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ON A SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH A TURNING POINT

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§ 1. Introduction.

1° The system of differential equations to be discussed here is

$$\varepsilon^{\sigma}\frac{dY}{dx}=A(x,\varepsilon)Y, \tag{1. 1}$$

in which σ is a positive integer, ε is a small complex parameter and $A(x,\varepsilon)$ is an n-by-n matrix function holomorphic in both variables in the domain $\mathfrak{D}$ defined by the inequalities

$$\mathfrak{D}:\quad |x|\leqq x_0,\qquad 0<|\varepsilon|\leqq\varepsilon_0,\qquad |\arg\varepsilon|\leqq\theta_0. \tag{1. 2}$$

We assume that the matrix $A(x,\varepsilon)$ is expressed by the asymptotic expansion such that

$$A(x,\varepsilon)\sim\sum_{r=0}^{\infty}A_r(x)\varepsilon^r,\qquad \varepsilon\to 0 \tag{1. 3}$$

is uniformly valid in $|x|\leqq x_0$ and $|\arg\varepsilon|\leqq\theta_0$. The coefficients $A_r(x)$ are then necessarily holomorphic in $|x|\leqq x_0$ (Wasow [12]).

Sibuya [8] has proved that the local asymptotic analysis, as $\varepsilon\to 0$, of such differential equations can be reduced to the study of the special case that $A(x,\varepsilon)$ satisfies the following hypothesis:

$A_0(0)$ is nilpotent and of a Jordan canonical form.

It will therefore be assumed, from now on, that $A_0(0)$ has this special property.

In this paper we investigate the case that $A_0(0)$ has n Jordan blocks. As $A_0(0)$ is nilpotent, which means that $A_0(0)=0$, the leading matrix $A_0(x)$ in (1. 3) must be of the form

$$A_0(x)=x^pG(x), \tag{1. 4}$$

with p a positive integer, $G(x)$ holomorphic at $x=0$, and $G(0)\neq 0$.

ASSUMPTION 1.

$$A_0(x)=x^pG(x),$$

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where p is a positive integer, $G(x)$ is holomorphic in $|x|\leqq x_0$, and all eigenvalues of $G(x)$ are distinct. That is to say, the origin is a turning point of order p.

In this case, without loss of generality (Sibuya [7], Wasow [12]) it can be assumed that $G(x)$ *has the diagonal form*:

$$G(x)=\begin{bmatrix} g_1(x) & & & \mathbf{0} \\ & g_2(x) & & \\ & & \ddots & \\ \mathbf{0} & & & g_n(x) \end{bmatrix} \tag{1.5}$$

with

$$g_j(x)\neq g_k(x) \qquad \text{in} \quad |x|\leqq x_0, \text{ for } j\neq k, \tag{1.6}$$

$$g_j(x)=\sum_{h=0}^{\infty} g_{jh}x^h \qquad \text{in} \quad |x|\leqq x_0. \tag{1.7}$$

Furthermore, we can assume that $A(x, \varepsilon)$ *is of a triangular form* (Iwano [4], Sibuya [8]):

$$\begin{aligned} A(x,\varepsilon)=[a_{jk}(x,\varepsilon)]\sim x^p &\begin{bmatrix} g_1(x) & & & \mathbf{0} \\ & g_2(x) & & \\ & & \ddots & \\ \mathbf{0} & & & g_n(x) \end{bmatrix} \\ &+\sum_{r=1}^{\infty}\begin{bmatrix} a_{11}^{(r)}(x) & & & & \mathbf{0} \\ a_{21}^{(r)}(x) & a_{22}^{(r)}(x) & & & \\ \cdots\cdots & & & & \\ a_{j1}^{(r)}(x) & a_{j2}^{(r)}(x)\cdots a_{jk}^{(r)}(x)\cdots a_{jj}^{(r)}(x) & & & \\ \cdots\cdots\cdots & & & & \\ a_{n1}^{(r)}(x) & a_{n2}^{(r)}(x)\cdots\cdots & & & a_{nn}^{(r)}(x) \end{bmatrix}\varepsilon^r, \end{aligned} \tag{1.8}$$

that is to say, $a_{jk}(x, \varepsilon)\sim 0$, $\varepsilon\to 0$ in $\mathfrak{D}$ for $j<k$, and

$$a_{jk}(x,\varepsilon)\sim \sum_{r=0}^{\infty} a_{jk}^{(r)}(x)\varepsilon^r, \qquad \varepsilon\to 0 \quad \text{in } \mathfrak{D} \text{ for } j\geqq k, \tag{1.9}$$

where

$$a_{jk}^{(r)}(x)=\sum_{h=m_{jk}^{(r)}}^{\infty} a_{jkh}^{(r)}x^h \qquad \text{in} \quad |x|\leqq x_0, \quad a_{jkm_{jk}^{(r)}}^{(r)}\neq 0, \tag{1.10}$$

in particular for $j=1, 2, \cdots, n$

$$a_{jj}(x,\varepsilon)\sim x^p g_j(x)+\sum_{r=1}^{\infty} a_{jj}^{(r)}(x)\varepsilon^r, \qquad \varepsilon\to 0 \quad \text{in } \mathfrak{D}. \tag{1.11}$$

2° For the simplicity we construct a characteristic polygon for the differential equation (1. 1) and we investigate the restricted case that it consists of one segment (Iwano [4]).

