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A SECOND THEOREM OF CONSISTENCY FOR ABSOLUTE SUMMABILITY BY DISCRETE RIESZ MEANS

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1. 1. Definitions and notations. Let $\sum a_n$ be any given infinite series, and let $\{\lambda_n\}$ be a monotonic increasing sequence of positive numbers, tending to infinity with n. Let us write

$$A_\lambda(\omega)=A_\lambda^0(\omega)=\sum_{\lambda_n\leqq\omega} a_n,$$

$$A_\lambda^r(\omega)=\sum_{\lambda_n<\omega}(\omega-\lambda_n)^r a_n, \qquad r>0.$$

Let us write $R_\lambda^r(\omega)=A_\lambda^r(\omega)/\omega^r$, $r\geqq 0$. $\sum a_n$ is said to be absolutely summable by Riesz means of *type* λ_n and *order* r, or summable $|R, \lambda_n, r|$, $r\geqq 0$, if

$$R_\lambda^r(\omega)\in BV(k, \infty),^{1)}$$

where k is some finite positive number.[2] We say that $\sum a_n$ is absolutely summable by *discrete Riesz means of type* λ_n and order r, or summable $|R^*, \lambda_n, r|$, $r\geqq 0$, if

$$\{\Omega_n\}\equiv\{R_\lambda^r(\lambda_n)\}\in BV.^{3)}$$

By definition, summability $|R, \lambda_n, 0|$ and summability $|R^*, \lambda_n, 0|$ are the same as absolute convergence.

Let P and Q be any two methods of summability. Then, by '$P\subset Q$' we mean that any series which is summable P is also summable Q. By '$P\sim Q$' we mean that $P\subset Q$ as well as $Q\subset P$.

It is easily seen that

$$|R, \lambda_n, r|\subset|R^*, \lambda_n, r|, \quad r\geqq 0.$$

Throughout, for any sequence $\{f_n\}$, we shall write $\Delta f_n=f_n-f_{n+1}$, and K will denote a positive constant, not necessarily the same at each occurrence.

1. 2. It is known that $|R, \lambda_n, 1|\sim|R^*, \lambda_n, 1|$.[4] For summability $|R, \lambda_n, r|$ the

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1) By '$f(x)\in BV(h, k)$' we mean that $f(x)$ is a function of bounded variation over (h, k).

2) Obrechkoff (4), (5).

3) By '$\{f_n\}\in BV$' we mean that $\sum_n|f_n-f_{n-1}|<\infty$.

4) A proof of this by the present author has been quoted in Iyer [2].

following 'second theorem of consistency' is known.

THEOREM A.[5] *If $\varphi(t)$ is a monotonic non-decreasing function of t for $t \geqq 0$, tending to infinity with t, and*

$$t^r \varphi^{(r)}(t)/\varphi(t) \in B(h, \infty),^{6)}$$

where h is some finite positive number, then $|R, \lambda_n, r| \subset |R, \mu_n, r|$, where r is a positive integer and $\mu_n = \varphi(\lambda_n)$.

In this theorem one assumes a functional relation: $\mu_n = \varphi(\lambda_n)$ between the two types. The object of the present paper is to demonstrate a second theorem of consistency for absolute summability by discrete Riesz means, in which we get the inclusion relation: $|R^*, \lambda_n, 1| \subset |R^*, \mu_n, 1|$, or equivalently $|R, \lambda_n, 1| \subset |R, \mu_n, 1|$, where μ_n and λ_n are related to each other in a simpler and more direct manner, without appealing to any such functional relation.

2. 1. We establish the following

THEOREM. *If $\{\lambda_n\}$ and $\{\mu_n\}$ be monotonic increasing sequences, diverging to ∞ with n, such that*

$$\Delta\mu_n/\Delta\lambda_n = O(\mu_n/\lambda_n), \qquad as \quad n \to \infty,$$

then $|R^*, \lambda_n, 1| \subset |R^*, \mu_n, 1|$, or, equivalently, $|R, \lambda_n, 1| \subset |R, \mu_n, 1|$.

2. 2. We require the following lemma.

LEMMA.[7] *If $P_n = p_1 + p_2 + \cdots + p_n$ $(n = 1, 2, 3, \cdots)$ and $p_n > 0$ for every n, then '$\{c_n\} \in BV$' implies:*

$$\left\{\frac{1}{P_n} \sum_{k=1}^{n} p_k c_k\right\} \in BV.$$

3. Proof of the Theorem. We are given that

$$\left\{\frac{1}{\lambda_n} \sum_{m=1}^{n} (\lambda_n - \lambda_m) a_m\right\} \in BV, \tag{3. 1}$$

and we are to show that, under the hypotheses of the theorem,

$$\left\{\frac{1}{\mu_n} \sum_{m=1}^{n} (\mu_n - \mu_m) a_m\right\} \in BV. \tag{3. 2}$$

