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CAPACITABILITY AND EXTREMAL RADIUS

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1. Introduction. Let Ω be a plane region and let α be its preassigned boundary component. In a previous paper of these reports [5] we constructed a circular and radial slit disc mapping of the region with respect to a partition, denoted by (α, A, B), of its boundary. In this construction, the coincidence and finiteness of the radii $\overline{R}(A)$ and $\underline{R}(B)$ defined below, were assumed. Then the following problem will arise: *When do the quantities* $\overline{R}(A)$ *and* $\underline{R}(B)$ *coincide?* We shall give an answer to this problem, making use of Choquet's theory of capacities [2]. The answer is as follows: Let the set A be generated by the Souslin operation from the class of closed set of boundary components in the Stoïlow compactification of the region less α. Then $\overline{R}(A)$ is equal to $\underline{R}(B)$.

We can see, as its consequence, that the univalent functions which correspond to a minimal sequence of $\overline{R}(A)$ and a maximal sequence of $\underline{R}(B)$, constructed in no. **4** are really circular and radial slit disc mappings.

So far as the construction of capacity functions concerns these results holds on open Riemann surfaces. The basic results for the partitions (α, A, B) in which A or B is closed were discussed by Marden and Rodin [3].

2. Preliminaries. Let Ω be a plane region which is not the extended plane. We denote by $\hat{\Omega}$ the Stoïlow compactification of Ω in which each boundary component is a point. Let α be a preassigned boundary component and let (α, A, B) denote a partition of the boundary $\partial\Omega=\hat{\Omega}-\Omega$.

A curve c is a continuous image of the closed interval [0, 1] into $\hat{\Omega}$. It is said to be locally rectifiable, if so is every component of $\Omega\cap c$. All quantities such as length, integral etc. are defined about the restriction of c on Ω.

Let a be a point of Ω. We denote by $\Gamma(\alpha, A, B)$ and $X(\alpha, A, B)$ the families of locally rectifiable curves separating α from a within $\hat{\Omega}-A$ and joining them within $\hat{\Omega}-B$ respectively. Let $\Gamma_q(\alpha, A, B)$ and $X_q(\alpha, A, B)$ denote the families in the difinitions of which the point a is replaced by a compact disc $|z-a|\leqq q$ in Ω. We define two quantities by

$$\log R_1=\lim_{q\to 0}(2\pi \bmod \Gamma_q(\alpha, A, B)+\log q) \tag{1}$$

and

$$\log R_2=\lim_{q\to 0}(2\pi\lambda(X_q(\alpha, A, B))+\log q), \tag{2}$$

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where the notations mod and λ denote module and extremal length respectively. If $R_1=R_2$ this quantity is called the *extremal radius* of α with respect to the partition (α, A, B) and denoted by $R(\alpha, A, B)$. The equality holds if A or B is closed in $\hat{\Omega}-\alpha$ [5]. In these cases, suppose $R(\alpha, A, B)<\infty$. A circular-radial or radial-circular slit disc mapping of Ω can be constructed if A or B is closed respectively [3, 5]. As to the properties of these functions the readers are referred to [5].

3. We define for an arbitrary partition

$$\overline{R}(A)=\inf_{A_*\subset A} R(\alpha, A_*, B^*)$$

for closed A_* in $\hat{\Omega}-\alpha$ and

$$\underline{R}(B)=\sup_{B_*\subset B} R(\alpha, A^*, B_*)$$

for closed B_* in it.

We remark that in the latter definition the class of closed sets can be replaced by that of compact sets. In fact, let $\{\Omega_n\}$ be an exhaustion of Ω towards α. Every closed set B_* is expressed as the union of at most a countable number of compact sets B_n, given by $B_*\cap\hat{\Omega}_n$. Then we get $\cup\Gamma(\alpha, A^n, B_n)=\Gamma(\alpha, A^*, B_*)$ and the assertion follows from a continuity lemma of extremal length stated in [5] (Lemma 1). It is worth mentioning that the same replacement can not be admitted in the former definition. This is shown by the following counterexample: Let α be unstable [4] and let A be $\partial\Omega-\alpha$. Then $\inf_{A_*}R(\alpha, A_*, B^*)$ for compact subsets A_* of A is infinite, since instability is a local property, while $\overline{R}(A)$ is finite.