The characteristic polygon, convex downward, can be constructed by joining the following points in the plane with a rectangular coordinate system (X, Y):

$$P_{jk}^{(r)}=\left(\frac{r}{j+1-k}, \frac{m_{jk}^{(r)}}{j+1-k}\right) \qquad (j\geqq k;\ r=0, 1, 2, \cdots),$$

$$R=(\sigma, -1).$$

In particular $P_{jj}^{(0)}=(0, p)$, and $P_{jk}^{(0)}=(0, \infty)$ for $j>k$ because $m_{jk}^{(0)}$ is, by definition, infinte for $a_{jk}^{(0)}(x)\equiv 0$.

For our requirement the coefficient of (1. 1) has to fulfill the following

Assumption 2.

$$m_{jj}^{(0)}=p\geqq 1 \qquad (j=1, 2, \cdots, n),$$

$$\frac{m_{jk}^{(r)}}{j+1-k}>p-\frac{p+1}{\sigma}\frac{r}{j+1-k} \qquad (j\geqq k;\ r=1, 2, \cdots).$$

3° Under these hypotheses, we can obtain two types of asymptotic representations of a fundamental solution of the differential equation (1. 1) whose domains of validity overlap each other in neighborhoods of the turning point for ε arbitrary small.

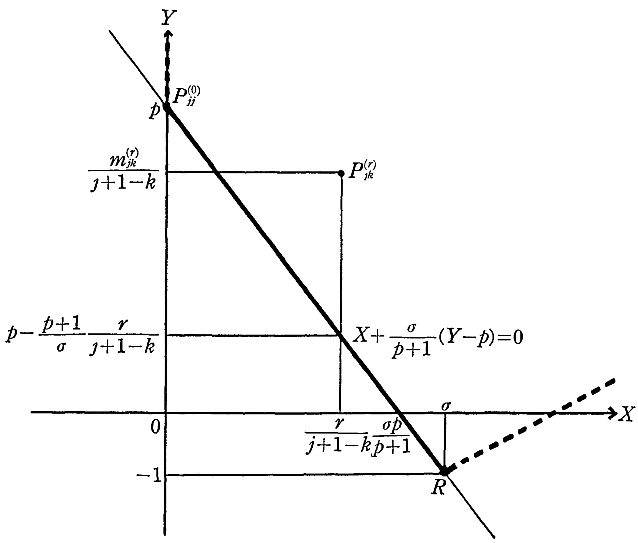

Fig. 1. Characteristic polygon

In 1966 Wasow [13] analyzed the simpler case of $p=1$ and $n=2$, in which the restriction $p=1$ seems to be essential.

§ 2. A formal outer solution.

In this section a formal solution for $x\neq 0$ will be obtained.

4° The linear transformation

$$Y=K(x)V, \tag{2.1}$$

where

$$K(x)=\begin{bmatrix} 1 & & & & 0 \\ & x^{q/n} & & & \\ & & x^{2q/n} & & \\ & & & \ddots & \\ 0 & & & & x^{(n-1)q/n} \end{bmatrix} \tag{2.2}$$

together with q some positive integer less than np, changes the equation (1. 1) into

$$\varepsilon^{\sigma}\frac{dV}{dx}=B^{*}(x,\varepsilon)V, \tag{2.3}$$

where

$$B^{*}(x,\varepsilon)=K^{-1}(x)A(x,\varepsilon)K(x)-\varepsilon^{\sigma}K^{-1}(x)\frac{dK(x)}{dx}. \tag{2.4}$$

In view of ASSUMPTION 2, the equation (2. 3) can be rewritten in

$$[x^{-1/a}\varepsilon]^{\sigma}x\frac{dV}{dx}=B(x,\varepsilon)V, \tag{2.5}$$

where $a=\sigma/(p+1)$, and

$$B(x,\varepsilon)=\begin{bmatrix} x^{1-\sigma/a}a_{11}(x,\varepsilon) & & & & 0 \\ x^{1-q/n-\sigma/a}a_{21}(x,\varepsilon) & x^{1-\sigma/a}a_{22}(x,\varepsilon) & & & \\ \cdots\cdots\cdots & & & & \\ x^{1-(j-1)q/n-\sigma/a}a_{j1}(x,\varepsilon) & \cdots\; x^{1-(j-k)q/n-\sigma/a}a_{jk}(x,\varepsilon) & \cdots\; x^{1-\sigma/a}a_{jj}(x,\varepsilon) & & \\ \cdots\cdots\cdots\cdots\cdots\cdots & & & & \\ x^{1-(n-1)q/n-\sigma/a}a_{n1}(x,\varepsilon) & \cdots\cdots\cdots & & & x^{1-\sigma/a}a_{nn}(x,\varepsilon) \end{bmatrix}$$

$$-\frac{q}{n}[x^{-1/a}\varepsilon]^{\sigma}\begin{bmatrix} 0 & & & & 0 \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ 0 & & & & n-1 \end{bmatrix}\sim\sum_{r=0}^{\infty}B_r(x)[x^{-1/a}\varepsilon]^{r}. \tag{2.6}$$

