KŌDAI MATH. SEM. REP.
20 (1968), 318—354

DIFFERENTIAL GEOMETRY OF TANGENT BUNDLES OF ORDER 2

BY KENTARO YANO AND SHIGERU ISHIHARA

Dedicated to Professor Shisanji Hokari on his Sixtieth Birthday

§ 0. Introduction.

The differential geometry of tangent bundles has been studied by Davies [13], Dombrowski [1], Kobayashi [15], Ledger [2], [3], [16], Morimoto [4], [5], Okumura [8], Sasaki [6], Tachibana [8], Tanno [9], Tondeur [10], the present authors [2], [3], [11], [13], [14], [15], [16], [17], [18] and others and that of cotangent bundles by Patterson [17], [18], Satô [7] and one of the present authors [12], [17], [18].[1]

The purpose of the present paper is to study the differential geometry of tangent bundles of order 2, the tangent bundle of order 2 $T_2(M)$ of a differentiable manifold M being defined as the set of all 2-jets of M determined by mappings of the real line R into M.

In § 1, we define the tangent bundles of order 2 and induced coordinates in it and fix the notations used throughout the paper.

In § 2, we study the lifts of functions and two vector fields A and B existing a priori in $T_2(M)$.

§ 3 is devoted to the study of lifts of vector fields, 1-forms and derivations, and § 4 to the study of lifts of tensor fields and two linear mappings α and β. In § 5, we give the local expressions of these lifts.

In § 6, we study in more detail the lifts of tensor fields of type (1, 1) and discuss lifts of torsion tensors and Nijenhuis tensors.

§ 7 is devoted to the study of lifts of affine connections and also of curvature tensor and torsion tensor of the connection.

We study lifts of infinitesimal transformations in § 8 and geodesics in $T_2(M)$ in the last § 9.

CONTENTS

Received November 27, 1967.

1) The numbers in brackets [ ] refer to Bibliography at the end of the paper.

§ 1. Tangent bundles of order 2.

Let M be a differentiable manifold[2)] of dimension n and R the real line. We introduce an equivalence relation $\sim$ in the set of all differentiable mappings F: $R\to M$ as follows. Let $r\geqq 1$ be a fixed integer. If two differentiable mappings F: $R\to M$ and G: $R\to M$ satisfy the conditions[3)]

$$(1.1)\qquad F^h(0)=G^h(0),\quad \frac{dF^h(0)}{dt}=\frac{dG^h(0)}{dt},\quad \cdots,\quad \frac{d^rF^h(0)}{dt^r}=\frac{d^rG^h(0)}{dt^r},$$

the mappings F and G being respectively represented by $x^h=F^h(t)$ and $x^h=G^h(t)$ $(t\in R)$ with respect to local coordinates (x^h) defined in a coordinate neighborhood containing the point $F^h(0)=G^h(0)$, then we say that the two mappings F and G are *equivalent* to each other and write $F\sim G$. Each equivalence class determined by the equivalence relation $\sim$ is called briefly an *r-jet* of M and denoted by $j^r_{\mathrm{P}}(F)$ if this class contains a mapping F: $R\to M$ such that $F(0)=\mathrm{P}$. The point P is called the *target* of the r-jet $j^r_{\mathrm{P}}(F)$. In the sequel, we shall restrict ourselves to the case $r=1$ or $r=2$.

If we denote by $T_2(M)$ the set of all 2-jets of M and topologize $T_2(M)$ in the natural way, the space $T_2(M)$ has the natural bundle structure over M, its bundle projection π_2: $T_2(M)\to M$ being defined by $\pi_2(j^2_{\mathrm{P}}(F))=\mathrm{P}$. The space $T_2(M)$ is called the *tangent bundle of order* 2 over M.

The set $T_1(M)$ of all 1-jets of M is nothing but the tangent bundle of M, if $T_1(M)$ is naturally topologized. The bundle projection π_1: $T_1(M)\to M$ of $T_1(M)$ is defined by $\pi_1(j^1_{\mathrm{P}}(F))=\mathrm{P}$. Each 1-jet of M is called a tangent vector of M. If we introduce a mapping π_{12}: $T_2(M)\to T_1(M)$ by $\pi_{12}(j^2_{\mathrm{P}}(F))=j^1_{\mathrm{P}}(F)$, F: $R\to M$ being an arbitrary differentiable mapping such that $F(0)=\mathrm{P}$, then $T_2(M)$ has a bundle structure over $T_1(M)$ with bundle projection π_{12}. It is easily verified that the relation

$$(1.2)\qquad \pi_2=\pi_1\circ\pi_{12}$$

holds.

Let U be a coordinate neighborhood of M and (x^h) certain coordinates defined in U. We call the set $(U, (x^h))$ simply a *coordinate neighborhood* of M. If we take an arbitrary 2-jet $j^2_{\mathrm{P}}(F)$ belonging to $\pi_2^{-1}(U)$ and put

2) Manifolds, mappings and objects we discuss are assumed to be differentiable and of class C^∞. Manifolds under consideration are supposed to be connected.

3) The indices $h, i, j, k, \cdots, m, t, s$ run over the range $\{1, 2, \cdots, n\}$ and the so-called Einstein's summation convention is used with respect to this system of indices.

(1. 3) $$y^h=\frac{dF^h(0)}{dt}, \qquad z^h=\frac{d^2F^h(0)}{dt^2},$$

then we see from (1. 1) that the 2-jet $j^2_P(F)$ is expressed in a unique way by the set (x^h, y^h, z^h), where x^h are the coordinates of the target P in $(U, (x^h))$. Thus a system of coordinates (x^h, y^h, z^h) is introduced in the open set $\pi_2^{-1}(U)$ of $T_2(M)$. We call (x^h, y^h, z^h) the *coordinates induced in* $\pi_2^{-1}(U)$ from $(U, (x^h))$, or, simply the *induced coordinates* in $\pi_2^{-1}(U)$. On putting

(1. 4) $$\xi^h=x^h, \qquad \xi^{\bar{h}}=y^h, \qquad \xi^{\bar{\bar{h}}}=z^h,$$

we denote the induced coordinates (x^h, y^h, z^h) by (ξ^A) in $\pi_2^{-1}(U)$.[4]

Let $(U, (x^h))$ and $(U', (x^{h'}))$ be two intersecting coordinate neighborhoods of M. Let $(\xi^A)=(x^h, y^h, z^h)$ and $(\xi^{A'})=(x^{h'}, y^{h'}, z^{h'})$ be the coordinates induced respectively from $(U, (x^h))$ and $(U', (x^{h'}))$. Then, denoting by $x^{h'}=x^{h'}(x^i)$ the coordinate transformation in $U\cap U'$, the transformation of the induced coordinates in $\pi_2^{-1}(U\cap U')$ is given by

(1. 5) $$x^{h'}=x^{h'}(x^i), \qquad y^{h'}=\frac{\partial x^{h'}}{\partial x^h}y^h, \qquad z^{h'}=\frac{\partial x^{h'}}{\partial x^h}z^h+\frac{\partial^2 x^{h'}}{\partial x^j\partial x^i}y^j y^i,$$

and its Jacobian matrix by

(1. 6) $$\begin{pmatrix} \dfrac{\partial x^{h'}}{\partial x^h} & 0 & 0 \\ \dfrac{\partial^2 x^{h'}}{\partial x^i\partial x^s}y^s & \dfrac{\partial x^{i'}}{\partial x^i} & 0 \\ \dfrac{\partial^2 x^{h'}}{\partial x^j\partial x^s}z^s+\dfrac{\partial^3 x^{h'}}{\partial x^j\partial x^t\partial x^s}y^t y^s & 2\dfrac{\partial^2 x^{i'}}{\partial x^j\partial x^s}y^s & \dfrac{\partial x^{j'}}{\partial x^j} \end{pmatrix}.$$