4. Suppose $\overline{R}(A)<\infty$. Let $\{A_n\}$ be a minimal sequence in the definition of $\overline{R}(A)$ and let f_{A_n} be the circular-radial slit disc mapping with respect to the partition (α, A_n, B^n) having the normalizations that $f_{A_n}(a)=0$ and $f'_{A_n}(a)=1$. Then the function f_{A_n} tends to a univalent function f_A in such a way that $\|f'_{A_n}/f_{A_n}-f'_A/f_A\|\to 0$. The limit function f_A is independent of particular minimal sequences. Similarly if $\underline{R}(B)<\infty$, for any maximal sequence $\{B_n\}$, the radialcircular slit disc mapping g_{B_n} tends to a unique univalent function g_B so that $\|g'_{B_n}/g_{B_n}-g'_B/g_B\|\to 0$. These were proved in [5].

We now state a fundamental result of circular and radial slit mappings [5].

THEOREM A. *Suppose $\overline{R}(A)=\underline{R}(B)<\infty$. Then $f_A=f_B$ and the function, denoted by $\varphi_{A,B}(z)$, possesses the following properties:*

i) *$\varphi_{A,B}(\alpha)$ is a circle $|\varphi_{A,B}|=R(\alpha, A, B)$ with possible radial incisions emanating from it, where $R(\alpha, A, B)=\overline{R}(A)$,*

ii) *$\varphi_{A,B}(\sigma)$, $\sigma\in A$, is a circular slit (possibly a point) with possible radial incision emanating from it,*

iii) *$\varphi_{A,B}(\sigma)$, $\sigma\in B$, is a radial slit (possibly a point) with possible circular incisions emanating from it,*

iv) *the area of $\varphi_{A,B}(\partial\Omega)$ vanishes,*

v) *the metric* $\rho_0=|\varphi_{A,B}'/(2\pi\varphi_{A,B})|$ *is extremal for the family* $\Gamma^q(\alpha, A, B)$ *which is the subfamily of* $\Gamma(\alpha, A, B)$ *consisting of curves separating* α *from a compact set* $|\varphi_{A,B}(z)|\leqq q$ *for sufficiently small* q *and* $\mathrm{mod}\, \Gamma^q(\alpha, A, B)=(2\pi)^{-1}\log(R(\alpha, A, B)/q)$ *and*

vi) *the metric* $\mu_0=|\varphi_{A,B}'/(\varphi_{A,B}\log(R(\alpha, A, B)/q))|$ *is extremal for the family* $X^q(\alpha, A, B)$ *whose module is equal to* $2/\log(R(\alpha, A, B)/q)$, *where* $X^q(\alpha, A, B)$ *is the family of curves joining* α *and the set* $|\varphi_{A,B}(z)|\leqq q$ *within* $\hat{\Omega}-B$.

The function $\varphi_{A,B}$ is called a *circular and radial slit disc mapping.* The circular-radial slit and the radial-circular slit disc mapping are both the circular and radial slit disc mappings [5].

5. Capacitability. Let A be a closed set in $\partial\Omega-\alpha$. Then $\tilde{A}=\alpha\cup A$ is compact in $\hat{\Omega}$. Let us assign every p-tuple $(n_1, n_2, \cdots, n_p)$ of positive integers to a compact set $A_{n_1n_2\cdots n_p}$. The operation generating a set

$$A=\bigcup_{n_1n_2\cdots} A_{n_1}\cap A_{n_1n_2}\cap\cdots\cap A_{n_1n_2\cdots n_p}\cap\cdots\cap\cdots,$$

where the n_p's run over all positive integers, is called the *Souslin operation.* The set A is called a *K-Souslin* set. We shall apply Choquet's theory [2] to the boundary of $\partial\Omega$ which is a compact Hausdorff space. We now mention a part of his results, following to Carleson [1].

Let V be a nonnegative set function defined only for all compact sets of $\partial\Omega$ containing α. We define for a set E containing α

$$V(E)=\sup_{K\subset E} V(K) \tag{3}$$

for compact K and

$$V^*(E)=\inf_{E\subset G} V(G)$$

for open G. Then E is said to be *capacitable,* if $V(E)=V^*(E)$. The following lemma will be needed later:

LEMMA [1]. *Suppose that the function* V *satisfies the following conditions*:

I) $V(K_1)\leqq V(K_2)$, *if* $K_1\subset K_2$ *for compact* K_1 *and* K_2.

II) *Let* $\{E_n\}$ *be an increasing sequence and let* $E_0=\cup E_n$. *Then* $\lim V^*(E_n)=V^*(E_0)$.