The coefficients $B_r(x)$ are holomorphic in $x^{1/an}$, and in particular

(2. 7) $$B_0(x)=G(x)=\operatorname{diag}[g_1(x), g_2(x), \cdots, g_n(x)].$$

In fact, for the diagonal elements of (2. 6) hold the equalities

$$\begin{aligned} x^{1-\sigma/a}a_{jj}(x, \varepsilon) &= x^{p+1-\sigma/a}g_j(x)+\sum_{r=1}^{\infty}\sum_{h=m_{jj}^{(r)}}^{\infty} a_{jjh}^{(r)}[x^{-1/a}\varepsilon]^r x^{h+1-\sigma/a+r/a} \\ &= g_j(x)+\sum_{r=1}^{\infty}\sum_{h=m_{jj}^{(r)}}^{\infty} a_{jjh}^{(r)}[x^{-1/a}\varepsilon]^r x^{h+1-\sigma/a+r/a} \end{aligned}$$

for $j=1, 2, \cdots, n$, because the value $h+1-\sigma/a+r/a$ vanishes for $r=0$, and it is not less than positive values $m_{jj}^{(r)}-p+r/a$ for $r\geqq 1$ by Assumption 2.

For the non-diagonal elements we have

$$x^{1-(j-k)q/n-\sigma/a}a_{jk}(x, \varepsilon)=\sum_{r=1}^{\infty}\sum_{h=m_{jk}^{(r)}}^{\infty}[x^{-1/a}\varepsilon]^r x^{h+1-(j-k)q/n-\sigma/a+r/a}a_{jkh}^{(r)},$$

where $h+1-(j-k)q/n-\sigma/a+r/a$ is not less than $(j+1-k)\{m_{jk}^{(r)}/(j+1-k)-p+r/a(j+1-k)\}+(j-k)(p-q/n)$, which is positive by Assumption 2 and the definition of q.

The relation (2. 6) means that

$$B(x, \varepsilon)-\sum_{r=0}^{m}B_r(x)[x^{-1/a}\varepsilon]^r=E_m(x, \varepsilon)[x^{-1/a}\varepsilon]^{m+1},$$

where $E_m(x, \varepsilon)$ is bounded in the domain $\mathfrak{D}$.

5° The transformation $x^{1/an}=t$ takes the equation (2. 5) into

(2. 8) $$[t^{-n}\varepsilon]^{\sigma}t\frac{dV}{dt}=C^*(t, \varepsilon)V,$$

where

$$C^*(t, \varepsilon)=[anB(x, \varepsilon)]_{x=t^{an}}=\sum_{r=0}^{\infty}C_r^*(t)[t^{-n}\varepsilon]^r,$$

and in particular

$$C_0^*(t)=an\begin{bmatrix} g_1(t^{an}) & & & 0 \\ & g_2(t^{an}) & & \\ & & \ddots & \\ 0 & & & g_n(t^{an}) \end{bmatrix}.$$

Then we can diagonalize the coefficient $C^*(t, \varepsilon)$ by the method introduced by Turrittin [10] as follows.

Let $T_k(t, \varepsilon)=I+[t^{-n}\varepsilon]^kQ_k(t)$, where I is the n-dimentional identity matrix and

$Q_k(t)$ are holomorphic matrix functions which are appropriately determined successively as follows.

The transformation $V=T_kZ^*$ changes (2. 8) into

$$[t^{-n}\varepsilon]^{\sigma}t\frac{dZ^*}{dt}=C^{(k)}(t,\varepsilon)Z^*,$$

where

$$\begin{aligned}C^{(k)}(t,\varepsilon)&=T_k^{-1}C^*T_k-[t^{-n}\varepsilon]^{\sigma+k}t\frac{dQ_k}{dt}\\&=(C_0^*+[t^{-n}\varepsilon]C_1^*+\cdots+[t^{-n}\varepsilon]^{k-1}C_{k-1}^*)\\&\quad+[t^{-n}\varepsilon]^k(C_k^*+C_0^*Q_k-Q_kC_0^*)+\cdots.\end{aligned}$$

Since the eigenvalues $ang_1(t^{an}),\cdots,ang_n(t^{an})$ of $C_0^*(t)$ are distinct, $C_k^*+C_0^*Q_k-Q_kC_0^*$ can be diagonalized by determining elements of Q_k.