Let φ: $M\to M$ be a differentiable transformation. The correspondence $j^2_P(F)\to j^2_{\varphi(P)}(\varphi\circ F)$, $j^2_P(F)\in T_2(M)$ determines a differentiable transformation φ^*: $T_2(M)\to T_2(M)$, called the *transformation induced in* $T_2(M)$ *from* φ. If we take a point P belonging to a coordinate neighborhood $(U, (x^h))$, and, if we suppose that the point φ(P) belongs to a coordinate neighborhood $(U', (x^{h'}))$, then we can express φ locally by equations

(1. 7) $$x^{h'}=\varphi^{h'}(x^h),$$

$\varphi^{h'}(x^h)$ being n differentiable functions of the variables x^h such that $|\partial\varphi^{h'}/\partial x^h|\neq 0$, where (x^h) are the coordinates of P in $(U, (x^h))$ and $(x^{h'})$ those of φ(P) in $(U', (x^{h'}))$. Then the induced transformation φ^* is expresssd locally by equations of the form

4) The indices A, B, C, D, E run over the symbols $\{1, 2, \cdots, n;\ \bar{1}, \bar{2}, \cdots, \bar{n};\ \bar{\bar{1}}, \bar{\bar{2}}, \cdots, \bar{\bar{n}}\}$ and the so-called Einstein's summation convention is used with respect to this system of indices.

(1. 8)
$$x^{h'}=\varphi^{h'}(x^h), \qquad y^{h'}=\frac{\partial\varphi^{h'}}{\partial x^h}y^h,$$
$$z^{h'}=\frac{\partial\varphi^{h'}}{\partial x^h}z^h+\frac{\partial^2\varphi^{h'}}{\partial x^j\partial x^i}y^jy^i,$$

where (x^h, y^h, z^h) are the induced coordinates of $j^2_P(F)$ in $\pi_2^{-1}(U)$ and $(x^{h'}, y^{h'}, z^{h'})$ those of $\varphi^*(j^2_P(F))$ in $\pi_2^{-1}(U')$.

Let X be an infinitesimal transformation (a vector field) in M. Then, taking account of (1. 8), we see easily that there naturally corresponds an infinitesimal transformation $\tilde{X}$ in $T_2(M)$ having components of the form

(1. 9)
$$\tilde{X}^h=X^h, \qquad \tilde{X}^{\bar{h}}=y^i\partial_iX^h, \qquad \tilde{X}^{\bar{\bar{h}}}=z^i\partial_iX^h+y^jy^i\partial_j\partial_iX^h$$

in $\pi_2^{-1}(U)$, the functions X^h being the components of X in $(U, (x^h))$ and ∂_i denoting the operator

$$\partial_i=\frac{\partial}{\partial x^i}.$$

Hence we have the relation

(1. 10)
$$(\exp(tX))^*=\exp(t\tilde{X}) \qquad (t\in R),$$

whenever $\exp(tX)$ is defined.

If we put $Y=fX$, f and X being respectively a function and a vector field in M, then we find in $T_2(M)$

$$\tilde{Y}=\tilde{f}\tilde{X}+2\tilde{g}\tilde{U}+\tilde{h}\tilde{V},$$

$\tilde{X}$ and $\tilde{Y}$ being constructed by (1. 9) respectively from X and Y, where $\tilde{U}$ and $\tilde{V}$ are vector fields having respectively components of the form

(1. 11)
$$\tilde{U}:\ \tilde{U}^h=0, \qquad \tilde{U}^{\bar{h}}=\frac{1}{2}y^i\partial_iX^h, \qquad \tilde{U}^{\bar{\bar{h}}}=X^h;$$

(1. 12)
$$\tilde{V}:\ \tilde{V}^h=0, \qquad \tilde{V}^{\bar{h}}=0, \qquad \tilde{V}^{\bar{\bar{h}}}=X^h$$

in $\pi_2^{-1}(U)$ and $\tilde{f}=f\circ\pi_{12}$, $\tilde{g}=y^i\partial_if$, $\tilde{h}=z^i\partial_if+y^jy^i\partial_j\partial_if$ with respect to the induced coordinates (x^h, y^h, z^h) in $\pi_2^{-1}(U)$. Therefore, given a vector field X in M, we obtain in $T_2(M)$ three vector fields $\tilde{X}$, $\tilde{U}$, and $\tilde{V}$ defined by (1. 9), (1. 11) and (1. 12) respectively.

Notations.

We list below notations used frequently in this paper.

1. $\mathcal{T}^r_s(M)$ is the space of all tensor fields of type (r, s), i.e., of contravariant degree r and covariant degree s, in a differentiable manifold M. An element of

$\mathcal{T}_0^0(M)$ is a function in M, an element of $\mathcal{T}_0^1(M)$ is a vector field in M, and an element of $\mathcal{T}_1^0(M)$ is a 1-form in M.

2.
$$\mathcal{T}(M)=\sum_{r,s}\mathcal{T}_s^r(M).$$

3. $\Lambda_*(M)$ is the space of all differential forms in M. Λ_s is the space of all s-forms in M.

$$\Lambda_*(M)=\sum_s\Lambda_s(M),\qquad \Lambda_s(M)=\Lambda_*(M)\cap\mathcal{T}_s^0(M).$$

4. A mapping φ: $\mathcal{T}(M)\to\mathcal{T}(M')$ is said to be *linear* if we have $\varphi(aS+bT)=a\varphi(S)+b\varphi(T)$ for any element $S, T\in\mathcal{T}(M)$, where a and b are constants.

§ 2. Lifts of functions.

Lifts of functions. Let f be a function in M. Then f is a mapping f: $M\to R$ and it gives a mapping $f\circ F$: $R\to R$. For the given function f a 2-jet $j_a^2(f\circ F)$ of R is completely determined by giving a 2-jet $j_{\mathrm{P}}^2(F)$, F being a mapping F: $R\to M$ such that $\mathrm{P}=F(0)$ and $a=f(\mathrm{P})$. Thus, if we put $f^*(j_{\mathrm{P}}^2(F))=j_a^2(f\circ F)$, there exists a mapping f^*: $T_2(M)\to T_2(R)$ corresponding to f. On the other hand, any element τ of $T_2(R)$ can be expressed canonically by a set $(A^0(\tau), A^{\mathrm{I}}(\tau), A^{\mathrm{II}}(\tau))$ of three numbers, which are the induced coordinates of τ in $T_2(R)$, because R is covered naturally by only one coordinate neighborhood R itself. Therefore, for a function f given in M, there corresponds in $T_2(M)$ three functions f^0, f^{I} and f^{II} respectively defined by

$$(2.1)\qquad f^0(\sigma)=A^0(f^*(\sigma)),\qquad f^{\mathrm{I}}(\sigma)=A^{\mathrm{I}}(f^*(\sigma)),\qquad f^{\mathrm{II}}(\sigma)=A^{\mathrm{II}}(f^*(\sigma)),$$

