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SOME EXTREMAL PROPERTIES IN THE UNIT BALL OF VON NEUMANN ALGEBRAS

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This paper is prepared to investigate some extremal properties in the unit ball of von Neumann algebras. Throughout this paper, by extremal point we mean the extremal point of the unit ball of the algebra considered. Theorem 1 is characterizations of extremal points. Theorems 2 and 6 are characterizations of finite von Neumann algebras. Theorem 3 gives a sufficient condition for a von Neumann algebra to be finite. Theorem 4 treats the extremal points of a von Neumann algebra which is induced into or reduced to the invariant subspace of the algebra or its commutant. Theorem 5 gives a necessary and sufficient condition for a von Neumann algebra to be a properly infinite factor. Theorems 6 and 7 treat the extremal points which appear in the tensor products. Theorems 1 and 2 are specializations of the results obtained by Kadison [2], Sakai [4], and Miles [3].

1. Notations and definitions. Let $\mathfrak{H}$ be a complex Hilbert space and $\mathfrak{L}(\mathfrak{H})$ be the full operator algebra on it. Let $\mathfrak{A}$ and $\mathfrak{B}$ be von Neumann algebras, and $\boldsymbol{C}$ the von Neumann algebra of all scalar multiples of the identity operator. For a projection E in $\mathfrak{A}$ or $\mathfrak{A}'$ the set $\{T_E : T\in\mathfrak{A}\}$ forms a von Neumann algebra $\mathfrak{A}_E$, where T_E is a restriction of ET to the range of E. For convenience, we shall denote by $\mathfrak{A}_1$ the unit ball of $\mathfrak{A}$, $\mathfrak{A}^e$ the set of extremal elements of $\mathfrak{A}_1$, $\mathfrak{A}^p$ the set of projections in $\mathfrak{A}$ and $\mathfrak{A}^{pi}$ the set of partially isometric operators in $\mathfrak{A}$. The operators 1 and 1_G stand for the identity of $\mathfrak{A}$ and $\mathfrak{A}_G$, where G is a projection belonging to the center of $\mathfrak{A}$. Furthermore, denote by $\mathfrak{A}^i$ the set of isometric operators in $\mathfrak{A}$, by $\mathfrak{A}^{i*}$ the set of A with $A^*\in\mathfrak{A}^i$ and $\mathfrak{A}^u$ the set of unitary operators in $\mathfrak{A}$. For E and $F\in\mathfrak{A}^p$, $E\sim F$ if and only if there is $A\in\mathfrak{A}$ with $A^*A=E$ and $AA^*=F$, and $E\prec F$ if and only if there is $A\in\mathfrak{A}$ with $A^*A=E$ and $AA^*\leqq F$. Let $\mathrm{Re}\,(x, y)$ be the real part of the inner product (x, y) for vectors x and y. Let $\mathfrak{A}\times\mathfrak{B}$ be the product von Neumann algebra of $\mathfrak{A}$ and $\mathfrak{B}$, and $A\times B$ be the product operator in $\mathfrak{A}\times\mathfrak{B}$ with $A\in\mathfrak{A}$ and $B\in\mathfrak{B}$. Let $\mathfrak{A}\otimes\mathfrak{B}$ be the tensor product of $\mathfrak{A}$ and $\mathfrak{B}$, and $\mathfrak{A}^e\otimes\mathfrak{B}^e$ denotes the set of tensor products $A\otimes B$ of all the pairs $A\in\mathfrak{A}^e$ and $B\in\mathfrak{B}^e$.

2. The following theorem due to Kadison plays an important role in this paper and the independent proof will be given.

THEOREM 1. *The following conditions*[1] *are equivalent*:

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1) Kadison has proved the mutual equivalence of (1) and (3) for C^*-algebra [2].

(1) $A\in\mathfrak{A}^e$;

(2) $A\in\mathfrak{A}^{pi}$ *and there exists a central projection* G *of* $\mathfrak{A}$ *such that* $G\leqq A^*A$ *and* $1-G\leqq AA^*$; *and*

(3) $A\in\mathfrak{A}^{pi}$ *and* $(1-AA^*)\mathfrak{A}(1-A^*A)=\{0\}$.