Let $P^*(t,\varepsilon)=\Pi_{k=1}^{\infty}T_k=I+\sum_{r=1}^{\infty}P_r^*(t)[t^{-n}\varepsilon]^r$ be a formal series obtained by products of T_k $(k=1,2,\cdots)$. Then, by virtue of the theorem of Borel-Ritt (Friedrichs [1] or Wasow [12]), there exists a holomorphic function $P(t,\varepsilon)$ having $P^*(t,\varepsilon)$ as its asymptotic expansion in the domain $\mathfrak{D}'$ defined by inequalities

(2. 9) $\mathfrak{D}'$: $|t|\leqq t_0,\quad |\arg t|\leqq t_1,\quad 0<|\varepsilon|\leqq\varepsilon_0,\quad |\arg\varepsilon|\leqq\theta_0,\quad 0<|t^{-n}\varepsilon|\leqq t_2,$

where t_0, which may be equal to $x_0^{1/an}$, and t_2 are sufficiently small constants, but t_1 is arbitrary.

By the transformation

$$V=PZ,$$

we get

$$[t^{-n}\varepsilon]^{\sigma}t\frac{dZ}{dt}=C(t,\varepsilon)Z, \tag{2. 10}$$

where $C(t,\varepsilon)$ is holomorphic in $\mathfrak{D}'$ and expanded asymptotically, as $t^{-n}\varepsilon$ tends to zero, in $\sum_{r=0}^{\infty}C_r(t)[t^{-n}\varepsilon]^r$, and $C_r(t)$ is diagonal and holomorpnic in t in $\mathfrak{D}'$, in particular

$$C_0(t)=an\begin{bmatrix}g_1(t^{an}) & & & 0\\ & g_2(t^{an}) & & \\ & & \ddots & \\ 0 & & & g_n(t^{an})\end{bmatrix}.$$

The result obtained above is summarized in

LEMMA 2. 1. *Assume that the condition* (1. 6) *is satisfied in the matrix* (2. 7) *and put* $x^{1/an}=t$. *Then the differential equation* (2. 5) *with a matrix coefficient, in general, not diagonal or not even asymptotically diagonal, can be reduced to the* *differential equation of the form* (2. 10) *with a coefficient holomorphic in* $\mathfrak{D}'$ *and asymptotically diagonal.*

Thus we can calculate a formal series solution of the differential equation (2. 10) and get the following theorem.

THEOREM 2. 1 [Formal outer solution]. *Assume that the differential equation* (1. 1) *satisfies* ASSUMPTIONS 1 *and* 2, *then for* $x \neq 0$ *it can be reduced to* (2. 10), *which possesses a formal sereis solution*

$$Z(t, \varepsilon) \sim \left[\sum_{r=0}^{\infty} Z_r(t)\varepsilon^r\right] \exp\left[\sum_{r=0}^{\sigma} F_r(t)\varepsilon^{r-\sigma}\right] \tag{2. 11}$$

with properties:

a) *the relation*

$$Z_r(t) = t^{-nr} Z_r^*(t)$$

holds, where $Z_r^*(t)$ *is a polynomial of degree* r, *at most, in* $\log t$ *with coefficients holomorphic in* $|t| \leqq t_0$, *and bounded in the damain* $\mathfrak{D}'$;

b) $F_r(t)$ *is diagonal and of the form*

$$\begin{cases} F_\sigma(t) = f_\sigma \log t + F_\sigma^*(t), \\ F_r(t) = t^{n(\sigma-r)} F_r^*(t) \qquad (0 \leqq r \leqq \sigma - 1), \end{cases} \tag{2. 12}$$

where $F_r^*(t)$ $(r=0, 1, 2, \cdots, \sigma)$ *are holomorphic in* $|t| \leqq t_0$ *and* f_σ *is a constant matrix.*

Proof. Since the coefficient of (2. 10) is diagonal, we obtain

$$Z(t, \varepsilon) = \exp\left[\sum_{r=0}^{\infty} \int^t \varepsilon^{r-\sigma} C_r(t) t^{n(\sigma-r)-1} dt\right].$$

Define the matrix functions $C_r(t)$ and $F_r(t)$ by

$$C_r(t) = \sum_{k=0}^{\infty} C_{rk} t^k \qquad \text{and} \qquad F_r(t) = \int^t C_r(t) t^{n(\sigma-1)-1} dt.$$

Then, for $r \geqq \sigma$ we have

$$\begin{aligned} F_r(t) &= \sum_{k=0}^{\infty} \int^t C_{rk} t^{k-n(r-\sigma)-1} dt \\ &= C_{r, n(r-\sigma)} \log t + \sum_{k \neq n(r-\sigma)} \frac{C_{rk}}{k - n(r-\sigma)} t^{k-n(r-\sigma)}, \end{aligned}$$

from which by changing notations we have

$$F_r(t) = f_r \log t + t^{-n(r-\sigma)} F_r^*(t), \tag{2. 13}$$

where f_r and $F_r^*(t)$ are holomorphic in $|t| \leqq t_0$. In particular,

$$F_\sigma(t)=C_{\sigma,0}\log t+\sum_{k=1}^{\infty}\frac{C_{\sigma k}}{k}t^k$$

can be written as (2. 12).