σ being an arbitrary element of $T_2(M)$. The three functions f^0, f^{I} and f^{II} thus defined in $T_2(M)$ is called respectively the *0-th*, the *1st* and the *2nd lifts* of f. *A function f in M is constant if and only if one of its lifts f^{I} and f^{II} vanishes identically in $T_2(M)$. A function f in M vanishes identically if and only if its lift f^0 does so in $T_2(M)$.*

The lifts f^0, f^{I} and f^{II} of a function f in M expressed by $f(x^h)$ in $(U,(x^h))$ are represented respectively as

$$(2.2)\qquad f^0\colon f(x^h),\qquad f^{\mathrm{I}}\colon y^i\partial_i f(x^h),\qquad f^{\mathrm{II}}\colon z^i\partial_i f(x^h)+y^jy^i\partial_j\partial_i f(x^h)$$

with respect to the induced coordinates $(\xi^A)=(x^h, y^h, z^h)$ in $\pi_2^{-1}(U)$. We note here that f^0 has in $\pi_2^{-1}(U)$ the same local representation as f has in $(U,(x^h))$.

Taking account of (2.2), we find

$$(2.3)\qquad f^0=f\circ\pi_2=(f^V)\circ\pi_{12},\qquad f^{\mathrm{I}}=(f^C)\circ\pi_{12}$$

for $f\in\mathcal{T}_0^0(M)$, where the functions f^V and f^C defined in $T_1(M)$ are respectively the vertical and the complete lifts of f in the sense of [14] and [15]. As consequences of (2. 2), we find the following formulas:

$$(fg)^0=g^0f^0, \qquad (fg)^{\mathrm{I}}=f^{\mathrm{I}}g^0+f^0g^{\mathrm{I}}, \tag{2. 4}$$
$$(fg)^{\mathrm{II}}=f^{\mathrm{II}}g^0+2f^{\mathrm{I}}g^{\mathrm{I}}+f^0g^{\mathrm{II}}$$

for $g, f\in\mathcal{T}_0^0(M)$.

REMARK. Let $\tilde{X}$ be a vector field in $T_2(M)$. Then $\tilde{X}$ vanishes identically in $T_2(M)$ if we have $\tilde{X}f^{\mathrm{II}}=0$ for any function f in M. In fact, if we take account of (2. 2) and denote by $(\tilde{X}^A)=(\tilde{X}^h, \tilde{X}^{\bar{h}}, \tilde{X}^{\bar{\bar{h}}})$ the components of $\tilde{X}$ with respect to the induced coordinates $(\xi^A)=(x^h, y^h, z^h)$, we see that the condition $\tilde{X}f^{\mathrm{II}}=0$ is expressed as

$$\tilde{X}^k(z^i\partial_k\partial_i f+y^jy^i\partial_k\partial_j\partial_i f)+2\tilde{X}^{\bar{k}}y^i\partial_k\partial_i f+\tilde{X}^{\bar{\bar{k}}}\partial_k f=0.$$

Thus, if we have $\tilde{X}f^{\mathrm{II}}=0$ for any element f of $\mathcal{T}_0^0(M)$, we find $\tilde{X}^h=\tilde{X}^{\bar{h}}=\tilde{X}^{\bar{\bar{h}}}=0$ by virtue of the continuity of $\tilde{X}$. Consequently, a vector field $\tilde{X}$ in $T_2(M)$ is completely determined by giving the values of $\tilde{X}f^{\mathrm{II}}$, f being arbitrary elements of $\mathcal{T}_0^0(M)$. In the sequel, this remark will be useful in determining values of vector fields given in $T_2(M)$.

Vector fields A and B. We now consider in each $\pi_2^{-1}(U)$ two local vector fields A and B respectively with components of the form

$$A\colon \begin{pmatrix} 0 \\ 0 \\ y^h \end{pmatrix}, \qquad B\colon \begin{pmatrix} 0 \\ \frac{1}{2}y^h \\ z^h \end{pmatrix} \tag{2. 5}$$

with respect to the induced coordinates (ξ^A), $(U, (x^h))$ being an arbitrary coordinate neighborhood of M. Taking account of (1. 5) and (1. 6), we can easily verify that both of the local vector fields A and B thus introduced determine respectively global vector fields in $T_2(M)$, which are also denoted by A and B respectively. We now obtain the following formulas:

$$Af^0=0, \qquad Af^{\mathrm{I}}=0, \qquad Af^{\mathrm{II}}=f^{\mathrm{I}}, \tag{2. 6}$$
$$Bf^0=0, \qquad Bf^{\mathrm{I}}=\frac{1}{2}f^{\mathrm{I}}, \qquad Bf^{\mathrm{II}}=f^{\mathrm{II}}$$

for $f\in\mathcal{T}_0^0(M)$ and

$$[A, B]=\frac{1}{2}A \tag{2. 7}$$

by virtue of (2. 2) and (2. 5).

§ 3. Lifts of vector fields, 1-forms and derivations.

Lifts of vector fields. Let X be a vector field in M. We introduce in each $\pi_2^{-1}(U)$ three local vector fields X^0, X^{I} and X^{II} having respective components of the form

$$X^0=\begin{pmatrix}0\\ 0\\ X^h\end{pmatrix},\qquad X^{\mathrm{I}}=\begin{pmatrix}0\\ \frac{1}{2}X^h\\ y^i\partial_i X^h\end{pmatrix},\qquad X^{\mathrm{II}}=\begin{pmatrix}X^h\\ y^i\partial_i X^h\\ y^i\partial_i X^h+y^j y^i\partial_j\partial_i X^h\end{pmatrix} \tag{3. 1}$$

with respect to the induced coordinates (ξ^A), where X^h denote the components of X in $(U,(x^h))$ (Cf. (1. 9), (1. 11) and (1. 12)). If we take account of (1. 5), (1. 6) and the transformation law $X^{h'}=(\partial x^{h'}/\partial x^h)X^h$ of the components of X, then we see that the local vector fields X^0, X^{I} and X^{II} above determine respectively global vector fields in $T_2(M)$, which are also denoted by X^0, X^{I} and X^{II} respectively. The vector fields X^0, X^{I} and X^{II} in $T_2(M)$ are called respectively the *0-th*, the *1st* and the *2nd lifts* of X. We find

$$\pi_{12}(X^0)=0,\qquad \pi_{12}(X^{\mathrm{I}})=\frac{1}{2}X^V,\qquad \pi_{12}(X^{\mathrm{II}})=X^C \tag{3. 2}$$

for $X\in\mathcal{T}_0^1(M)$ because of (3. 1), π_{12} denoting the differential mapping of the projection π_{12}: $T_2(M)\to T_1(M)$, where the vector fields X^V and X^C defined in $T_1(M)$ denote respectively the vertical and the complete lifts of X in the sense of [14] and [15]. According to (3. 1), *a vector field X in M vanishes identically if and only if one of X^0, X^{I} and X^{II} does so in $T_2(M)$.*