Proof. (1) *implies* (2): It is obvious that $A\neq 0$ for $A\in\mathfrak{A}^e$. Suppose that an operator $A\in\mathfrak{A}^e$ is not partially isometric. Then there is a vector $x\in\mathfrak{H}$ in the carrier of A such that $||Ax||<||x||$. Let $A=U|A|$ be the polar decomposition of A and let P be the projection onto the subspace of $\mathfrak{H}$ spanned by $\{Tx: T\in\mathfrak{A}\cup\mathfrak{A}'\}$. Then P is a central projection of $\mathfrak{A}$. Using these operators, put $V=A(1-P)+UP$ and $W=A(1-P)+(2A-U)P$. Then $V\in\mathfrak{A}_1$ and $W\in\mathfrak{A}_1$. Since $||Ax||<||x||$ and $x\in U^*U\mathfrak{H}$

$$||APx||=||Ax||<||x||=||Ux||=||UPx||$$

and so $AP\neq UP$, then $V\neq W$. But $A=A(1-P)+AP=(V+W)/2$, so this contradicts the assumption that $A\in\mathfrak{A}^e$. Thus $A\in\mathfrak{A}^{pi}$. Let $E=1-A^*A$ and $F=1-AA^*$ and apply the theorem of comparability [1] to these E and F, then there exists a central projection G of $\mathfrak{A}$ such that $EG\prec FG$ and $F(1-G)\prec E(1-G)$. If $EG\neq 0$, then $B^*B=EG$ and $BB^*\leqq FG$ for some non zero $B\in\mathfrak{A}^{pi}$. Define A^+ and A^- by $A^\pm=A\pm B$, then $A=(A^++A^-)/2$ and $A^\pm\in\mathfrak{A}_1$. This contradicts $A\in\mathfrak{A}^e$ and hence $EG=0$. Similarly $F(1-G)=0$. Therefore $(1-E)G=G$ and $(1-F)(1-G)=1-G$, that is $G\leqq 1-E=A^*A$ and $1-G\leqq 1-F=AA^*$.

(2) *implies* (3): By the condition (2), $1-A^*A\leqq 1-G$ and $1-AA^*\leqq G$, then $(1-AA^*)T(1-A^*A)=(1-AA^*)GT(1-G)(1-A^*A)=0$ for every $T\in\mathfrak{A}$.

(3) *implies* (1): If an operator A given in (3) is not in $\mathfrak{A}^e$, then there are two different operators S^+ and S^- in $\mathfrak{A}_1$ such that $A=(S^++S^-)/2$. For every $z\in\mathfrak{H}$

$$\begin{aligned}||(S^+-S^-)z||^2&=2(||S^+z||^2+||S^-z||^2)-||(S^++S^-)z||^2\\&=2(||S^+z||^2+||S^-z||^2)-4||Az||^2,\end{aligned}$$

so that for $x\in A^*A\mathfrak{H}$ $S^+x=S^-x$,[2] hence $S^\pm x=(S^+x+S^-x)/2=Ax$ and for $y\in(1-A^*A)\mathfrak{H}$ $(S^++S^-)y=2Ay$, hence $S^+y=-S^-y$. Define $B=S^+(1-A^*A)$. Then $S^\pm=A\pm B$ and so $B\neq 0$. For every $x\in A^*A\mathfrak{H}$ and $y\in(1-A^*A)\mathfrak{H}$

$$\begin{aligned}||x||^2+||y||^2=||x+y||^2&\geqq||S^\pm(x+y)||^2=||(A\pm B)(x+y)||^2\\&=||Ax\pm By||^2=||Ax||^2\pm 2\,\mathrm{Re}\,(Ax, By)+||By||^2.\end{aligned}$$

Hence $||y||^2-||By||^2=\pm 2\,\mathrm{Re}\,(Ax, By)$ for every $x\in A^*A\mathfrak{H}$ and $y\in(1-A^*A)\mathfrak{H}$. Therefore $(Ax, By)=0$. Thus $B\mathfrak{H}\subset(1-AA^*)\mathfrak{H}$ and so $(1-AA^*)B(1-A^*A)\neq 0$, which is contrary to (3). Q.E.D.