For $0\leqq r\leqq\sigma$, we have

$$F_r(t)=\sum_{k=0}^{\infty}C_{rk}\frac{t^{k+n(\sigma-r)}}{k+n(\sigma-r)}=t^{n(\sigma-r)}F_r^*(t),$$

where $F_r^*(t)$ are holomorphic in $|t|\leqq t_0$.

As for $Z_r(t)$, it follows from

$$Z(t,\varepsilon)=\exp\left[\sum_{r=\sigma+1}^{\infty}F_r(t)\varepsilon^{r-\sigma}\right]\exp\left[\sum_{r=0}^{\sigma}F_r(t)\varepsilon^{r-\sigma}\right].$$

In fact, in view of (2. 13) we have

$$\exp\left[\sum_{r=\sigma+1}^{\infty}F_r(t)\varepsilon^{r-\sigma}\right]=I+t^{-n}Z_1^*(t)\varepsilon+t^{-2n}Z_2^*(t)\varepsilon^2+\cdots,$$

where $Z_r^*(t)$ is a polynomial in $\log t$ of degree r at most. Q.E.D.

§ 3. A formal inner solution.

This section is devoted to a formal solution in neighborhoods of $x=0$.

6° The transformations

(3. 1) $$\begin{cases} x=\varepsilon^a t & \text{(stretching transformation)},\\ Y=K(\varepsilon^a)U, \end{cases}$$

where $a=\sigma/(p+1)$ and $K(\varepsilon^a)$ is defined by (2. 2) with ε^a instead of x, take the original equation (1. 1) into

(3. 2) $$\frac{dU}{dt}=D^*(x,\varepsilon)U,$$

where

(3. 3) $$D^*(x,\varepsilon)=\varepsilon^{a-\sigma}K^{-1}(\varepsilon^a)A(x,\varepsilon)K(\varepsilon^a)$$

$$=\begin{bmatrix} a_{11}(x,\varepsilon)\varepsilon^{a-\sigma} & & & & \\ a_{21}(x,\varepsilon)\varepsilon^{-aq/n+a-\sigma} & a_{22}(x,\varepsilon)\varepsilon^{a-\sigma} & & \mathbf{0} & \\ \cdots & \cdots & & & \\ a_{j1}(x,\varepsilon)\varepsilon^{-(j-1)aq/n+a-\sigma} & \cdots & a_{jk}(x,\varepsilon)\varepsilon^{-(j-k)aq/n+a-\sigma} & \cdots & a_{jj}(x,\varepsilon)\varepsilon^{a-\sigma} \\ \cdots & \cdots & & \cdots & \\ a_{n1}(x,\varepsilon)\varepsilon^{-(n-1)aq/n+a-\sigma} & \cdots & \cdots & \cdots & a_{nn}(x,\varepsilon)\varepsilon^{a-\sigma} \end{bmatrix}.$$

Let $\varepsilon^{1/n(p+1)}=\rho$. Then the equation (3. 2) can be rewritten in the form

$$\frac{dU}{dt}=D(t,\rho)U, \tag{3. 4}$$

where

$$D(t,\rho)\sim\sum_{r=0}^{\infty}D_r(t)\rho^r. \tag{3. 5}$$

The coefficient $D(t,\rho)$ of (3. 4) is clearly of the same triangular type as $A(x,\varepsilon)$,

$$D_0(t)=t^pG=t^p\begin{bmatrix} g_{10} & & & 0 \\ & g_{20} & & \\ & & \ddots & \\ 0 & & & g_{n0}\end{bmatrix}, \tag{3. 6}$$

and for $r\geqq 1$ $D_r(t)$ is a polynomial in t of degree, at most,

$$m_r=r/n\sigma+(n-1)q/n+p-(p+1)/\sigma, \tag{3. 7}$$

hence $D_r(t)$ is of the form

$$D_r(t)=t^{r/n\sigma+(n-1)q/n+p-(p+1)/\sigma}D_r^*(t)$$

with $D_r^*(t)$ bounded at $t=\infty$.

Therefore (3. 5) is written in

$$D(t,\rho)-\sum_{r=0}^{m}D_r(t)\rho^r=t^{(n-1)q/n+p-(p+1)/\sigma}E_{m+1}(t,\rho)[t^{1/n\sigma}\rho]^{m+1} \tag{3. 8}$$

with E_{m+1} (t,ρ) bounded in the domain under consideration for $m=0,1,2,\cdots$, $|t|\geqq t_0'>0$.

The relations (3. 5), (3. 6) can be proved as follows.