Taking account of (3. 1), we find the following formulas:

$$\begin{gathered}(fX)^0=f^0X^0,\qquad (fX)^{\mathrm{I}}=f^{\mathrm{I}}X^0+f^0X^{\mathrm{I}},\\ (fX)^{\mathrm{II}}=f^{\mathrm{II}}X^0+2f^{\mathrm{I}}X^{\mathrm{I}}+f^0X^{\mathrm{II}}\end{gathered} \tag{3. 3}$$

for $f\in\mathcal{T}_0^0(M)$, $X\in\mathcal{T}_0^1(M)$. As immediate consequences of (2. 2) and (3. 1), we have the following formulas:

$$\begin{array}{lll} X^0f^0=0, & X^0f^{\mathrm{I}}=0, & X^0f^{\mathrm{II}}=(Xf)^0,\\ X^{\mathrm{I}}f^0=0, & X^{\mathrm{I}}f^{\mathrm{I}}=\frac{1}{2}(Xf)^0, & X^{\mathrm{I}}f^{\mathrm{II}}=(Xf)^{\mathrm{I}},\\ X^{\mathrm{II}}f^0=(Xf)^0, & X^{\mathrm{II}}f^{\mathrm{I}}=(Xf)^{\mathrm{I}}, & X^{\mathrm{II}}f^{\mathrm{II}}=(Xf)^{\mathrm{II}}\end{array} \tag{3. 4}$$

for $f\in\mathcal{T}_0^0(M)$, $X\in\mathcal{T}_0^1(M)$.

Lifts of 1-forms. Let ω be a 1-form in M. We introduce in each $\pi_2^{-1}(U)$ three local 1-forms ω^0, ω^{I} and ω^{II} having respective components of the form

$$\omega^0:\quad (\omega_i, 0, 0),$$
$$(3.5)\qquad \omega^{\mathrm{I}}:\quad (y^k\partial_k\omega_i, \omega_i, 0),$$
$$\omega^{\mathrm{II}}:\quad (z^k\partial_k\omega_i + y^k y^j\partial_k\partial_j\omega_i, 2y^j\partial_j\omega_i, \omega_i)$$

with respect to the induced coordinates (ξ^A), where ω_i denote the components of ω in $(U, (x^h))$. Taking account of (1. 5), (1. 6) and the transformation law $\omega_{i'} = (\partial x^i/\partial x^{i'})\omega_i$ of components of ω, we can easily verify that the local 1-forms ω^0, ω^{I} and ω^{II} above determine respectively global 1-forms in $T_2(M)$, which are also denoted respectively by ω^0, ω^{I} and ω^{II}. These 1-forms ω^0, ω^{I} and ω^{II} are respectively called the *0-th*, the *1st* and the *2nd lifts* of ω. From (3. 5) we find

$$(3.6)\qquad \omega^0 = \omega\circ\pi_2 = \omega^V\circ\pi_{12},\qquad \omega^{\mathrm{I}} = \omega^C\circ\pi_{12}$$

for $\omega\in\mathcal{T}_0^1(M)$, where the 1-forms ω^V and ω^C defined in $T_2(M)$ are respectively the vertical and the complete lifts of ω in the sense of [14] and [15]. According to (3. 5), *a 1-form ω vanishes identically in M if and only if one of ω^0, ω^{I} and ω^{II} does so in $T_2(M)$.*

Taking account of (3. 5), we obtain the formulas

$$(3.7)\qquad (f\omega)^0 = f^0\omega^0,\qquad (f\omega)^{\mathrm{I}} = f^{\mathrm{I}}\omega^0 + f^0\omega^{\mathrm{I}},$$
$$(f\omega)^{\mathrm{II}} = f^{\mathrm{II}}\omega^0 + 2f^{\mathrm{I}}\omega^{\mathrm{I}} + f^0\omega^{\mathrm{II}}$$

for $f\in\mathcal{T}_0^0(M)$, $\omega\in\mathcal{T}_1^0(M)$. As immediate consequences of (3. 1) and (3. 5), we find the following formulas:

$$(3.8)\qquad \begin{array}{lll} \omega^0(X^0)=0, & \omega^0(X^{\mathrm{I}})=0 & \omega^0(X^{\mathrm{II}})=(\omega(X))^0, \\ \omega^{\mathrm{I}}(X^0)=0, & \omega^{\mathrm{I}}(X^{\mathrm{I}})=\frac{1}{2}(\omega(X))^0, & \omega^{\mathrm{I}}(X^{\mathrm{II}})=(\omega(X))^{\mathrm{I}}, \\ \omega^{\mathrm{II}}(X^0)=(\omega(X))^0, & \omega^{\mathrm{II}}(X^{\mathrm{I}})=(\omega(X))^{\mathrm{I}}, & \omega^{\mathrm{II}}(X^{\mathrm{II}})=(\omega(X))^{\mathrm{II}} \end{array}$$

for $X\in\mathcal{T}_0^1(M)$, $\omega\in\mathcal{T}_1^0(M)$.

Formulas. We have here the following formulas:

$$(3.9)\qquad \begin{array}{ll} [X^0, Y^0]=0, & [X^{\mathrm{I}}, Y^{\mathrm{I}}]=\frac{1}{2}[X, Y]^0, \\ [X^{\mathrm{I}}, Y^0]=0, & [X^{\mathrm{II}}, Y^{\mathrm{I}}]=[X, Y]^{\mathrm{I}}, \\ [X^{\mathrm{II}}, Y^0]=[X, Y]^0, & [X^{\mathrm{II}}, Y^{\mathrm{II}}]=[X, Y]^{\mathrm{II}} \end{array}$$

for $X, Y\in\mathcal{T}_0^1(M)$. In fact, taking account of (3. 4), we have

$$[X^{\mathrm{II}}, Y^{\mathrm{I}}]f^{\mathrm{II}} = X^{\mathrm{II}}(Y^{\mathrm{I}}f^{\mathrm{II}}) - Y^{\mathrm{I}}(X^{\mathrm{II}}f^{\mathrm{II}}) = (X(Yf)^{\mathrm{I}} - (Y(Xf))^{\mathrm{I}}$$
$$= ([X, Y]f)^{\mathrm{I}} = [X, Y]^{\mathrm{I}}f^{\mathrm{II}},$$

$$[X^{\mathrm{II}}, Y^{\mathrm{II}}]f^{\mathrm{II}}=([X, Y]f)^{\mathrm{II}}=[X, Y]^{\mathrm{II}}f^{\mathrm{II}}$$

for any element f of $\mathcal{T}_0^0(M)$. Therefore, if we take account of the Remark stated in § 2, we obtain $[X^{\mathrm{II}}, Y^{\mathrm{I}}]=[X, Y]^{\mathrm{I}}$ and $[X^{\mathrm{II}}, Y^{\mathrm{II}}]=[X, Y]^{\mathrm{II}}$. Applying similar devices, we can prove the other formulas given in (3. 9).

The correspondences $X\rightarrow X^0$, $X\rightarrow X^{\mathrm{I}}$ and $X\rightarrow X^{\mathrm{II}}$ $(X\in\mathcal{T}_0^1(M))$ determine respectively one-to-one linear mappings of $\mathcal{T}_0^1(M)$ into $\mathcal{T}_0^1(T_2(M))$. We have, from the last formulas given in (3. 9),

PROPOSITION 3. 1. *The correspondence* $X\rightarrow X^{\mathrm{II}}$ $(X\in\mathcal{T}_0^1(M))$ *determines an isomorphism of the Lie algebra* $\mathcal{T}_0^1(M)$ *into the Lie algebra* $\mathcal{T}_0^1(T_2(M))$.