Then, if every compact set is capacitable, so are all K-Souslin sets. Here all the sets are assumed to contain α

Although this result was established in the Euclidean space in [1], the proof will be achieved word for word in the space $\partial\Omega$ under the above assumption.

6. As a direct result of this lemma we have

THEOREM 1. *Let $\alpha \cup A$ be a K-Souslin set generated by compact sets containing α. Then we have $\overline{R}(A)=\underline{R}(B)$.*

Proof. Let (α, A, B) be a partition such that A is closed in $\partial\Omega-\alpha$. Then $\tilde{A}=\alpha\cup A$ is compact. Put $V(\tilde{A})=1/R(\alpha, A, B)$. We can deduce from (1) that $V(\tilde{A})$ is nonnegative and increasing, since $R(\alpha, A, B)$ is decreasing with respect to A.

In order to prove II), suppose that B is compact, whence $\alpha\cup A$ is open. Then we have $V(\alpha\cup A)=1/R(\alpha, A, B)$ by (3). In fact, taking an exhaustion $\{\Omega_n\}$ of Ω towards B such that $\alpha\in\hat{\Omega}_1$, we set $A_n=\hat{\Omega}_n\cap A$ and $B^n=\partial\Omega-(\alpha\cup A_n)$. Clearly A_n is closed in $\partial\Omega-\alpha$, increasing and $\cup A_n=A$. Hence $X(\alpha, A, B)=\cup X(\alpha, A_n, B^n)$, because every curve of $X(\alpha, A, B)$ is running through $\hat{\Omega}_n-B_n$ for an n, where B_n is the relative boundary of Ω_n. If $R(\alpha, A, B)<\infty$, we may assume, from the continuity of module, that $R(\alpha, A_n, B^n)<\infty$. Let $f_{A_n}(z)$ be the circular-radial slit disc mapping of Ω with respect to the partition (α, A_n, B^n) and let $g_B(z)$ be the radial-circular slit disc mapping with respect to the partition (α, A, B), which are all circular and radial slit disc mappings. Then we can deduce, from the continuity lemma of extremal length [5] that $\|g_B'/g_B-f_{A_n}'/f_{A_n}\|\to 0$, which implies the above relation, since $R(\alpha, A_n, B^n)\to R(\alpha, A, B)$. When $R(\alpha, A, B)=\infty$, clearly $R(\alpha, A_n, B^n)=\infty$.

We verify the condition II). Let E_n $(n\geqq 1)$ be containing α and $E_n\subset E_{n+1}$. Put $E_0=\cup E_n$. Then there exists an open set G_n containing E_n and satisfying

$$V(G_n)\leqq V^*(E_n)+\varepsilon$$

for given $\varepsilon>0$. Put $G=\cup G_n$, which contains E_0. We have as above $V(G)=\lim V(G_n)$, since they are the reciprocals of the extremal radii. We get

$$V^*(E_0)\leqq V(G)=\lim V(G_n)\leqq \lim V^*(E_n)+\varepsilon,$$

which implies II).

Finally we show the capacitability of every compact $\alpha\cup A$. Let (α, A, B) be the partition determined by the A. Using an exhaustion of Ω towards $\alpha\cup A$, we can express the set B as the union of an increasing sequence of compact B_n's. Let (α, A^n, B_n) denote the partition determined by B_n. Then we have

$$\lim R(\alpha, A^n, B_n)=R(\alpha, A, B),$$

since $\Gamma(\alpha, A, B)=\cup\Gamma(\alpha, A^n, B_n)$. Since $\alpha\cup A^n$ is open, we get $V^*(\tilde{A})=V(\tilde{A})$. Thus we have proved Theorem 1 by the lemma, because $V^*(\tilde{A})=\underline{R}(B)^{-1}$ and $V(\tilde{A})=\overline{R}(A)^{-1}$ for an arbitrary partition (α, A, B).

7. We can show immediately

COROLLARY. *Let $\{A_n\}$ be a minimal sequence to define $\overline{R}(A)$ for an arbitrary partition (α, A, B). If $\overline{R}(A)<\infty$, the function f_A in no. 4 is the circular and radial slit disc mapping with respect to the partition determined by $A_0=\cup A_n$, which has* *the properties in Theorem* A.

Similarly the function g_B *is also a circular and radial slit disc mapping with respect to a partition* (α, A^0, B_0), *if* $R(B)<\infty$. *Here* B_0 *is the union* $\cup B_n$ *of a maximal sequence* $\{B_n\}$.