For the main diagonal elements of (3. 3), namely, for

$$a_{jj}(x,\varepsilon)\varepsilon^{a-\sigma}=\sum_{h=0}^{\infty}g_{jh}t^{h+p}\varepsilon^{ah+ap+a-\sigma}+\sum_{r=1}^{\infty}\sum_{h=m_{jj}^{(r)}}^{\infty}a_{jjh}^{(r)}t^h\varepsilon^{r+a-\sigma+ah},\qquad (j=1,2,3,\cdots,n),$$

we can show by ASSUMPTION 2 that $ah+ap+a-\sigma$ vanishes for $h=0$, and it takes positive values for $h\geqq 1$, and $r+a-\sigma+ah$ takes positive values for $r\geqq 1$ and $h\geqq m_{jj}^{(r)}$. This means that $a_{jj}(x,\varepsilon)\varepsilon^{a-\sigma}$ contains none of the terms of negative powers of ε, and that the constant term with respect to ε of $a_{jj}(x,\varepsilon)\varepsilon^{a-\sigma}$ is only $g_{j0}t^p$, which proves (3. 6).

Similarly for the off-diagonal elements

$$a_{jk}(x,\varepsilon)\varepsilon^{a-\sigma-(j-k)aq/n}=\sum_{r=1}^{\infty}\sum_{h=m_{jk}^{(r)}}^{\infty}a_{jkh}^{(r)}t^h\varepsilon^{a-\sigma-(j-k)aq/n+ah+r}\qquad (j>k),$$

we can see easily that the exponent of ε takes positive values only by ASSUMPTION 2 and the definition of q (see (2. 2)).

Next, the validity of the equality (3. 7) will be shown.

Since $\rho=\varepsilon^{1/n(p+1)}$, every element of (3. 3) is rewritten in

$$a_{jk}(x, \varepsilon)\varepsilon^{a-\sigma-(j-k)aq/n}=\sum_{r=1}^{\infty}\sum_{h=m_{jk}^{(r)}}^{\infty} a_{jkh}^{(r)}t^h\rho^{\sigma[nh-np-(j-k)q]+n(p+1)r}$$

$$=\sum_{s=1}^{\infty}\left(\sum_{l=\min\limits_{r>0} h}^{\max\limits_{r>0} h} d_{jk}^{(l)}t^l\right)\rho^s=\sum_{s=1}^{\infty} D_s(t)_{jk}\rho^s.$$

The possibly highest degree of $D_s(t)_{jk}$ is the maximum of h with respect to $r\geqq 1$. Hence, the possibly highest degree of $D_s(t)$ is the maximum of h with respect to r, j and k:

$$m_s=\underset{r>0,n\geqq j\geqq k\geqq 1}{\text{maximum}}\, h=[s/n\sigma-r/a+(j-k)q/n+p]_{r=1,\,j-k=n-1}$$

$$=s/n\sigma-1/a+(n-1)q/n+p,$$

which proves (3. 7).

7° In order to obtain a solution of the equation (3. 4) together with (3. 5), we will construct the recursion formulas of integral equations.

Let

$$U\sim\sum_{r=0}^{\infty} U_r(t)\rho^r$$

be a formal solution of (3. 4). The recursive conditions for this to be the case are

$$\frac{dU_0}{dt}=D_0(t)U_0, \tag{3. 9a}$$

$$\frac{dU_r}{dt}=D_0(t)U_r+\sum_{b=1}^{r} D_b(t)U_{r-b} \qquad (r\geqq 1). \tag{3. 9b}$$

Since the coefficient $D_0(t)=t^pG$ is diagonal, the solution of (3. 9a) is

$$U_0(t)=\exp\left[\frac{t^{p+1}}{p+1}G\right]=\exp[G^*(t)].$$

From the variation of constants formula, solutions of the equation (3. 9b) are

$$\begin{aligned} U_r(t)&=U_0(t)\int_{\Gamma_r(t)} U_0^{-1}(s)\sum_{b=1}^{r} D_b(s)U_{r-b}(s)ds \\ &=\int_{\Gamma_r(t)} e^{G^*(t)-G^*(s)}\sum_{b=1}^{r} D_b(s)U_{r-b}(s)ds, \end{aligned} \tag{3. 10}$$

where $\gamma_r(t)$ denotes the n^2 paths of integration terminating at t.

8° It will be shown that there exist solutions of the integral equations (3.10) of the form

$$U_r(t)=t^{rm^*}U_r^*(t)\exp[G^*(t)], \tag{3.11}$$

where $m^*=m_1+1$ (see (3.7)) and $U_r^*(t)$ is bounded in some domain $\mathfrak{S}(t_0')$ defined by the inequalities:

$$\mathfrak{S}(t_0'):\quad |t|>t_0',\qquad \alpha'\leqq\arg t\leqq\alpha'+\frac{\pi}{p+1}.$$

In particular, since $U_0(t)=I\exp[G^*(t)]$,

$$U_0^*(t)=I.$$

Insertion of $U_r(t)=Y_r(t)\cdot\exp[G^*(t)]$ into (3.10) implies

$$Y_r(t)=\int_{\gamma_r(t)}e^{G^*(t)-G^*(s)}\sum_{b=1}^{r}D_b(s)Y_{r-b}(s)e^{G^*(s)-G^*(t)}ds. \tag{3.12}$$