According to (3. 1) and (3. 5), we find in each neighborhood $\pi_2^{-1}(U)$ the formulas

$$\left(\frac{\partial}{\partial x^i}\right)^0=\frac{\partial}{\partial z^i},\qquad \left(\frac{\partial}{\partial x^i}\right)^{\mathrm{I}}=\frac{1}{2}\frac{\partial}{\partial y^i},\qquad \left(\frac{\partial}{\partial x^i}\right)^{\mathrm{II}}=\frac{\partial}{\partial x^i};$$
$$(3.10)\qquad (dx^h)^0=dx^h,\qquad (dx^h)^{\mathrm{I}}=dy^h,\qquad (dx^h)^{\mathrm{II}}=dz^h$$

with respect to the induced coordinateg $(\xi^A)=(x^h, y^h, z^h)$, where $(U, (x^h))$ is a coordinate neighborhood of M.

REMARK. If we take account of (3. 1) and (3. 5), we see that a tensor field K, say, of type (1, 2) in $T_2(M)$ is completely determined by giving values $K(X^{\mathrm{II}}, Y^{\mathrm{II}}, \omega^{\mathrm{II}})$, X and Y being arbitray elements of $\mathcal{T}_0^1(M)$ and ω an arbitrary element of $\mathcal{T}_1^0(M)$.

Lifts of derivations. In this paper we mean by a *derivation* in M a linear mapping D: $\mathcal{T}(M)\rightarrow\mathcal{T}(M)$ satisfying the conditions:

(3. 11)

(a) D: $\mathcal{T}_s^r(M)\rightarrow\mathcal{T}_s^r(M)$,

(b) $D(S\otimes T)=(DS)\otimes T+S\otimes(DT)$ for $S, T\in\mathcal{T}(M)$,

(c) $DI=0$,

where I denotes the identity tensor field of type (1, 1) in M.

For a given derivation D in M, there exists a vector field P in M such that

$$(3.12)\qquad Pf=Df,$$

f being an arbitrary element of $\mathcal{T}_0^0(M)$. In each coordinate neighborhood $(U, (x^h))$ of M, taking account of (3. 11, a), we can put

$$(3.13)\qquad D\left(\frac{\partial}{\partial x^i}\right)=Q_i{}^h\frac{\partial}{\partial x^h},$$

$Q_i{}^h$ being certain functions in U. Thus, taking account of (3. 11, b), (3. 12) and

(3.13), we obtain

$$D\left(X^h \frac{\partial}{\partial x^h}\right) = (P^i \partial_i X^h + Q_i{}^h X^i) \frac{\partial}{\partial x^h}$$

in $(U, (x^h))$ for any element $X = X^h(\partial/\partial x^h)$ of $\mathcal{T}_0^1(M)$. That is to say, for any element X of $\mathcal{T}_0^1(M)$, DX has components of the form

$$(DX)^h = P^i \partial_i X^h + Q_i{}^h X^i \tag{3.14}$$

in $(U, (x^h))$, if X has components X^h in $(U, (x^h))$. According to (3.11), we have $D(\omega(X)) = (D\omega)(X) + \omega(DX)$ for any element X of $\mathcal{T}_0^1(M)$ and any element ω of $\mathcal{T}_1^0(M)$. Thus, as a consequence of (3.14), $D\omega$ has components of the form

$$(D\omega)_i = P^j \partial_j \omega_i - Q_i{}^h \omega_h \qquad \text{for} \quad \omega \in \mathcal{T}_1^0(M) \tag{3.15}$$

in $(U, (x^h))$, if ω has components ω_i in $(U, (x^h))$. The set $(P^h, Q_i{}^h)$ is called the *components of the derivation* D in $(U, (x^h))$.

We suppose that a derivation D has components $(P^h, Q_i{}^h)$ and $(P^{h'}, Q_{i'}{}^{h'})$ respectively in $(U, (x^h))$ and in $(U', (x^{h'}))$. Then, as a consequence of (3.14) and the transformation law $X^{h'} = (\partial x^{h'}/\partial x^h) X^h$ of the components X^h of X, we obtain the transformation law

$$\begin{aligned} P^{h'} &= \frac{\partial x^{h'}}{\partial x^h} P^h, \\ Q_{i'}{}^{h'} &= \frac{\partial x^{h'}}{\partial x^h}\left(\frac{\partial x^i}{\partial x^{i'}} Q_i{}^h + P^j \frac{\partial x^{j'}}{\partial x^j} \frac{\partial_2 x^h}{\partial x^{j'} \partial x^{i'}}\right) \end{aligned} \tag{3.16}$$

of the components of a derivation D in $U \cap U'$.

If we are given a derivation D in M, we introduce in $\pi_2^{-1}(U)$ three local vector fields D^0, D^{I} and D^{II} having components of the form

$$D^0: \begin{pmatrix} 0 \\ 0 \\ P^h \end{pmatrix}, \qquad D^{\mathrm{I}}: \begin{pmatrix} 0 \\ \frac{1}{2} P^h \\ -y^i Q_i{}^h \end{pmatrix}, \qquad D^{\mathrm{II}}: \begin{pmatrix} P^h \\ \frac{1}{2} y^i (\partial_i P^h - Q_i{}^h) \\ -(z^i Q_i{}^h + y^j y^i \partial_j Q_i{}^h) \end{pmatrix} \tag{3.17}$$

with respect to the induced coordinates (ξ^A), where $(P^h, Q_i{}^h)$ denote the components of the given derivation D in $(U, (x^h))$. Thus, taking account of (1.5), (1.6), (3.16) and (3.17), we see that all of the local vector fields D^0, D^{I} and D^{II} above determine respectively global vector fields in $T_2(M)$, which are denoted also by D^0, D^{I} and D^{II} respectively. These three vector fields D^0, D^{I} and D^{II} in $T_2(M)$ are called respectively the *0-th*, the *1st* and the *2nd lifts* of the derivation D.

We now find for any derivation D the following formulas:

$$
(3.18)\qquad
\begin{array}{lll}
D^0 f^0 = 0, & D^{\mathrm{I}} f^0 = 0, & D^{\mathrm{II}} f^0 = (Df)^0, \\
D^0 f^{\mathrm{I}} = 0, & D^{\mathrm{I}} f^{\mathrm{I}} = \dfrac{1}{2}(Df)^0, & D^{\mathrm{II}} f^{\mathrm{I}} = \alpha(Ddf), \\
D^0 f^{\mathrm{II}} = (Df)^0, & D^{\mathrm{I}} f^{\mathrm{II}} = \alpha(Ddf), & D^{\mathrm{II}} f^{\mathrm{II}} = \beta(Ddf)
\end{array}
$$

for $f \in \mathcal{T}^0_0(M)$, where $\alpha\omega$ and $\beta\omega$ for any element ω of $\mathcal{T}^0_1(M)$ are functions in $T_2(M)$ having respectively local representations $\alpha\omega = y^i\omega_i$ and $\beta\omega = z^i\omega_i + y^j y^i \partial_j \omega_i$ in $\pi_2^{-1}(U)$ with respect to the induced coordinates (ξ^A), the functions ω_i being the components of ω in $(U, (x^h))$ (Cf. § 5).