REMARK 1.1. In Theorem 1, (2) follows directly from (3) without assuming

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2) It is easily seen that $\mathfrak{A}^i\subset\mathfrak{A}^e$ and $\mathfrak{A}^{i*}\subset\mathfrak{A}^e$ by the parallelogram law,

the partial isometry of the operator. Taking $A\in\mathfrak{A}$ with $(1-AA^*)\mathfrak{A}(1-A^*A)=\{0\}$, then

$$0=A^*(1-AA^*)A(1-A^*A)=(A^*-A^*AA^*)(A-AA^*A)=(A-AA^*A)^*(A-AA^*A)$$

and so $A=AA^*A$, therefore

$$AA^*=(AA^*)^*=(AA^*)^2 \qquad \text{and} \qquad A^*A=(A^*A)^*=(A^*A)^2$$

hence $A\in\mathfrak{A}^{pi}$. Let G be the projection onto the subspace $\{Tx: T\in\mathfrak{A}\cup\mathfrak{A}'$ and $x\in(1-A^*A)\mathfrak{H}\}^{\perp}$. Then G is central and $1-A^*A\leqq 1-G$, so $G\leqq A^*A$. But $(1-AA^*)\mathfrak{A}(1-A^*A)=\{0\}$ and so $(1-AA^*)(1-G)=0$. Thus $1-AA^*\leqq G$.

REMARK 1. 2. (1) follows directly from (2). Suppose now that the condition (2) holds. If $A=(B+C)/2$ with $B\in\mathfrak{A}_1$ and $C\in\mathfrak{A}_1$, then for each $x\in\mathfrak{H}$

$$||(B-C)x||^2=2(||Bx||^2+||Cx||^2)-||(B+C)x||^2=2(||Bx||^2+||Cx||^2)-4||Ax||^2$$

and so for $x\in A^*A\mathfrak{H}$, $Bx=Cx=Ax$. Since $G\leqq A^*A$, $BG=CG=AG$. Since $A^*=(B^*+C^*)/2$, by repeating the similar argument, $B^*(1-G)=C^*(1-G)=A^*(1-G)$, so that $B(1-G)=C(1-G)=A(1-G)$. Thus $A=B=C$ and so $A\in\mathfrak{A}^e$.

THEOREM 2. *The following conditions are equivalent*:
(1) $\mathfrak{A}$ *is finite*; *and*
(2) $\mathfrak{A}^e=\mathfrak{A}^u$.

Proof. (1) *implies* (2): Being stated above[2] that $\mathfrak{A}^u\subset\mathfrak{A}^e$, it is sufficient to show the converse inclusion. Suppose $A\in\mathfrak{A}^e$, then by Theorem 1, both A^*A and AA^* are projections with $A^*A\sim AA^*$. Since $\mathfrak{A}$ is finite, $1-A^*A\sim 1-AA^*$ and so there is a partially isometric operator $B\in\mathfrak{A}$ such that $1-A^*A=B^*B$ and $1-AA^*=BB^*$. Let $A^{\pm}=A\pm B$, then $A^{\pm}\in\mathfrak{A}_1$. Since $A\in\mathfrak{A}^e$, $A=A^+=A^-$, therefore $B=0$. Thus $A^*A=AA^*=1$, that is, $A\in\mathfrak{A}^u$.
(2) *implies* (1): If $A\in\mathfrak{A}$ with $A^*A=1$, then $A\in\mathfrak{A}^i$. Hence $A\in\mathfrak{A}^e$ and so $AA^*=1$. Thus according to the algebric characterization of a finite von Neumann algebra [1], $\mathfrak{A}$ is finite.

REMARK 2. 1. In the last theorem that (1) implies (2) is obtained somewhat different methods. If $A\in\mathfrak{A}^e$, then $A\in\mathfrak{A}^{pi}$ and $(1-AA^*)\mathfrak{A}(1-A^*A)=\{0\}$. Since $\mathfrak{A}$ is finite, $B^*B=1-A^*A$ and $BB^*=1-AA^*$ for some $B\in\mathfrak{A}^{pi}$, hence $0=(1-AA^*)B(1-A^*A)=BB^*BB^*B=B$, therefore $A^*A=AA^*=1$, which implies $A\in\mathfrak{A}^u$.

