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AUTOMORPHISMS OF A FREE NILPOTENT ALGEBRA

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Let F be a finite dimensional nilpotent algebra over a field K with index of nilpotency ρ: $F\supset F^2\supset F^3\supset\cdots\supset F^{\rho-1}\supset F^\rho=0$. Let $u_1, u_2, \cdots, u_n$ be a system of generators of F such that u's are linearly independent over K modulo F^2. We shall call F a free nilpotent algebra if the generators $u_1, u_2, \cdots, u_n$ satisfy only relations $u_{i_1}u_{i_2}\cdots u_{i_\rho}=0$ $(1\leqq i_1, i_2, \cdots, i_\rho\leqq n)$; we shall denote it by $F=F(u_1, u_2, \cdots, u_n; \rho)$.

Let N be a nilpotent algebra over K with index of nilpotency ρ generated by n elements $a_1, a_2, \cdots, a_n$ and let F be as above. Then we can find a homomorphism φ of F onto N defined by $\varphi(u_i)=a_i$, so that N is isomorphic to the residue class ring $F/\mathfrak{p}$ where $\mathfrak{p}$ is the kernel of φ. Thus we may say that the study of nilpotent algebras can be reduced to that of free nilpotent algebras and their ideals.

In this note we shall consider a free nilpotent algebra and its automorphism groups. The first section is preliminary and we make some considerations about the relations between nilpotent algebras and free nilpotent algebras. In the second, we study automorphisms of a free nilpotent algebra. Throughout the note, we assume that the characteristic of the ground field K is 0, and algebras mean associative finite dimensional algebras over K.

1. Preliminaries. The following theorem is well known.

THEOREM 1. *Let N be a nilpotent algebra over a field K with index of nilpotency ρ, then N is generated by a system of elements $a_1, a_2, \cdots, a_n$ whicn form a basis of N modulo N^2. And any such system of elements generates N.*

We call such a system of elements a *minimal generating system of N*. From theorem 1, we get

COROLLARY. *Every nilpotent algebra N over K with index of nilpotency ρ is isomorphic to a residue class ring of a free nilpotent algebra F with index ρ by a two-sided ideal $\mathfrak{p}$ which is contained in F^2.*

If $a_1, a_2, \cdots, a_n$ generate N, the isomorphism mentioned in the above corollary is induced by the mapping of F onto N defined by

$$F\ni u_i\rightarrow a_i\in N, \qquad i=1, 2, \cdots, n.$$

If we take another minimal generating system $a_1', a_2', \cdots, a_n'$, we have the following

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expressions

$$a_i'=f_i\ (a_1, a_2, \cdots, a_n)=f_i(a), \qquad i=1, 2, \cdots, n, \tag{1}$$

where $f_i(x_1, x_2, \cdots)$ are non-commutative polynomials in $x_1, x_2, \cdots, x_n$ over K. Thus, if we define a new generating system of F corresponding to the above formula

$$u_i'=f_i(u), \qquad i=1, 2, \cdots, n, \tag{2}$$

then $u_1', u_2', \cdots, u_n'$ are linearly independent modulo F^2. So, they form a minimal generating system of F.

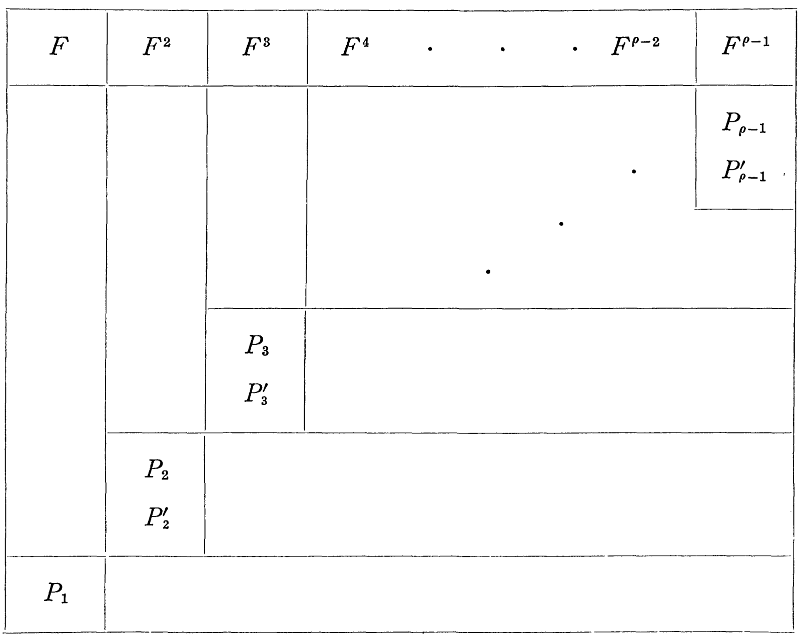
If an automorphism $\bar{\sigma}$ of N is given by the formula (1), then we can define the corresponding automorphism σ of F by the formula (2). Let $p(u)$ be a polynomial belonging to the kernel of the original mapping of F onto N, then $p(u)$ is transformed to $p(u^\sigma)$ by means of the automorphism σ. And,

$$p(a^{\bar{\sigma}})=(p(a))^{\bar{\sigma}}=0$$

So, the kernel of the mapping is invariant under the automorphism σ.