The j,k-element $Y_r(t)_{jk}$ of $Y_r(t)$ is, if denoting $\sum_{b=1}^{r}D_bY_{r-b}=F^{(r)}=[f_{jk}^{(r)}]$, of the form

$$\begin{aligned} Y_r(t)_{jk}&=\int_{\gamma_r(t)_{jk}}\exp[(g_{j0}-g_{k0})(t^{p+1}-s^{p+1})/(p+1)]f_{jk}^{(r)}(s)ds \\ &=\int_{\gamma_r^*(t^*)_{jk}}s^{*-p/(p+1)}\exp[(g_{j0}-g_{k0})(t^*-s^*)]f_{jk}^{*(r)}(s^*)ds^*, \end{aligned} \tag{3.13}$$

where asterisked letters are defined by

$$t^*=\frac{t^{p+1}}{p+1},\qquad s^*=\frac{s^{p+1}}{p+1}. \tag{3.14}$$

The arguments of the leading terms g_{j0} of $g_j(x)$ may be assumed to take their values in

$$-\pi<\arg g_{j0}\leqq\pi\qquad (j=1,2,\cdots,n).$$

The paths of integration $\gamma_r^*(t^*)_{jk}$ are defined to be straight lines from t^* to infinity parallel to the straight lines passing through the origin

$$\mathrm{Re}[-(g_{j0}-g_{k0})s^*]<0\qquad (j\neq k) \tag{3.15}$$

in the domain

$$\mathfrak{S}^*:\quad -\alpha\leqq\arg s^*\leqq-\alpha+\pi\qquad (0\leqq\alpha\leqq\pi/2), \tag{3.16}$$

where α should be chosen so that none of lines $\mathrm{Re}[-(g_{j0}-g_{k0})s^*]=0$ is on the boundaries of $\mathfrak{S}^*$.

Therefore $\gamma_r^*(t^*)_{jk}$ is of the form

$$\gamma_r^*(t^*)_{jk}:\quad s^*=t^*+\delta_{jk}r \qquad (r\geqq 0,\ |\delta_{jk}|=1).$$

The paths $\gamma_r^*(t^*)_{jj}$ are segments from t^* to arbitrary bounded points t_j^* in $\mathfrak{S}^*$:

$$\gamma_r^*(t^*)_{jj}:\quad s^*=t^*+\delta_{jj}r \qquad (0\leqq r\leqq r(t^*)<\infty,\ |\delta_{jj}|=1).$$

The original paths $\gamma_r(t)_{jk}$ are the inverse images of $\gamma_r^*(t^*)_{jk}$ under the transformation (3. 14).

The domain $\mathfrak{S}$ may be taken to be the inverse image of $\mathfrak{S}^*$ under (3. 14):

(3. 17) $$\mathfrak{S}:\quad -\alpha/(p+1)\leqq \arg s\leqq(-\alpha+\pi)/(p+1),$$

and the domain $\mathfrak{S}(t_0')$ is a subset of $\mathfrak{S}$ for $|t|>t_0'$.

Under the above conditions the next lemma is valid.

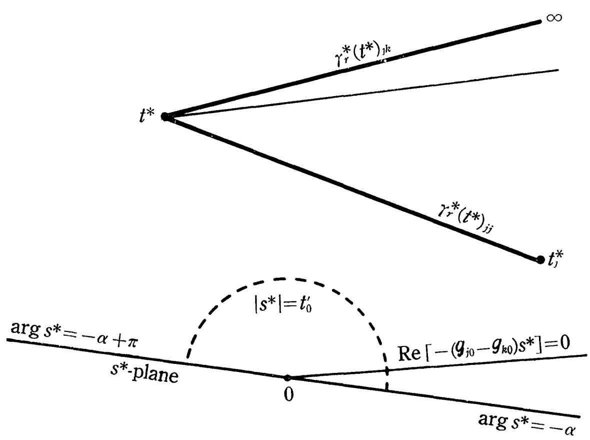

FIG. 2. Paths of integration

LEMMA 3. 1. *If the matrix function* $F^{(r)}(t)t^{-c}$ $(c>0)$ *is bounded as* t *tends to infinity in* $\mathfrak{S}$, *then* $Y_r(t)t^{-(c+1)}$ *is bounded as* t *tends to infinity in* $\mathfrak{S}$. *In other words,*

(3. 18) $$F^{(r)}(t)=O(t^c) \text{ in } \mathfrak{S}(t_0') \text{ implies } Y_r(t)=O(t^{c+1}) \text{ in } \mathfrak{S}(t_0').$$

Proof. Let

$$F^{(r)}(s)=O(s^c) \quad \text{in } \mathfrak{S}(t_0')$$
$$=O(s^{*c/(p+1)}) \quad \text{in } \mathfrak{S}^*(t_0'^*).$$