Lifts of Lie derivations. The Lie derivation $\mathcal{L}_X$ with respect to a vector field X is a derivation having components of the form

$$
(3.19)\qquad \mathcal{L}_X:\quad P^h = X^h, \qquad Q_i{}^h = -\partial_i X^h,
$$

where X^h denote the components of X. Thus, substituting (3. 19) in (3. 17), we have

PROPOSITION 3. 2. *The formulas*

$$
(\mathcal{L}_X)^0 = X^0, \qquad (\mathcal{L}_X)^{\mathrm{I}} = X^{\mathrm{I}}, \qquad (\mathcal{L}_X)^{\mathrm{II}} = X^{\mathrm{II}}
$$

hold for $X \in \mathcal{T}^1_0(M)$.

Lifts of covariant derivations. Let ∇ be an affine connection in M. Then the covariant differentiation ∇_X with respect to a vector field X is a derivation in M, which has components of the form

$$
(3.20)\qquad \nabla_X:\quad P^h = X^h, \qquad Q_i{}^h = X^j \Gamma_j{}^h{}_i,
$$

$\Gamma_j{}^h{}_i$ denoting the coefficients of ∇ and X^h the components of X. The covariant derivative $\nabla_X Z$ has components of the form

$$
(\nabla_X Z)^h = X^j(\partial_j Z^h + \Gamma_j{}^h{}_i Z^i)
$$

for any vector field Z with components Z^h. Substituting (3. 20) in (3. 17), we see that the lifts $(\nabla_X)^0$, $(\nabla_X)^{\mathrm{I}}$ and $(\nabla_X)^{\mathrm{II}}$ have respectively components of the form

$$
(3.21)\qquad
\begin{array}{l}
(\nabla_X)^0:\ \begin{pmatrix} 0 \\ 0 \\ X^h \end{pmatrix}, \qquad (\nabla_X)^{\mathrm{I}}:\ \begin{pmatrix} 0 \\ \dfrac{1}{2} X^h \\ -X^j y^i \Gamma_j{}^h{}_i \end{pmatrix}, \\[2ex]
(\nabla_X)^{\mathrm{II}}:\ \begin{pmatrix} X^h \\ \dfrac{1}{2} y^i(\partial_i X^h - X^j \Gamma_j{}^h{}_i) \\ -(X^j z^i \Gamma_j{}^h{}_i + y^j y^i \partial_j (X^k \Gamma_k{}^h{}_i) \end{pmatrix}
\end{array}
$$

for any element X of $\mathcal{T}_0^1(M)$. Therefore we have, from (3. 1) and (3. 21),

PROPOSITION 3. 3. *The formulas*

$$(\nabla_X)^0 = X^0, \qquad (\nabla_X)^{\mathrm{I}} = X^{\mathrm{I}} - \alpha(\hat{\nabla} X), \qquad (\nabla_X)^{\mathrm{II}} = X^{\mathrm{II}} - \beta(\hat{\nabla} X)$$

hold for any element X *of* $\mathcal{T}_0^1(M)$.

In Proposition 3. 3, $\hat{\nabla}$ is an affine connection in M defined by

$$\hat{\nabla}_X Y = \nabla_Y X + [X, Y] \qquad \text{for} \quad X, Y \in \mathcal{T}_0^1(M),$$

and αF and βF for any element F of $\mathcal{T}_1^1(M)$ are vector fields in $T_2(M)$ having respectively components

$$(3.\,22) \qquad \alpha F: \begin{pmatrix} 0 \\ 0 \\ y^i F_i{}^h \end{pmatrix}, \qquad \beta F: \begin{pmatrix} 0 \\ \frac{1}{2} y^i F_i{}^h \\ z^i F_i{}^h + y^j y^i \partial_j F_i{}^h \end{pmatrix}$$

with respect to the induced coordinates (ξ^A) in $\pi_2^{-1}(U)$, the functions $F_i{}^h$ being components of F in $(U, (x^h))$ (Cf. § 4 or § 5). We see easily that the affine connection $\hat{\nabla}$ has coefficients $\hat{\Gamma}_j{}^h{}_i = \Gamma_i{}^h{}_j$, $\Gamma_j{}^h{}_i$ being the coefficients of ∇. As an immediate consequence of Proposition 3. 3, we have

PROPOSITION 3. 4. *For any element* X *of* $\mathcal{T}_0^1(M)$

$$(\nabla_X)^{\mathrm{I}} = X^{\mathrm{I}}, \qquad (\nabla_X)^{\mathrm{II}} = X^{\mathrm{II}}$$

hold if and only if $\hat{\nabla} X = 0$.

Derivation determined by a tensor field of type (1, 1). When a derivation D satisfies the condition $Df = 0$ for $f \in \mathcal{T}_0^0(M)$, D determines an element F of $\mathcal{T}_1^1(M)$ such that $DX = FX$ for any element X of $\mathcal{T}_0^1(M)$. In such a case, we denote D by D_F and call it the *derivation determined by a tensor field* F *of type* (1, 1). The derivation D_F has components of the form

$$(3.\,23) \qquad D_F: \quad P^h = 0, \qquad Q_i{}^h = F_i{}^h,$$

$F_i{}^h$ being components of F. Substituting (3. 23) in (3. 17), we find

$$(3.\,24) \qquad (D_F)^0 = 0, \qquad (D_F)^{\mathrm{I}} = -\alpha F, \qquad (D_F)^{\mathrm{II}} = -\beta F,$$

αF and βF being defined by (3. 22).

§ 4. Lifts of tensor fields.

Lifts of tensor fields. We have introduced in § 2 and § 3 three kinds of lifts for functions, vector fields and 1-forms given in M. The operations taking these lifts are linear mappings $\mathcal{T}_0^0(M)\to\mathcal{T}_0^0(T_2(M))$, $\mathcal{T}_0^1(M)\to\mathcal{T}_0^1(T_2(M))$ and $\mathcal{T}_1^0(M)\to\mathcal{T}_1^0(T_2(M))$ respectively. Thus we can now define for any element K of $\mathcal{T}_s^r(M)$ its lifts K^0, K^{I} and K^{II}, which are elements of $\mathcal{T}_s^r(T_2(M))$, in such a way that the correspondence $K\to K^0$, $K\to K^{\mathrm{I}}$ and $K\to K^{\mathrm{II}}$ all define linear mappings $\mathcal{T}_s^r(M)\to\mathcal{T}_s^r(T_2(M))$, which are characterized by the properties

$$
\begin{aligned}
&(S\otimes T)^0=S^0\otimes T^0,\\
(4.1)\qquad &(S\otimes T)^{\mathrm{I}}=S^{\mathrm{I}}\otimes T^0+S^0\otimes T^{\mathrm{I}},\\
&(S\otimes T)^{\mathrm{II}}=S^{\mathrm{II}}\otimes T^0+2S^{\mathrm{I}}\otimes T^{\mathrm{I}}+S^0\otimes T^{\mathrm{II}}
\end{aligned}
$$

for $S, T\in\mathcal{T}(M)$. The conditions (4. 1) are compatible with the conditions (2. 4), (3. 3) and (3. 7). The tensor fields K^0, K^{I} and K^{II} are called respectively the 0-*th*, the 1*st* and the 2*nd lifts* of K. We see that *a tensor field* K, *not belonging to* $\mathcal{T}_0^0(M)$, *vanishes identically in* M *if and only if one of its lifts* K^0, K^{I} *and* K^{II} *does so in* $T_2(M)$.

