX}\tilde{F})BY+\tilde{F}\{B\nabla_X Y+CH(X, Y)\}\\ &=B\{\eta(Y)X-g(X, Y)\xi\}+BF\nabla_X Y+\tilde{F}CH(X, Y)\end{aligned}$$

for $X, Y\in\mathcal{T}_0^1(V)$. Therefore we have

$$CH(X, FY)=\tilde{F}CH(X, Y)) \tag{4. 8}$$

for $X, Y\in\mathcal{T}_0^1(V)$, which implies

$$H(X, FY)=H(Y, FX) \tag{4. 9}$$

for $X, Y\in\mathcal{T}_0^1(V)$. Substituting $Y=\xi$ in (4. 8), we obtain for $X\in\mathcal{T}_0^1(V)$

$$H(X, \xi)=0 \tag{4. 10}$$

because of $F\xi=0$. If we take account of (4. 9) and (4. 10) and use a similar device as used in Case I, we see that the mean curvature vector A vanishes identically, i.e., that V *is a minimal submanifold in* M. Summing up, we have

PROPOSITION 4. 2. *An invariant submanifold* V *imbedded in a normal contact* *metric manifold M in such a way that the vector field $\tilde{\xi}$ is always tangent to $\iota(V)$ is a normal contact metric manifold with the induced structure (F, g, ξ, η) and is a minimal submanifold in M.*

From Propositions 4. 1 and 4. 2, we have

PROPOSITION 4. 3. *Any invariant submanifold imbedded in a normal contact metric manifold is a minimal submanifold.*

§ 5. Invariant submanifolds in a normal, regular almost contact manifold.

Let $\tilde{M}$ be an n-dimensional almost contact manifold with structure $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$, which is normal, and assume that any orbit of the vector field $\tilde{\xi}$ is a closed set in $\tilde{M}$. In such a case, $\tilde{M}$ is said to be *regular*. Then the set of all orbits of $\tilde{\xi}$ forms a differentiable manifold M of dimension $n-1$ if it is naturally topologized, and the natural projection π: $\tilde{M} \to M$ is differentiable and of the maximum rank everywhere. Denoting by $d\pi$ the differential mapping of π, we have

(5. 1) $$d\pi\tilde{\xi}=0.$$

Since the contact metric structure $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$ is normal, we have

(5. 2) $$\mathcal{L}\tilde{F}=0, \qquad \mathcal{L}\tilde{\eta}=0$$

by virtue of (2. 5) and (2. 6), where $\mathcal{L}$ denotes the Lie derivation with respect to $\tilde{\xi}$. Thus, by virtue of the second equation of (5. 2), the set $(\tilde{M}, M, \pi; \tilde{\xi}, \tilde{\eta})$ forms a *fibred space* in the sense of [7] and [8], where $\tilde{M}$ and M are called respectively the *total space* and the *base space*.

In the fibred space $(\tilde{M}, M, \pi; \tilde{\xi}, \tilde{\eta})$, a vector field $\tilde{X}$ given in $\tilde{M}$ is said to be *horizontal* if $\tilde{\eta}(\tilde{X})=0$. An element $\tilde{T}$ of $\mathcal{T}^r_s(M)$ is said to be *invariant* if $\mathcal{L}\tilde{T}=0$, $\mathcal{L}$ denoting the Lie derivation with respect to $\tilde{\xi}$. For an invariant function $\tilde{f}$ in $\tilde{M}$, its *projection* $\hat{f}=p\tilde{f}$ is a function in M such that $\tilde{f}=\hat{f}\circ\pi$. For an invariant vector field $\tilde{X}$ in $\tilde{M}$, its *projection* $\hat{X}$ is a vector field in M such that $\hat{X}=d\pi\tilde{X}$. For a vector field $\hat{X}$ in M, a horizontal vector field $\tilde{X}$ in $\tilde{M}$ is called the *lift* of $\hat{X}$ and denoted by $\hat{X}^L$ if $\hat{X}=p\hat{X}^L$, where $\hat{X}^L$ is horizontal and invariant in $\tilde{M}$. For an invariant 1-form $\tilde{\omega}$, its *projection* $\hat{\omega}=p\tilde{\omega}$ is a 1-form in M such that $\hat{\omega}(\hat{X})=p\tilde{\omega}(\hat{X}^L)$ for $\hat{X}\in\mathcal{T}^1_0(M)$. For an invariant tensor field $\tilde{P}$ of type (1, 1) in $\tilde{M}$, its projection $\hat{P}=p\tilde{P}$ is a tensor field of the same type in M such that $\hat{P}\hat{X}=p(\tilde{P}\hat{X}^L)$ for $\hat{X}\in\mathcal{T}^1_0(M)$. For an invariant tensor field $\tilde{Q}$ of type (0, 2) in $\tilde{M}$, its projection $\hat{Q}=p\tilde{Q}$ is a tensor field of the same type such that $\hat{Q}(\hat{X}, \hat{Y})=p\tilde{Q}(\hat{X}^L, \hat{Y}^L)$ for $\hat{X}, \hat{Y}\in\mathcal{T}^1_0(M)$. For an invariant tensor field $\tilde{R}$ of type (1, 2) in $\tilde{M}$, its projection $\hat{R}=p\tilde{R}$ is a tensor field of the same type such that $\hat{R}(\hat{X}, \hat{Y})=p\tilde{R}(\hat{X}^L, \hat{Y}^L)$ for $\hat{X}, \hat{Y}\in\mathcal{T}^1_0(M)$ (Cf. [7], [8]). We have now

(5. 3) $$p\tilde{\xi}=0, \qquad p\tilde{\eta}=0$$

directly from the definitions (Cf. (5. 1) and (5. 2)).