THEOREM 3. (1) *If* $A\in\mathfrak{A}^e$. *then* $A^*A\sim AA^*\sim 1$; *and*
(2) *If the converse of* (1) *holds, then* $\mathfrak{A}$ *is finite.*

Proof. (1) Suppose $A\in\mathfrak{A}^e$, then there is a central projection G with $G\leqq A^*A$ and $1-G\leqq AA^*$, and hence

$$G=A^*AG=(AG)^*(AG)\sim(AG)(AG)^*=AA^*G$$

and $1-G=AA^*(1-G)$. Thus

$$1=G+(1-G)\sim AA^*G+AA^*(1-G)=AA^*.$$

(2) If $\mathfrak{A}$ is not finite, then there is a projection $A\in\mathfrak{A}$ with $A\neq 1$ and $A\sim 1$. Define A^+ and A^- by $A^{\pm}=A\pm(1-A)$. Then $A=(A^++A^-)/2$, therefore $A\notin\mathfrak{A}^e$ and $A^*A\sim AA^*\sim 1$.

It follows immediately from the last theorem that (1) implies (2) in Theorem 2.

REMARK 3. 1. An alternative proof of (1) in Theorem 3. In the case where $A^*A=1$ or $AA^*=1$, by Theorem 1, $A^*A\sim AA^*\sim 1$. Therefore it suffices to prove (1) in the case where $A^*A\neq 1$ and $AA^*\neq 1$. Let $E=1-A^*A$ and $F=1-AA^*$. By Theorem 1, E and F are projections with $EF=0$, so that $F\leqq 1-E=A^*A$. Define $F_1=F$ and $F_{n+1}=A^nFA^{*n}$ for $n=1, 2, 3, \cdots$. Then $F_n\in\mathfrak{A}^p$ such that $F_n\leqq A^*A$, $F_n\sim F_{n+1}$ and $F_nF_m=0$ for $n, m=1, 2, 3, \cdots$ and $n\neq m$. Put $F_0=F_1+F_2+F_3+\cdots$, then F_0 is also a projection with $F_0\leqq A^*A$ and $A^*A-F_0\sim 1-F_0$. Thus

$$A^*A=(A^*A-F_0)+F_0\sim(1-F_0)+F_0=1.$$

LEMMA. *For any $E\in\mathfrak{A}^p$ or $\mathfrak{A}'^p$, $(\mathfrak{A}_1)_E=(\mathfrak{A}_E)_1$.*

Proof. It is clear that $(\mathfrak{A}_1)_E\subset(\mathfrak{A}_E)_1$, so that it suffices to show the converse inclusion. i) If $E\in\mathfrak{A}$ and $B=A_E\in(\mathfrak{A}_E)_1$ with $A\in\mathfrak{A}$, then $\|EAE\|\leqq 1$. Let $C=EAE+(1-E)$. Then $C\in\mathfrak{A}_1$ and $C_E=A_E=B$. Hence $(\mathfrak{A}_E)_1\subset(\mathfrak{A}_1)_E$. ii) Let E be a projection in $\mathfrak{A}'$ and G its central carrier. Since $\mathfrak{A}_E$ and $\mathfrak{A}_G$ are isomorphic, for any B in $(\mathfrak{A}_E)_1$, there exists an element C of $(\mathfrak{A}_G)_1$ such that $C_E=B$. But since $G\in(\mathfrak{A}\cap\mathfrak{A}')^p\subset\mathfrak{A}^p$, by i) there is $A\in\mathfrak{A}_1$ with $A_G=C$. Hence $A_E=(A_G)_E=C_E=B$, which implies $(\mathfrak{A}_E)_1\subset(\mathfrak{A}_1)_E$.

THEOREM 4. (1) *For any $E\in\mathfrak{A}^p$ or $\mathfrak{A}'^p$, $(\mathfrak{A}_E)^e\subset(\mathfrak{A}^e)_E$;*

(2) *for any $E\in\mathfrak{A}'^p$, $(\mathfrak{A}^e)_E\subset(\mathfrak{A}_E)^e$. In particular, for any $E\in\mathfrak{A}'^p$, $(\mathfrak{A}^e)_E=(\mathfrak{A}_E)^e$, which may be denoted by $\mathfrak{A}^e_E$; and*

(3) *for any $E\in(\mathfrak{A}\cap\mathfrak{A}')^p$, $\mathfrak{A}^e_E\times\mathfrak{A}^e_{1-E}=\mathfrak{A}^e$.*