Linear mappings γ_X. Let T be an element of $\mathcal{T}_s^r(M)$ $(s\geqq 1)$. Then it is a correspondence

$$T:\ (X_1, \cdots, X_s)\to T(X_1, \cdots, X_s)\in\mathcal{T}_0^r(M),$$

$X_1, \cdots, X_s$ being arbitrary elements of $\mathcal{T}_0^1(M)$. If for an element X of $\mathcal{T}_0^1(M)$ we define an element $\gamma_X T$ of $\mathcal{T}_{s-1}^r(M)$ by

$$(\gamma_X T)(X_2, \cdots, X_s)=T(X, X_2, \cdots, X_s),$$

$X_2, \cdots, X_s$ being arbitrary elements of $\mathcal{T}_0^1(M)$, then the correspondence $T\to\gamma_X T$ determines a mapping γ_X: $\mathcal{T}_s^r(M)\to\mathcal{T}_{s-1}^r(M)$ such that $\gamma_X(fT+gS)=f(\gamma_X T)+g(\gamma_X S)$ for $f, g\in\mathcal{T}_0^0(M)$ and $T, S\in\mathcal{T}_s^r(M)$. If T has components of the form $T_{i_1 i_2\cdots i_s}{}^{h_1\cdots h_r}$, then $\gamma_X T$ has the components $X^k T_{k i_2\cdots i_s}{}^{h_1\cdots h_r}$, X^k being components of X. We have the formula

$$\gamma_{X_s}\cdots\gamma_{X_1} T=T(X_1, \cdots, X_s)\in\mathcal{T}_0^r(M)$$

for any elements $X_1, \cdots, X_s$ of $\mathcal{T}_0^1(M)$.

We now have the following formulas:

$$
\begin{aligned}
&\gamma_{X^0} K^0=0, &&\gamma_{X^{\mathrm{I}}} K^0=0, &&\gamma_{X^{\mathrm{II}}} K^0=(\gamma_X K)^0,\\
(4.2)\qquad &\gamma_{X^0} K^{\mathrm{I}}=0, &&\gamma_{X^{\mathrm{I}}} K^{\mathrm{I}}=\frac{1}{2}(\gamma_X K)^0, &&\gamma_{X^{\mathrm{II}}} K^{\mathrm{I}}=(\gamma_X K)^{\mathrm{I}},\\
&\gamma_{X^0} K^{\mathrm{II}}=(\gamma_X K)^0, &&\gamma_{X^{\mathrm{I}}} K^{\mathrm{II}}=(\gamma_X K)^{\mathrm{I}}, &&\gamma_{X^{\mathrm{II}}} K^{\mathrm{II}}=(\gamma_X K)^{\mathrm{II}}
\end{aligned}
$$

for $X\in\mathcal{T}_0^1(M)$, $K\in\mathcal{T}(M)$. In fact, if we suppose that $K=\omega\otimes S$, $\omega\in\mathcal{T}_1^0(M)$, $S\in\mathcal{T}(M)$, then we have

$$\gamma_{X^0}K^0=\gamma_{X^0}(\omega^0\otimes S^0)=\omega^0(X^0)S^0=0,$$

$$\gamma_{X^{\mathrm{I}}}K^{\mathrm{I}}=\gamma_{X^{\mathrm{I}}}(\omega^{\mathrm{I}}\otimes S^0+\omega^0\otimes S^{\mathrm{I}})=\frac{1}{2}(\omega(X))^0S^0$$
$$=\frac{1}{2}(\gamma_X(\omega\otimes S))^0=\frac{1}{2}(\gamma_X K)^0,$$

$$\gamma_{X^{\mathrm{II}}}K^{\mathrm{II}}=\gamma_{X^{\mathrm{II}}}(\omega^{\mathrm{II}}\otimes S^0+2\omega^{\mathrm{I}}\otimes S^{\mathrm{I}}+\omega^0\otimes S^{\mathrm{II}})$$
$$=((\omega(X))^{\mathrm{II}}S^0+2(\omega(X))^{\mathrm{I}}S^{\mathrm{I}}+(\omega(X))^0S^{\mathrm{II}})$$
$$=(\omega(X)S)^{\mathrm{II}}=(\gamma_X(\omega\otimes S))^{\mathrm{II}}=(\gamma_X K)^{\mathrm{II}}$$

by virtue of (3. 8) and (4. 1). Thus, according to $\gamma_X(fS+gT)=f\gamma_X S+g\gamma_X T$ for $S, T\in\mathcal{T}_s^r(M)$ and $f, g\in\mathcal{T}_0^0(M)$, we can prove these three formulas for any element K of $\mathcal{T}(M)$. In a similar way, we can prove the other formulas given in (4. 2).

Lifts of differential forms. We now obtain the following formulas:

$$(4.\ 3)\qquad (\omega\wedge\pi)^0=\omega^0\wedge\pi^0,\qquad (\omega\wedge\pi)^{\mathrm{I}}=\omega^{\mathrm{I}}\wedge\pi^0+\omega^0\wedge\pi^{\mathrm{I}},$$
$$(\omega\wedge\pi)^{\mathrm{II}}=\omega^{\mathrm{II}}\wedge\pi^0+2\omega^{\mathrm{I}}\wedge\pi^{\mathrm{I}}+\omega^0\wedge\pi^{\mathrm{II}}$$

for $\omega, \pi\in\Lambda_*(M)$. Moreover we have the following formulas:

$$\omega^0(X^{\mathrm{II}}, Y^{\mathrm{II}}, \cdots, Z^{\mathrm{II}})=(\omega(X, Y, \cdots, Z))^0,$$
$$(4.\ 4)\qquad \omega^{\mathrm{I}}(X^{\mathrm{II}}, Y^{\mathrm{II}}, \cdots, Z^{\mathrm{II}})=(\omega(X, Y, \cdots, Z))^{\mathrm{I}},$$
$$\omega^{\mathrm{II}}(X^{\mathrm{II}}, Y^{\mathrm{II}}, \cdots, Z^{\mathrm{II}})=(\omega(X, Y, \cdots, Z))^{\mathrm{II}}$$

for $\omega\in\Lambda_*(M)$, $X, Y, \cdots, Z$ being arbitrary element of $\mathcal{T}_0^1(M)$. The formulas (4. 4) are immediate consequences of (4. 2).

We obtain directly from (2. 2) and (3. 5)

$$(4.\ 5)\qquad (df)^0=d(f^0),\qquad (df)^{\mathrm{I}}=d(f^{\mathrm{I}}),\qquad (df)^{\mathrm{II}}=d(f^{\mathrm{II}})$$

for $f\in\mathcal{T}_0^0(M)$. We next have the following formulas:

$$(4.\ 6)\qquad (d\omega)^0=d(\omega^0),\qquad (d\omega)^{\mathrm{I}}=d(\omega)^{\mathrm{I}},\qquad (d\omega)^{\mathrm{II}}=d(\omega^{\mathrm{II}})$$

for $\omega\in\mathcal{T}_1^0(M)$. In fact, taking account of (3. 4), (3. 8) and (3. 9), we have

$$2(d\omega^0)(X^{\mathrm{II}}, Y^{\mathrm{II}})=X^{\mathrm{II}}\omega^0(Y^{\mathrm{II}})-Y^{\mathrm{II}}\omega^0(X^{\mathrm{II}})-\omega^0([X^{\mathrm{II}}, Y^{\mathrm{II}}])$$
$$=(X\omega(Y)-Y\omega(X)-\omega([X, Y]))^0$$
$$=2((d\omega)(X, Y))^0=2(d\omega)^0(X^{\mathrm{II}}, Y^{\mathrm{II}}).$$

Therefore, according to the Remark stated in § 3, we have $(d\omega)^0=d(\omega^0)$. By similar devices, we have the other formulas given in (4.6).