Conversely, an automorphism σ of F leaving the kernel invariant induces an automorphism of N.

Next, we shall consider ideals of F. Let $\mathfrak{p}$ be an ideal of F. Then we can take a kind of generating systems S of $\mathfrak{p}$ (in the sense $\mathfrak{p}=SF+FS$) by means of the following diagram.



Here, $P_1, P_2, \cdots$ are (K-)subspaces of F and $P_2', P_3', \cdots$ are sets of (K-)linearly independent elements of $\mathfrak{p}$; they are chosen as follows. Set $Q_1=\{x\in F \mid xF\subset\mathfrak{p},\ Fx\subset\mathfrak{p}\}$, which is a subspace of F. P_1 is a subspace of F such that $Q_1=Q_1\cap F^2\oplus P_1$. P_2' is a system of elements in $\mathfrak{p}$ which forms a basis of $\mathfrak{p}$ modulo $F^3+[P_1]$ ($[P_1]$ means the ideal generated by P_1). P_2 is a subspace of F such that $Q_1\cap F^2=(\mathfrak{p}\cap F^2+F^3)\cap Q_1\oplus P_2$, and so on.

Thus, if we choose a linearly independent system of elements in $P_1, P_2', P_2, \cdots, P_{\rho-1}', P_{\rho-1}$, then we obtain a generating system of $\mathfrak{p}$ as an ideal. And the K-module $(P_1, P_2, \cdots, P_{\rho-1})$ consists of elements which are two-sided zero divisors of F modulo $\mathfrak{p}$. The K-module $F_1=(P_1, P_2, \cdots, P_{\rho-1}, \mathfrak{p})$ corresponds to the ideal whose elements are all two-sided zero divisors of N. The corresponding ideal

$$N_1=\{b;\ bx=xb=0 \quad \text{for all elements} \quad x\in N\}$$

is called two-sided zero ideal of N.

Similarly, we make the same considerations in the residue class ring F/F_1, then we obtain F_2 whose elements are all two-sided zero divisors modulo F_1, and so on.

Thus, we have the following series of ideals in F.

$$F_1\subset F_2\subset F_3\subset\cdots\subset F_{\rho-2}\subset F_{\rho-1}=F.$$

Correspondingly, in N we have

$$N_1\subset N_2\subset N_3\subset\cdots\subset N_{\rho-2}\subset N_{\rho-1}=N.$$

Each ideal F_i of the series is invariant with respect to automorphisms leaving $\mathfrak{p}$ invariant, because in N the corresponding ideal is invariant with respect to all automorphisms of N.

2. Automorphism groups of a free nilpotent algebra.

Let F be a free nilpotent algebra over a field K with minimal generating system $u_1, u_2, \cdots, u_n$ and index ρ. Then we can choose the following basis of F over K which will be called the normal basis of F:

$$u_1, u_2, \cdots, u_n;\ u_1^2, u_1u_2, \cdots, u_n^2: \cdots;\ u_1^{\rho-1}, u_1^{\rho-2}u_2, \cdots, u_n^{\rho-1}.$$

The normal basis of F is briefly written in the following form:

$$[U^1;\ U^2;\ \cdots;\ U^{\rho-1}].$$

Let σ be an automorphism of F over K. We shall give the representation of σ by means of the normal basis. (In this representation, we shall write automorphisms as left operators.)