Denoting by $\hat{F}$ the projection $p\tilde{F}$ of the invariant tensor field $\tilde{F}$ (Cf. (5. 2)), we have

$$p(\tilde{F}\hat{X}^L)=\hat{F}\hat{X} \tag{5. 4}$$

for $\hat{X}\in\mathcal{T}_0^1(M)$. If we take the projection of the identity $F^2=-I+\tilde{\eta}\otimes\tilde{\xi}$, we find

$$\hat{F}^2=-I$$

because of (5. 3). Thus *the base space M is an almost complex manifold with structure* $\hat{F}$.

We have obtained in [7] the formula

$$p[\hat{X}^L, \hat{Y}^L]=[\hat{X}, \hat{Y}] \tag{5. 5}$$

for $\hat{X}, \hat{Y}\in\mathcal{T}_0^1(M)$. Denoting by $\tilde{N}$ and $\hat{N}$ respectively the Nijenhuis tensors of $\tilde{F}$ and $\hat{F}$, we have

$$p\tilde{N}=\hat{N}. \tag{5. 6}$$

In fact, taking account of $\tilde{F}\tilde{\xi}=0$, we have by virtue of (2. 2), (5. 4) and (5. 5)

$$\begin{aligned}p\tilde{N}(\hat{X}^L, \hat{Y}^L)&=p\{[\tilde{F}\hat{X}^L, \tilde{F}\hat{Y}^L]-\tilde{F}[\tilde{F}\hat{X}^L, \hat{Y}^L]-\tilde{F}[\hat{X}^L, \tilde{F}\hat{Y}^L]+\tilde{F}^2[\hat{X}^L, \hat{Y}^L]\}\\&=[\hat{F}\hat{X}, \hat{F}\hat{Y}]-\hat{F}[\hat{F}\hat{X}, \hat{Y}]-\hat{F}[\hat{X}, \hat{F}\hat{Y}]+\hat{F}^2[\hat{X}, \hat{Y}]\\&=\hat{N}(\hat{X}, \hat{Y})\end{aligned}$$

for $\hat{X}, \hat{Y}\in\mathcal{T}_0^1(M)$, which implies (5. 6).

Denoting by $\tilde{S}$ the tensor field defined by (2. 3) in terms of $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$, we find by virtue of (5. 3) and (5. 6)

$$p\tilde{S}=p\tilde{N}=N. \tag{5. 7}$$

Summing up, we have

PROPOSITION 5. 1. *Let $\tilde{M}$ be a normal almost contact manifold with structure $(\tilde{F}, \tilde{\xi}, \tilde{\eta})$, which is regular. Then the base space M is a complex manifold with structure $\hat{F}=p\tilde{F}$.*

Let $\tilde{V}$ be an m-dimensional invariant submanifold imbedded in $\tilde{M}$ and denote its imbedding by $\iota\colon \tilde{V}\to\tilde{M}$. In Case I, the vector field $\tilde{\xi}$ is never tangent to $\iota(\tilde{V})$ and hence the tangent space $T_P(\iota(\tilde{V}))$ of $\iota(\tilde{V})$, at each point P of $\iota(\tilde{V})$, is contained in the horizontal plane, i.e., in the set of all horizontal vectors at P. In Case II, the vector field $\tilde{\xi}$ is always tangent to $\iota(\tilde{V})$ and hence the tangent space $T_P(\iota(\tilde{V}))$ of $\iota(\tilde{V})$, at each point P of $\iota(\tilde{V})$, is a direct sum of a certain subspace of the horizontal plane and a 1-dimensional subspace spanned by $\tilde{\xi}$. Therefore we see that $\dim(T_P(\iota(\tilde{V}))=m$ (or respectively $m-1$) in Case I (or respectively in Case II) for any point P of $\iota(\tilde{V})$. Thus, in each case, the projection π restricted to $\iota(\tilde{V})$ is differentiable and of constant rank. Consequently the image $V=\pi(\iota(\tilde{V}))$ is a submanifold immersed differentiablly in the base space M.