Proof. (1) For a projection E in $\mathfrak{A}$ or $\mathfrak{A}'$, let φ be a linear mapping $A\rightarrow A_E$ of $\mathfrak{A}$ onto $\mathfrak{A}_E$. Then φ is weakly continuous and $\varphi(\mathfrak{A}_1)=(\mathfrak{A}_E)_1$ by Lemma. Now, suppose $B\in(\mathfrak{A}_E)^e$ and put $\varphi^{-1}(B)=\{A\in\mathfrak{A}_1; \varphi(A)=B\}$. Then by the continuity of φ, $\varphi^{-1}(B)$ is a weakly closed and convex set in the weakly compact unit sphere $\mathfrak{A}_1$. Hence by the Krein-Milman's theorem there exists an extremal point A in $\varphi^{-1}(B)$ and this A is also an extremal point of $\mathfrak{A}_1$. Because, if A is not extremal in $\mathfrak{A}_1$, then different operators A_1 and A_2 in $\mathfrak{A}_1$ exist and $A=(A_1+A_2)/2$, while $(\varphi(A_1)+\varphi(A_2))/2=\varphi(A)=B$. Hence $\varphi(A_1)=\varphi(A_2)=B$ follows from the extremality of B and so $A_1, A_2\in\varphi^{-1}(B)$, which is impossible since A is extremal in $\varphi^{-1}(B)$. Consequently $B\in(\mathfrak{A}^e)_E$, thus $(\mathfrak{A}_E)^e\subset(\mathfrak{A}^e)_E$.

(2) If $A\in\mathfrak{A}^e$, then there is a central projection G such that $G\leqq A^*A$ and

$1-G \leqq AA^*$. Hence for every $E \in \mathfrak{A}'$, $GE \leqq EA^*AE = (AE)^*(AE)$ and $(1-G)E \leqq EAA^*E = (AE)(AE)^*$. Since G_E is a central projection of $\mathfrak{A}_E$, $A_E \in (\mathfrak{A}_E)^e$. Thus $(\mathfrak{A}^e)_E \subset (\mathfrak{A}_E)^e$.

(3) If $A \in \mathfrak{A}_E^e$ and $B \in \mathfrak{A}_{1-E}^e$, then there exist central projections F of $\mathfrak{A}_E$ and G of $\mathfrak{A}_{1-E}$ such that $F \leqq A^*A$, $1-F \leqq AA^*$, $G \leqq B^*B$ and $1-G \leqq BB^*$. Then $F \times G$ is also a central projection of $\mathfrak{A}$,

$$F \times G \leqq A^*A \times B^*B = (A \times B)^*(A \times B),$$

$$(1-F) \times (1-G) \leqq AA^* \times BB^* = (A \times B)(A \times B)^*$$

and

$$F \times G + (1-F) \times (1-G) = 1.$$

Therefore $A \times B \in \mathfrak{A}^e$. Thus $\mathfrak{A}_E^e \times \mathfrak{A}_{1-E}^e \subset \mathfrak{A}^e$. Conversely, if $A \in \mathfrak{A}^e$, then there is a central projection G such that $G \leqq A^*A$ and $1-G \leqq AA^*$. Hence for any central projection E of $\mathfrak{A}$

$$GE \leqq A^*AE = (AE)^*(AE), \qquad (1-G)E \leqq AA^*E = (AE)(AE)^*,$$

$$G(1-E) \leqq A^*A(1-E) = (A(1-E))^*(A(1-E))$$

and

$$(1-G)(1-E) \leqq AA^*(1-E) = (A(1-E))(A(1-E))^*.$$

Thus $A_E \in \mathfrak{A}_E^e$ and $A_{1-E} \in \mathfrak{A}_{1-E}^e$.

COROLLARY. *For any* $E \in \mathfrak{A}'^p$, $\mathfrak{A}^e \subset \mathfrak{A}_E^e \times \mathfrak{A}_{1-E}^e$. *The equality holds if and only if* $E \in \mathfrak{A} \cap \mathfrak{A}'$.