Let

$$\sigma(u_i)=\sum_{p=1}^{\rho-1} U^p a_{pi} \qquad i=1, 2, \cdots, n,$$

where a_{pi} are matrices of type $(n^p, 1)$. Or, briefly,

$$\sigma(U^1)=\sum_{p=1}^{\rho-1} U^p A_p.$$

Then, from the properties of automorphism,

$$\sigma(u_i u_j)=\sigma(u_i)\sigma(u_j)=\left(\sum_{p=1}^{\rho-1} U^p a_{pi}\right)\left(\sum_{q=1}^{\rho-1} U^q a_{qj}\right)=\sum_{l=2}^{\rho-1} U^l \sum_{p+q=l} a_{pi}\times a_{qj}$$
$$(i, j=1, 2, \cdots, n),$$

where $a_{pi}\times a_{qj}$ is a right direct product as in [9]. In the present note we shall use the symbol $\times$ in this sense. In matrix form, we have

$$\sigma(U^2)=\sum_{l=2}^{\rho-1} U^l \sum_{p+q=l} A_p\times A_q.$$

Similarly, we have

$$\sigma(U^i)=\sum_{l=i}^{\rho-1} U^l \sum A_{p_1}\times A_{p_2}\times\cdots\times A_{p_i}.$$

Therefore, the automorphism of F is represented by the following matrix:

$$(3)\quad \begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_2 & A_1\times A_1 & 0 & 0 \\ A_3 & A_1\times A_2+A_2\times A_1 & A_1\times A_1\times A_1 & \cdot \\ \cdot & & \cdot & 0 \\ \cdot & & \cdot & 0 \\ \cdot & & \cdot & 0 \\ A_{\rho-1} & A_1\times A_{\rho-2}+A_2\times A_{\rho-3}+\cdots+A_{\rho-2}\times A_1 & \cdots\ A_1\times A_1\times\cdots & \times A_1 \end{bmatrix}$$

in which the i-j-block is the sum of j-th direct products $A_{p_1}\times A_{p_2}\times\cdots\times A_{p_j}$, where $\sum p_k=i$.

In the expression (3), if $A_1=E$, we call the corresponding automorphism monic; and automorphism corresponding to the following type of matrix pseudo-diagonal:

$$\begin{bmatrix} A_1 & 0 & 0 & & 0 \\ 0 & A_1\times A_1 & 0 & & 0 \\ 0 & 0 & A_1\times A_1\times A_1 & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \\ & & & & 0 \\ 0 & & & & A_1\times A_1\times\cdots\times A_1 \end{bmatrix}.$$

Then, we have the following proposition immediately.

PROPOSITION 1. *Any automorphism σ of a free nilpotent algebra F can be written as a product of a pseudo-diagonal automorphism and a monic one, and the expression is unique.*

In general, a pseudo-diagonal automorphism and a monic one are not commutative to each other. For instance

$$\begin{bmatrix} E & 0 \\ B & E\times E \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A\times A \end{bmatrix}=\begin{bmatrix} A & 0 \\ BA & A\times A \end{bmatrix}.$$

So, if commutative, we obtain the equation

$$BA=(A\times A)B.$$

But this can not be true in general, since B is arbitrarily given.

We have defined monic automorphisms of a free nilpotent algebra by means of representation, but we can define them directly for a nilpotent algebra N as follows.

DEFINITION. An automorphism σ of a nilpotent algebra N is called *monic* in case $a^\sigma-a$ lies in N^{r+1} whenever a lies in N^r for $r=1, 2, \cdots$.

Then, we have

PROPOSITION 2. *Let F be a free nilpotent algebra over K with index ρ, then the following three conditions are equivalent to each others.*

(1) *σ is monic as in the first meaning.*

(2) *σ is as in the definition.*

(3) *σ is an automorphism such that $a^\sigma-a\in F^2$ for all $a\in F$.*

Proof. We shall prove (3)$\rightarrow$(2). Others are evident. It is sufficient to prove it when a is a monomial in F^r. In that case,

$$(u_{i_1}u_{i_2}\cdots u_{i_r})^\sigma-u_{i_1}u_{i_2}\cdots u_{i_r}=u_{i_1}^\sigma u_{i_2}^\sigma\cdots u_{i_r}^\sigma-u_{i_1}u_{i_2}^\sigma\cdots u_{i_r}^\sigma+u_{i_1}u_{i_2}^\sigma\cdots u_{j_r}^\sigma-\cdots-u_{i_1}u_{i_2}\cdots u_{i_l}.$$

Therefore, $a^{\sigma}-a\in F^{r+1}$.

The following theorem is due to Dubisch and Perlis [5].