For off-diagonal elements, i.e., for $j>k$,

$$Y_r(t)_{jk}=\int_{\tau_r^*(t^*)_{jk}} s^{*-p/(p+1)} \exp\left[(g_{j0}-g_{k0})(t^*-s^*)\right] f_{jk}^{*(r)}(s^*)ds^*$$

$$=t^{*(c-p)/(p+1)}\int_0^{\infty}(1+\delta_{jk}r/t^*)^{(c-p)/(p+1)} \exp\left[-(g_{j0}-g_{k0})\delta_{jk}r\right]$$

$$\times\left[f_{jk}^{*(r)}(s^*)s^{*-c/(p+1)}\right]\delta_{jk}dr$$

$$=O(t^{*(c-p)/(p+1)}) \quad \text{in } \mathfrak{S}^*(t_0'^*)$$

$$=O(t^{c-p}) \quad \text{in } \mathfrak{S}(t_0'),$$

and for diagonal elements

$$Y_r(t)_{jj}=\int_{\tau_r^*(t^*)_{jj}} s^{*-p/(p+1)} f_{jj}^{*(r)}(s^*)ds^*$$

$$=\int_0^r (t^*+\delta_{jj}r)^{(c-p)/(p+1)}\left[f_{jj}^{*(r)}(s^*)s^{*-c/(p+1)}\right]\delta_{jj}dr$$

$$=O(t^{*(c-p)/(p+1)+1}) \quad \text{in } \mathfrak{S}^*(t_0'^*)$$

$$=O(t^{c+1}) \quad \text{in } \mathfrak{S}(t_0'),$$

by which the validity of (3. 18) has been shown. Q.E.D.

Now, it will be shown that

(3. 19) $$Y_r(t)=O(t^{rm^*}) \quad \textit{in } \mathfrak{S}(t_0'),$$

namely,

$$Y_r(t)=t^{rm^*}U_r^*(t),$$

where $U_r^*(t)$ *is bounded in* $\mathfrak{S}(t_0')$ *and* $m^*=m_1+1$.

Indeed, (3. 18) shows that $F^{(1)}(s)=D_1(s)Y_0(s)=D_1(s)=O(s^{m_1})$ in $\mathfrak{S}(t_0')$ implies $Y_1(t)=O(t^{m^*})$ in $\mathfrak{S}(t_0')$, and the rest of the statement can be proved by induction.

Suppose $Y_j(t)=O(t^{jm^*})$, $j\leqq r-1$, are known. Then

$$F^{(r)}(s)=\sum_{b=1}^{r} D_b(s)Y_{r-b}(s)$$

(3. 20) $$=D_1Y_{r-1}+D_2Y_{r-2}+\cdots+D_rY_0$$

$$=O(s^{m_1+(r-1)m^*})+O(s^{m_2+(r-2)m^*})+\cdots+O(s^{m_j+(r-j)m^*})+\cdots+O(s^{m_r}).$$

Since, however, the value of the difference $[m_j+(r-j)m^*]-[m_{j+1}+(r-j-1)m^*]$ is $(n-1)q/n+p+1-(p+1)/\sigma$, which is positive, the maximum of $m_j+(r-j)m^*$ $(1\leqq j\leqq r)$ can be attained for $j=1$, and its value is $m_1+(r-1)m^*$. This fact is followed by $F^{(r)}(s)=O(s^{m_1+(r-1)m^*})$. By applying (3. 18) once more we have $Y_r(t)=O(t^{rm^*})$, which shows the validity of (3. 19), and hence (3. 11) has been proved.

The above results are summarized in the following

LEMMA 3. 2. *The integral equation* (3. 10) *possesses a formal solution*

$$U \sim \sum_{r=0}^{\infty} U_r(t)\rho^r,$$

where $U_r(t)$ *is represented by* (3. 11).

Thus, finally, we can get the formal solution of the differential equation (3. 4), which is stated in

THEOREM 3. 1 [Formal inner solution]. *Let* $k(t)$ *be defined by*

$$k(t)=\begin{cases}0 & if \ |t|\leqq t_0', \\ 1 & if \ |t|>t_0'.\end{cases}$$

Then the differential equation (3. 4) *possesses a formal matrix solution* U *of the form*

$$U(t,\rho)\sim\left(\sum_{r=0}^{\infty}[t^{k(t)m^*}\rho]^r\, U_r^*(t)\right)\cdot\exp[G^*(t)], \tag{3. 21}$$

where $U_r^*(t)$ *are bounded in the domain* $\mathfrak{D}$ *and* $|t|\leqq t_0'$ *together with* $k(t)=0$, *and in the domain* $\mathfrak{D}$ *and* $|t|>t_0'$ *together with* $k(t)=1$, *and* $m^*=m_1+1=1/n\sigma+(n-1)q/n+p-(p+1)/\sigma+1$.

9° It can be shown that the analytic theory corresponding to the formal theory obtained above is valid, i.e., there exist holomorphic solutions asymptotically expansible in the outer and the inner solutions respectively and that there exists a domain in which two different types of solutions are valid for ε arbitrarily small, and so two solutions can be *matched* at any point in that domain.

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