REMARK 4.1. In (2) of Theorem 4 the inclusion does not necessarily hold for $E \in \mathfrak{A}$. Because, in the most case, $A_E = 0$ or $A_E \notin (\mathfrak{A}_E)^{pi}$ even if $A \in \mathfrak{A}^{pi}$. Such a concrete example can be given as follows: let $\mathfrak{A} = \mathfrak{L}(\mathfrak{H})$ where $\mathfrak{H}$ is of two dimension. Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{A} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{A}^e.$$

Then $0 = A_E \in (\mathfrak{A}^e)_E$, but $0 \notin (\mathfrak{A}_E)^e$. Thus $(\mathfrak{A}_E)^e \subsetneqq (\mathfrak{A}^e)_E$.

From the last theorem, $\mathfrak{A}^e \times \mathfrak{B}^e = (\mathfrak{A} \times \mathfrak{B})^e$ follows immediately. In general, the fact that $\mathfrak{A}^e = \mathfrak{A}^i \cup \mathfrak{A}^{i*}$ whenever $\mathfrak{A}$ is a factor has been proved by Kadison [2]. And also, combining Theorem 1 and (3) of Theorem 4, in the case of general von Neumann algebra $\mathfrak{A}$, $\mathfrak{A}^e = \cup (\mathfrak{A}_G^i \cup \mathfrak{A}_G^{i*}) \times (\mathfrak{A}_{1-G}^i \cup \mathfrak{A}_{1-G}^{i*})$ where the union run through all the central projections G of $\mathfrak{A}$.[3] In particular, in the case where $\mathfrak{A}$ is properly

3) Miles has previously shown this fact [3].

infinite, the former relation will be sufficient for $\mathfrak{A}$ to be a factor.

THEOREM 5. *Let $\mathfrak{A}$ be properly infinite. Then the following conditions are equivalent*:

(1) *$\mathfrak{A}$ is a factor; and*

(2) $\mathfrak{A}^e=\mathfrak{A}^i\cup\mathfrak{A}^{i*}$.

Proof. (1) *implies* (2): Since $\mathfrak{A}^i\cup\mathfrak{A}^{i*}\subset\mathfrak{A}^e$, it suffices to show the converse inclusion. If $A\in\mathfrak{A}^e$, then there is a central projection G such that $G\leqq A^*A$ and $1-G\leqq AA^*$. Since $\mathfrak{A}$ is a factor, $G=0$ or $G=1$. Hence either $A^*A=1$ or $AA^*=1$. Thus $A\in\mathfrak{A}^i\cup\mathfrak{A}^{i*}$.

(2) *implies* (1): If $\mathfrak{A}$ is not a factor, then a central projection G of $\mathfrak{A}$ exists with $0<G<1$. Since $\mathfrak{A}$ is properly infinite, G and $1-G$ is not finite. Hence there exist A and B such that $A\in\mathfrak{A}_G^e$ with $AA^*\neq 1$ and $B\in\mathfrak{A}_{1-G}^e$ with $B^*B\neq 1$. Therefore by Theorem 4, $A\times B\in\mathfrak{A}_G^e\times\mathfrak{A}_{1-G}^e=\mathfrak{A}^e$, $(A\times B)^*(A\times B)=A^*A\times B^*B\neq 1$ and $(A\times B)(A\times B)^*=AA^*\times BB^*\neq 1$. Thus $A\times B\notin\mathfrak{A}^i\cup\mathfrak{A}^{i*}$. Consequently $\mathfrak{A}^i\cup\mathfrak{A}^{i*}\not\supset\mathfrak{A}^e$.

THEOREM 6. *The following conditions are equivalent*:

(1) *$\mathfrak{A}$ is finite;*

(2) $\mathfrak{A}^e\otimes\mathfrak{B}^e\subset(\mathfrak{A}\otimes\mathfrak{B})^e$ *for any $\mathfrak{B}$; and*

(3) $\mathfrak{A}^e\otimes\mathfrak{L}(\mathfrak{H})^e\subset(\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{H}))^e$ *for any $\mathfrak{H}$ of infinite dimension.*

Proof. (1) *implies* (2): If $A\in\mathfrak{A}^e$ and $B\in\mathfrak{B}^e$, then A is a unitary operator and there is a central projection G with $G\leqq B^*B$ and $1-G\leqq BB^*$. Hence $1\otimes G$ is also a central projection, $1\otimes G\leqq 1\otimes B^*B=A^*A\otimes B^*B=(A\otimes B)^*(A\otimes B)$, $1\otimes(1-G)\leqq 1\otimes BB^*=AA^*\otimes BB^*=(A\otimes B)(A\otimes B)^*$ and $1\otimes G+1\otimes(1-G)=1$. Thus $A\otimes B\in(\mathfrak{A}\otimes\mathfrak{B})^e$.