THEOREM 2. *The totality of monic automorphisms of a free nilpotent algebra F form a nilpotent normal subgroup $\mathfrak{M}$ of the automorphism group $\mathfrak{G}$ of F.*

Proof. Let $\mathfrak{M}_i$ be the subset of all the monic automorphisms leaving all elements of F^i invariant, then these automorphisms are represented as the following form

$$\begin{bmatrix} E & & & & & \\ 0 & E\times E & 0 & & & \\ \cdot & & E\times E\times E & 0 & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & & & & & \\ A_i & 0 & & & & \\ \cdot & A_i\times E+E\times A_i & 0 & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ A_{\rho-1} & & & 0 & 0 & E\times E\times\cdots\times E \end{bmatrix}.$$

Then, it is easily verified that the following series is a central series of $\mathfrak{M}$.

$$\mathfrak{M}=\mathfrak{M}_{\rho}\supset\mathfrak{M}_{\rho-1}\supset\cdots\supset\mathfrak{M}_2\supset\mathfrak{M}_1=e.$$

Any monic automorphism is expressed in the following form

(4)

$$M=\begin{bmatrix} E & 0 & 0 & 0 \\ A_2 & E\times E & 0 & 0 \\ A_3 & A_2\times E+E\times A & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & 0 \\ A_{\rho-1} & & & E\times\cdots\times E \end{bmatrix}=\begin{bmatrix} E & 0 \\ A_2 & E\times E \\ A_3 & A_2\times E+E\times A_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ A_{\rho-2} & \\ 0 & \end{bmatrix}\begin{bmatrix} E & 0 & 0 & \\ 0 & E\times E & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & & & \\ A_{\rho-1} & 0 & \cdots & E\times\cdots\times E \end{bmatrix}.$$

In brief,

$$M=M_1M_2, \qquad M_2\in\mathfrak{M}_2.$$

In the above expression, M_1 and M_2 are uniquely determined from M, and they are commutative to each other. The totality of M_2 form the abelian group $\mathfrak{M}_2$.

In case of a free nilpotent algebra F, every element a of F is quasi-regular, and there is an element b such that

$$a+b+ab=a+b+ba=0.$$

Then an inner automorphism J of F is defined by the formula

$$x^J=x+bx+xa+bxa \qquad \text{for} \qquad x\in F. \tag{5}$$

The totality of inner automorphisms $J=J_a$ given by (5) form, as a varies over all elements of F, a group $\mathfrak{J}$; and from (5), it is evident that $\mathfrak{J}\subset\mathfrak{M}$ and $\mathfrak{J}$ is a normal subgroup of $\mathfrak{G}$.

Nil automorphisms are defined as such automorphisms fixing all the absolute zero divisors of F in [5], but in case of a free nilpotent algebra they coincide with monic automorphisms.

Now let $\mathfrak{A}$ be an algebra over K with a unity element 1 possessing F as its radical. Then from the assumption on K, $\mathfrak{A}$ splits, $\mathfrak{A}=S+F$, where S is a semisimple subalgebra of $\mathfrak{A}$ which is isomorphic to the residue class ring $\mathfrak{A}/F$. Then the totality of regular elements of $\mathfrak{A}$ form a multiplicative group $\mathfrak{A}^\times=S^\times(1+F)$, where $S^\times$ is the multiplicative group consisting of all regular elements of S. In the followings we shall consider the effect of inner automorphisms of $\mathfrak{A}$ (by regular elements of $\mathfrak{A}$) on the radical F.

Let a be a regular element of S, then the left and right multiplications a_l and a_r induce endomorphisms of F^+. Now, we consider, formally, the representation of multiplication endomorphisms by means of normal basis of F.

Let a be a regular element of S and let the effect of a_r on U be as follows:

$$u_ia=\alpha_{11}u_1+\alpha_{12}u_2+\cdots+\alpha_{1m}u_n^{\rho-1}, \qquad i=1, 2, \cdots, n.$$

In matrix form,

$$U^1a=[U^1;\ U^2;\ \cdot\cdot\cdot;\ U^{\rho-1}]\begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{\rho-1} \end{bmatrix}.$$

Then, we have

$$u_iU^1a=[u_iU^1;\ u_iU^2;\ \cdots\ u_iU^{\rho-1}]\begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{\rho-1} \end{bmatrix}$$

$$=[U^1;\ U^2;\ \cdots;\ U^{\rho-1}]\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ A_1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

Therefore,

$$U^2a=[U^1;\ U^2;\ \cdots;\ U^{\rho-1}]\begin{bmatrix} 0 \\ E\times A_1 \\ E\times A_2 \\ \cdot \\ \cdot \\ \cdot \\ E\times A_{\rho-2} \end{bmatrix}.$$

Hence, with respect to the normal basis of F, a_r is represented by the matrix

$$(6)\qquad \begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_2 & E\times A_1 & 0 & \\ A_3 & E\times A_2 & E\times E\times A_1 & \\ \cdot & \cdot & \cdot & 0 \\ A_{\rho-1} & E\times A_{\rho-2} & E\times E\times A_{\rho 3} & E\times E\times E\times\cdots\times A^1 \end{bmatrix}.$$

Similarly, let b_l be as follows.