Take a vector $\hat{v}$ tangent to $V=\pi(\iota(\tilde{V}))$ at a point Q of V. Then there exists a unique horizontal vector $\tilde{v}$ tangent to $\iota(\tilde{V})$ at each point P belonging to $\pi^{-1}(Q)\cap\iota(\tilde{V})$ such that $d\pi\tilde{v}=\hat{v}$. Thus, we have by virtue of (5. 4)

$$d\pi(\tilde{F}\tilde{v})=\hat{F}\hat{v}, \tag{5. 8}$$

where $\hat{F}=p\tilde{F}$ is the complex structure of the base space M. The submanifold $\iota(\tilde{V})$ being invariant in the total space $\tilde{M}$, $\tilde{F}\tilde{v}$ is tangent to $\iota(\tilde{V})$. Therefore we see from (5. 8) that $\hat{F}\hat{v}$ is tangent to V, i.e., that the submanifold V is invariant in the complex space M. Thus, taking account of Proposition 5. 1, we have

PROPOSITION 5. 2. *Let $\tilde{V}$ be an invariant submanifold imbedd in a normal almost contact manifold $\tilde{M}$ with structure $(\tilde{F},\tilde{\xi},\tilde{\eta})$, which is regular. Then the image V of the $\tilde{V}$ by the projection π: $\tilde{M}\to M$ is, in the base space M, a complex submanifold with respect to the complex structure $\hat{F}=p\tilde{F}$ of M.*

We assume now that $\tilde{M}$ is a normal contact metric manifold with structure $(\tilde{F}, \tilde{G}, \tilde{\xi}, \tilde{\eta})$, which is regular. Following the notations used in the above and putting

$$\hat{g}=p\tilde{G},$$

we find by virtue of (2. 7) and (5. 4)

$$\hat{g}(\hat{X}, \hat{Y})=\hat{g}(\hat{F}\hat{X}, \hat{F}\hat{Y}) \tag{5. 9}$$

for $\hat{X}, \hat{Y}\in\mathcal{T}_0^1(M)$. Denoting by $\tilde{\nabla}$ the Riemannian connection determined by $\tilde{G}$, we proved in [8] the fact that the Riemannian connection determined by $\hat{g}$ satisfies the condition

$$p(\tilde{\nabla}_{\hat{X}^L}\hat{Y}^L)=\hat{\nabla}_{\hat{X}}\hat{Y} \tag{5. 10}$$

for $\hat{X}, \hat{Y}\in\mathcal{T}_0^1(M)$. Thus we have by virtue of (2. 12), (5. 4) and (5. 10)

$$\hat{\nabla}_{\hat{X}}\hat{F}\hat{Y}=p(\tilde{\nabla}_{\hat{X}^L}(\hat{F}\hat{Y})^L)=p(\tilde{\nabla}_{\hat{X}^L}(F\hat{X}^L)), \qquad \text{i.e.,}$$

$$\begin{aligned}(\hat{\nabla}_{\hat{X}}\hat{F})\hat{Y}&=p((\tilde{\nabla}_{\hat{X}^L}F)\hat{Y}^L)\\&=p(\tilde{\eta}(\hat{X}^L)\hat{Y}^L-\tilde{G}(\hat{Y}^L, \hat{X}^L)\tilde{\xi})=0\end{aligned}$$

for $\hat{X}, \hat{Y}\in\mathcal{T}_0^1(M)$, because of $\tilde{\eta}(\hat{X}^L)=0$ and $p\tilde{\xi}=0$. Therefore we obtain

$$\hat{\nabla}\hat{F}=0,$$

which implies together with (5. 9) that *the base space M is a Kählerian manifold with structure $(\hat{F}, \hat{g})$*. Consequently, taking account of Proposition 5. 2 and [5], we have

PROPOSITION 5. 3. *Let* $\tilde{V}$ *be an invariant submanifold imbedded in a normal contact metric manifold* $\tilde{M}$ *with structure* $(\tilde{F}, \tilde{G}, \tilde{\xi}, \tilde{\eta})$, *which is regular. The image* V *of* $\tilde{V}$ *by the projection* π: $\tilde{M} \rightarrow M$ *is, in the base space* M, *a Kählerian submanifold with respect to the Kählerian structure* $(\hat{F}, \hat{g}) = (p\tilde{F}, p\tilde{G})$ *of* M *and is a minimal submanifold in* M.

REFERENCES

[1] OKUMURA, M., On contact metric immersion. To appear.

[2] SASAKI, S., On differentiable manifolds with certain structures which are closely related to almost contact structure I. Tôhoku Math. J. **12** (1960), 459–476.

[3] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II. Tôhoku Math. J. **13** (1961), 281–294.

[4] SASAKI, S. AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures. J. Math. Soc. Japan **14** (1962), 249–271.

[5] SCHOUTEN, J. A., AND K. YANO, On invariant subspaces in the almost complex X_{2n}. Ind. Math. **17** (1955), 261–269.

[6] YANO, K., Differential geometry on complex and almost complex spaces. Pergamon Press (1955).

[7] YANO, K., AND S. ISHIHARA Differential geometry of fibred spaces. Kōdai Math. Sem. Rep. **19** (1967), 257–288.

[8] YANO, K., AND S. ISHIHARA, Fibred spaces with invariant Riemannian metric. Kōdai Math. Sem. Rep. **19** (1967), 317–360.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.