(2) *implies* (3) is obvious.

(3) *implies* (1): If $\mathfrak{A}$ is not finite and $\mathfrak{H}$ is infinite dimensional, then there exist A and B such that $A\in\mathfrak{A}^e$ with $AA^*\neq 1$ and $B\in\mathfrak{L}(\mathfrak{H})^e$ with $B^*B\neq 1$. Now define C^+ and C^- by $C^\pm=A\otimes B\pm(1-AA^*)\otimes(1-B^*B)$. Then

$$(C^\pm)^*C^\pm=A^*A\otimes B^*B+(1-AA^*)\otimes(1-B^*B)$$

and so $C^\pm\in(\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{H}))_1$ with $C^+\neq C^-$. But $A\otimes B=(C^++C^-)/2$, hence $A\otimes B\notin(\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{H}))^e$.

THEOREM 7. *The following conditions are equivalent*:

(1) $\mathfrak{A}=\boldsymbol{C}$;

(2) $\mathfrak{A}^e\otimes\mathfrak{B}^e=(\mathfrak{A}\otimes\mathfrak{B})^e$ *for any $\mathfrak{B}$; and*

(3) $\mathfrak{A}^e\otimes\mathfrak{L}(\mathfrak{H})^e=(\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{H}))^e$ *for two dimensional $\mathfrak{H}$.*

Proof. (1) *implies* (2): If $B\in(\boldsymbol{C}\otimes\mathfrak{B})^e$ such that $B=1\otimes B_1$ with $B_1\in\mathfrak{B}$, then there exists a central projection G of $\boldsymbol{C}\otimes\mathfrak{B}$ such that $G=1\otimes G_1$ with $G_1\in(\mathfrak{B}\cap\mathfrak{B}')^p$, $G\leqq B^*B$ and $1-G\leqq BB^*$. Since $0\leqq A$ whenever $0\leqq 1\otimes A$, there exists G_1 in $\mathfrak{B}\cap\mathfrak{B}'$ such that $G_1\leqq B_1^*B_1$ and $1-G\leqq B_1B_1^*$. Thus $B_1\in\mathfrak{B}^e$, i.e. $(\boldsymbol{C}\otimes\mathfrak{B})^e\subset\boldsymbol{C}^e\otimes\mathfrak{B}^e$. The converse inclusion is also similar.

(2) *implies* (3) is obvious.

(3) *implies* (1): If $A \in (\mathfrak{A} \otimes \mathfrak{L}(\mathfrak{H}))^e$, then it may be expressed in the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad A_{ij} \in \mathfrak{A} \quad (i, j = 1, 2).$$

However, since $\mathfrak{A}^e \otimes \mathfrak{L}(\mathfrak{H})^e = (\mathfrak{A} \otimes \mathfrak{L}(\mathfrak{H}))^e$, A must be expressed by a suitable operator $U \in \mathfrak{A}^e$ and a matrix $a = (a_{ij}) \in \mathfrak{L}(\mathfrak{H})^e$ in the tensor product

$$U \otimes a = \begin{pmatrix} a_{11}U & a_{12}U \\ a_{21}U & a_{22}U \end{pmatrix}.$$

Now, suppose that $\mathfrak{A} \neq \boldsymbol{C}$, then a projection E with $0 < E < 1$ exists in $\mathfrak{A}$. Define B by

$$B = \begin{pmatrix} E & 1-E \\ 1-E & E \end{pmatrix}.$$

Then $B^*B = BB^* = 1$ and so $B \in (\mathfrak{A} \otimes \mathfrak{L}(\mathfrak{H}))^e$, which is a contradiction to $E \notin \mathfrak{A}^e$.

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REFERENCES

[1] DIXMIER, J., Les algèbres d'opérateurs dans l'espace hilbertien. (1957).
[2] KADISON, R. V., Isometries of operator algebras. Ann. Math. **54** (1951), 325–338.
[3] MILES, P., B^*-algebra unit ball extremal points. Pacific J. Math. **14** (1964), 627–637.
[4] SAKAI, S., The theory of W^*-algebras. Lecture note, Yale Univ. (1962).

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