$$bU=[U^1;\ U^2;\ \cdot\cdot\cdot;\ U_\rho^{-1}]\begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{\rho-1} \end{bmatrix}.$$

Then, b_l is represented by the matrix

$$(7)\qquad \begin{bmatrix} B_1 & 0 & 0 & & & & 0 \\ B_2 & B_1\times E & 0 & & & & \\ B_3 & B_2\times E & B_1\times E\times E & 0 & & & \\ \cdot & \cdot & \cdot & & & & \cdot \\ B_{\rho-1} & B_{\rho-2}\times E & B_{\rho-3}\times E\times E & \cdot & \cdot & \cdot & B_1\times E\times E\times\cdots\times E \end{bmatrix}.$$

Now, we can define left and right multiplication operators a_r, b_l of F by the formula (6), (7) respectively. Then we get

THEOREM 3. *Let a_r and b_l be the multiplication operators of F ($F^2\neq 0$) defined by* (6) *and* (7) *respectively, and if the product $b_l\ a_r$ becomes automorphism of F, then, in the representation* (6) *and* (7), *A_1 and B_1 are scalar matrices and $A_1B_1=E$. That is, b_la_r is a monic automorphism. And B_i are uniquely determined by A for $i=1, 2, \cdots, \rho-2$.*

Proof. We use the representation (6) and (7) directly. Then

$$\begin{bmatrix} B_1 & 0 & 0 & & 0 \\ B_2 & B_1\times E & 0 & & \cdot \\ B_3 & B_2\times E & B_2\times E\times E & & \cdot \\ \cdot & \cdot & \cdot\ \cdot & & \cdot \\ \cdot & \cdot & \cdot\ \ \cdot & & \\ \cdot & \cdot & \cdot & \cdot & 0 \\ B_{\rho-1} & B_{\rho 2}\times E & B_{\rho-3}E\times E & & B_1\times E\times\cdots\times E \end{bmatrix}$$

$$\cdot\begin{bmatrix} A_1 & 0 & 0 & & 0 \\ A_2 & E\times A_1 & 0 & & 0 \\ A_3 & E\times A_2 & E\times E\times A_1 & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot \\ A_{\rho-1} & E\times A_{\rho-2} & E\times E\times A_{\rho-3} & \cdot\;\cdot & E\times E\times\cdots\times A_1 \end{bmatrix}=[D_{ij}].$$

In order that the product D corresponds to an automorphism, D must be of form as in (3). Therefore, the matrix

$$(B_1\times E)(E\times A_1)=B_1\times A_1$$

must be equal to the matrix $B_1A_1\times B_1A_1$. Hence,

$$A_1\times A_1^{-1}BA_1=E.$$

Consequently, A_1 and B_1 are scalar matrices and the product of them is equal to E. Next, we shall prove that B_i are uniquely determined by A for $i=1, 2, \cdots, \rho-2$. For $i=1$, we have proved above.

$$A_1=\lambda E \quad\text{and}\quad B_1=\frac{1}{\lambda}.$$

We assume that for all numbers $i<s$, B_i is uniquely determined by $A_1, A_2, \cdots, A_i$. Then,

$$D_{s,1}=B_sA_1+g(A_1, A_2, \cdots, A_{s-1}),$$

$$D_{s+1,2}=(B_s\times E)(E\times A_1)+f(A_1, A_2, \cdots, A_{s-1}).$$

Comparing with (3), we have

$$(9)\qquad D_{s+1,2}=(B_sA_1+g(A_1, A_2, \cdots, A_{s-1}()\times E+E\times(BA+g(A, A, \cdots, A_{s-1}))$$
$$+h(A_1\, A_2, \cdots, A_{s-2}).$$

From (9) and (10), the assertion is proved.

COROLLARY. *A bound algebra* $\mathfrak{A}$ *over* K *possessing a free nilpotent algebra* F *as its radical and unity element* 1 *must be* $K+F$, *if* $F^2\neq 0$.

Proof. From our assumption on K, $\mathfrak{A}$ splits: $\mathfrak{A}=S+F$. Let us consider the regular representations of $\mathfrak{A}$ by use of a basis of S and the normal basis of F. Then for any regular element a of S, by theorem 3, there exists an element λ of K such that $(a-\lambda)_r$ and $(a-\lambda)_l$ induce nilpotent multiplicative operations on F.

The set of elements in S which are nilpotent multiplicative operators on F form a nilpotent ideal of S. This follows from the assumption that $\mathfrak{A}$ is bound to F. Hence we have $a-\lambda=0$. Since S is generated by regular elements we get $\mathfrak{A}=K+F$.

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