We observe that (3. 1) can be re-written as:

5) Guha [1].
6) By '$f(t) \in B(h, k)$' we mean that $f(t)$ is bounded over the interval (h, k).
7) Mohanty (3).

$$\left\{\frac{-1}{\lambda_n}\sum_{m=1}^{n-1}\Delta\lambda_m S_m\right\}\in BV,$$

or, what is the same thing,

$$\{\sigma_n\}\equiv\left\{\frac{1}{\lambda_{n+1}}\sum_{m=1}^{n}\Delta\lambda_m S_m\right\}\in BV. \qquad (\sigma_0=0)$$

Similarly, (3. 2) has the equivalent form:

$$\{\tau_n\}\equiv\left\{\frac{1}{\mu_{n+1}}\sum_{m=1}^{n}\Delta\mu_m S_m\right\}\in BV.$$

Now

$$\begin{aligned}\tau_n &= \frac{1}{\mu_{n+1}}\sum_{m=1}^{n}\frac{\Delta\mu_m}{\Delta\lambda_m}\Delta\lambda_m S_m \\ &= -\frac{1}{\mu_{n+1}}\sum_{m=1}^{n}\frac{\Delta\mu_m}{\Delta\lambda_m}\Delta(\lambda_m\sigma_{m-1}) \\ &= -\frac{1}{\mu_{n+1}}\sum_{m=1}^{n}\frac{\Delta\mu_m}{\Delta\lambda_m}\lambda_m\Delta\sigma_{m-1}-\frac{1}{\mu_{n+1}}\sum_{m=1}^{n}\Delta\mu_m\sigma_m \\ &= \tau_n^{(1)}+\tau_n^{(2)}, \qquad \text{say.}\end{aligned}$$

We write

$$\tau_n^{(2)}=-\frac{\sum_{m=1}^{n}\Delta\mu_m\sigma_m}{\sum_{m=1}^{n}\Delta\mu_m}\cdot\frac{\sum_{m=1}^{n}\Delta\mu_m}{\mu_{n+1}}$$

We observe that, by the lemma, the first factor is a sequence of bounded variation, since, by hypothesis, $\{\sigma_n\}\in BV$. Also, the second factor is the sequence

$$\left\{\frac{\mu_1}{\mu_{n+1}}-1\right\},$$

which is a sequence of bounded variation since $\{\mu_n\}$ is monotonic increasing and $\mu_n\to\infty$, as $n\to\infty$. Thus $\{\tau_n^{(2)}\}\in BV$.

We proceed to show that $\{\tau_n^{(1)}\}\in BV$. Now

$$\begin{aligned}\Delta\tau_{n-1}^{(1)} &= \frac{1}{\mu_{n+1}}\sum_{m=1}^{n}\frac{\Delta\mu_m}{\Delta\lambda_m}\lambda_m\Delta\sigma_{m-1}-\frac{1}{\mu_n}\sum_{m=1}^{n-1}\frac{\Delta\mu_m}{\Delta\lambda_m}\lambda_m\Delta\sigma_{m-1} \\ &= -\Delta\left(\frac{1}{\mu_n}\right)\sum_{m=1}^{n}\frac{\Delta\mu_m}{\Delta\lambda_m}\lambda_m\Delta\sigma_{m-1}+\frac{1}{\mu_n}\frac{\Delta\mu_n}{\Delta\lambda_n}\lambda_n\Delta\sigma_{n-1}.\end{aligned}$$

Hence

$$\sum_n |\varDelta\tau_{n-1}^{(1)}| \leqq \sum_n \varDelta\left(\frac{1}{\mu_n}\right)\sum_{m=1}^{n} \frac{\varDelta\mu_m}{\varDelta\lambda_m}\lambda_m|\varDelta\sigma_{m-1}| + \sum_n \frac{1}{\mu_n}\frac{\lambda_n}{\varDelta\lambda_n}\varDelta\mu_n|\varDelta\sigma_{n-1}|$$

$$= \sum\nolimits_1 + \sum\nolimits_2, \qquad \text{say.}$$

Now

$$\sum\nolimits_2 \leqq K\sum_{n=1}^{\infty}|\varDelta\sigma_{n-1}| \leqq K,$$

by hypothesis. And

$$\sum\nolimits_1 = \sum_{m=1}^{\infty}\frac{\varDelta\mu_m}{\varDelta\lambda_m}\lambda_m|\varDelta\sigma_{m-1}|\sum_{n=m}^{\infty}\varDelta\left(\frac{1}{\mu_n}\right)$$

$$= \sum_{m=1}^{\infty}|\varDelta\sigma_{m-1}|\frac{\varDelta\mu_m}{\varDelta\lambda_m}\frac{\lambda_m}{\mu_m}$$

$$\leqq K\sum_{1}^{\infty}|\varDelta\sigma_{m-1}| \leqq K,$$

by hypothesis.

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