Proof. We may assume that the minimal sequence is increasing [5]. The set $\alpha \cup A_0$ is a K_σ set (the union of at most a countable number of compact sets) which is a K-Souslin set. For g_B, we can select an increasing maximal sequence $\{B_n\}$ consisting of compact B_n by the remark in no. **3**. Let (α, A^n, B_n) be the partition determined by B_n. Put $B_0=\cup B_n$ and $A^0=\cap A^n$. Since A^0 is a G_δ set and every open set is a K_σ set in $\partial\Omega$, the set $\alpha\cup A^0$ is a K-Soulin set.

8. **Concluding remark.** If we set a capacity $V(B)=R(\alpha, A, B)$ for compact B in $\partial\Omega-\alpha$, we can show a corresponding result without proof:

THEOREM 2. *If* B *is a* K*-Soulin set contained in a fixed compact set in* $\partial\Omega-\alpha$, *then* $\bar{R}(A)=\underline{R}(B)$.

In this case one of Choquet's results is applicable directly. In order to remark it, we say that a subset of $\partial\Omega$ is *K-analptic* if it is the continuous image of a $K_{\sigma\delta}$ set of a compact space, where a $K_{\sigma\delta}$ set is the intersection of at most a countable number of K_σ sets. It is known that the class of K-analytic sets contains every K-Souslin set [2]. Then by Choquet's theorem ([2], 30. 1), we know that $\bar{R}(A)=\underline{R}(B)$ for K-analytic B.

9. The circular and radial slit annulus or plane mappings can be similarly discussed. We now state a result of a capacity function corresponding to the latter case on an open Riemann surface.

Let W be an arbitrary open Riemann surface and let $\hat{W}$ be its Stoïlow's compactification. Let a_j $(j=1, 2)$ be two distinct points in W, denoted by local variables. Denoting by (A, B) a partition of $\partial W=\hat{W}-W$ such that $\partial W=A\cup B$ and $A\cap B=\phi$, we have

THEOREM 3. *If* A *or* B *is a* K*-Souslin* (K*-analytic*) *set, then there exists a harmonic function in* W *less* a_j*'s such that* $v_{A,B}(z)+(-1)^j \log|z-a_j|$ *is harmonic at* a_j *and satisfies that*

i) *the metric* $\rho_0|dz|=(2\pi)^{-1}|\text{grad}\, v_{A,B}||dz|$ *is extremal for the family of curves separating two compact sets* $v_{A,B}(z)\geqq M$ *and* $v_{A,B}(z)\leqq -N$ *within* $\hat{W}-A$ *for suficiently large* M *and* N, *whose module is equal to* $(M+N)/2\pi$ *and*

ii) *the metric* $\mu_0|dz|=(M+N)^{-1}|\text{grad}\, v_{A,B}||dz|$ *is extremal for the family of curves joining them within* $\hat{W}-B$, *whose module is equal to* $2\pi/(M+N)$.

Conversely the condition i) *or* ii) *for an* M *and* N *characterizes the function* $v_{A,B}$ *except for an additive constant under the same assumption.*

Proof. We first define capacities. Let $\Gamma_{M,N}(A, B)$ be the family of curves separating the compact sets $\log|z-a_1|\leqq -M$ and $\log|z-a_2|\leqq -N$ and let $X_{M,N}(A,B)$

be the family of curves joining them. Then the quantities

$$\log Q_1(A, B)=\sum_{M,N\to\infty}\left(2\pi \bmod \Gamma_{M,N}(A, B)+\log\frac{N}{M}\right)$$

and

$$\log Q_2(A, B)=\sum_{M,N\to\infty}\left(2\pi\lambda(X_{M,N}(A, B))+\log\frac{N}{M}\right)$$

are the limits of monotone increasing sequences which are positive and finite. If A or B is compact, $Q_1=Q_2$, which is denoted by $Q(A, B)$. We put the set functions $V(A)=Q(A, B)^{-1}$ and $W(B)=Q(A, B)$ for compact A and B respectively. The capacitabilities are as before. The construction of $v_{A,B}$ is analogous to [5].

Roughly speaking, the function $v_{A,B}$ is such that $v_{A,B}$=const on $\sigma\in A$ and $\int_\sigma dv_{A,B}{}^*=0$ and that $dv_{A,B}^*=0$ along $\sigma\in B$, which can be formulated in terms of extremal lengths (cf. [3]).

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