If we consider a differential form ω which has the local expression $\omega=fdx^{i_1}\wedge \cdots\wedge dx^{i_s}$, $f\in\mathcal{T}_0^0(U)$ in $(U,(x^h))$, we obtain

$$d\omega=df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_s}$$

and hence, taking account of (4.3), (4.5) and (4.6),

$$\begin{aligned}(d\omega)^{\mathrm{I}}&=(df)^{\mathrm{I}}\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_s})^0+(df)^0\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_s})^{\mathrm{I}}\\&=(df^{\mathrm{I}})\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_s})^0+(df^0)\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_s})^{\mathrm{I}}\\&=d(f^{\mathrm{I}}(dx^{i_1}\wedge\cdots\wedge dx^{i_s})^0+f^0(dx^{i_1}\wedge\cdots\wedge dx^{i_s})^{\mathrm{I}})\\&=d(fdx^{i_1}\wedge\cdots\wedge dx^{i_s})^{\mathrm{I}}=d(\omega^{\mathrm{I}})\end{aligned}$$

by virtue of (3.10). Therefore, taking account of the identity $(\omega+\pi)^{\mathrm{I}}=\omega^{\mathrm{I}}+\pi^{\mathrm{I}}$ for $\omega,\pi\in\Lambda_*(M)$, we have $(d\omega)^{\mathrm{I}}=d(\omega^{\mathrm{I}})$ for any element ω of $\Lambda_*(M)$. Similarly, we obtain $(d\omega)^0=d(\omega^0)$ and $(d\omega)^{\mathrm{II}}=d(\omega^{\mathrm{II}})$ for any element ω of $\Lambda_*(M)$. Thus we have

PROPOSITION 4.1. *The formulas*

$$(d\omega)^0=d(\omega^0),\qquad (d\omega)^{\mathrm{I}}=d(\omega^{\mathrm{I}}),\qquad (d\omega)^{\mathrm{II}}=d(\omega^{\mathrm{II}})$$

hold for any element ω *of* $\Lambda_*(M)$.

Lie derivatives with respect to lifts. Denoting by $\mathcal{L}_X$ the operator of Lie derivation with respect to a vector field X, we have directly from (3.4)

$$\begin{aligned}&\mathcal{L}_{X^0}f^0=0, &&\mathcal{L}_{X^0}f^{\mathrm{I}}=0, &&\mathcal{L}_{X^0}f^{\mathrm{II}}=(\mathcal{L}_Xf)^0,\\ &\mathcal{L}_{X^{\mathrm{I}}}f^0=0, &&\mathcal{L}_{X^{\mathrm{I}}}f^{\mathrm{I}}=\frac{1}{2}(\mathcal{L}_Xf)^0, &&\mathcal{L}_{X^{\mathrm{I}}}f^{\mathrm{II}}=(\mathcal{L}_Xf)^{\mathrm{I}},\\ &\mathcal{L}_{X^{\mathrm{II}}}f^0=(\mathcal{L}_Xf)^0, &&\mathcal{L}_{X^{\mathrm{II}}}f^{\mathrm{I}}=(\mathcal{L}_Xf)^{\mathrm{I}}, &&\mathcal{L}_{X^{\mathrm{II}}}f^{\mathrm{II}}=(\mathcal{L}_Xf)^{\mathrm{II}}\end{aligned}\tag{4.7}$$

for $f\in\mathcal{T}_0^0(M)$, $X\in\mathcal{T}_0^1(M)$ and directly from (3.9)

$$\begin{aligned}&\mathcal{L}_{X^0}Y^0=0, &&\mathcal{L}_{X^0}Y^{\mathrm{I}}=0, &&\mathcal{L}_{X^0}Y^{\mathrm{II}}=(\mathcal{L}_XY)^0,\\ &\mathcal{L}_{X^{\mathrm{I}}}Y^0=0, &&\mathcal{L}_{X^{\mathrm{I}}}Y^{\mathrm{I}}=\frac{1}{2}(\mathcal{L}_XY)^0, &&\mathcal{L}_{X^{\mathrm{I}}}Y^{\mathrm{II}}=(\mathcal{L}_XY)^{\mathrm{I}},\\ &\mathcal{L}_{X^{\mathrm{II}}}Y^0=(\mathcal{L}_XY)^0, &&\mathcal{L}_{X^{\mathrm{II}}}Y^{\mathrm{I}}=(\mathcal{L}_XY)^{\mathrm{I}}, &&\mathcal{L}_{X^{\mathrm{II}}}Y^{\mathrm{II}}=(\mathcal{L}_XY)^{\mathrm{II}}\end{aligned}\tag{4.8}$$

for $X, Y\in\mathcal{T}_0^1(M)$. We have now the following formulas:

$$\begin{aligned}(4.9)\qquad &\mathcal{L}_{X^0}\,\omega^0=0, && \mathcal{L}_{X^0}\,\omega^{\mathrm{I}}=0, && \mathcal{L}_{X^0}\,\omega^{\mathrm{II}}=(\mathcal{L}_X\,\omega)^0,\\ &\mathcal{L}_{X^{\mathrm{I}}}\,\omega^0=0, && \mathcal{L}_{X^{\mathrm{I}}}\,\omega^{\mathrm{I}}=\frac{1}{2}(\mathcal{L}_X\,\omega)^0, && \mathcal{L}_{X^{\mathrm{I}}}\,\omega^{\mathrm{II}}=(\mathcal{L}_X\,\omega)^{\mathrm{I}},\\ &\mathcal{L}_{X^{\mathrm{II}}}\,\omega^0=(\mathcal{L}_X\,\omega)^0, && \mathcal{L}_{X^{\mathrm{II}}}\,\omega^{\mathrm{I}}=(\mathcal{L}_X\,\omega)^{\mathrm{I}}, && \mathcal{L}_{X^{\mathrm{II}}}\,\omega^{\mathrm{II}}=(\mathcal{L}_X\,\omega)^{\mathrm{II}}\end{aligned}$$

for $X\in\mathcal{T}_0^1(M)$, $\omega\in\mathcal{T}_1^0(M)$. In fact, taking an arbitrary vector field Y in M, we have

$$\begin{aligned}(\mathcal{L}_{X^0}\,\omega^0)(Y^{\mathrm{II}})&=\mathcal{L}_{X^0}\,(\omega^0(Y^{\mathrm{II}}))-\omega^0(\mathcal{L}_{X^0}\,Y^{\mathrm{II}})=0,\\ (\mathcal{L}_{X^{\mathrm{I}}}\,\omega^{\mathrm{I}})(Y^{\mathrm{II}})&=\mathcal{L}_{X^{\mathrm{I}}}\,(\omega^{\mathrm{I}}(Y^{\mathrm{II}}))-\omega^{\mathrm{I}}(\mathcal{L}_{X^{\mathrm{I}}}\,Y^{\mathrm{II}})=\frac{1}{2}(\mathcal{L}_X\,(\omega(Y))-\omega(\mathcal{L}_X\,Y))^0\\ &=\frac{1}{2}((\mathcal{L}_X\,\omega)(Y))^0=\frac{1}{2}(\mathcal{L}_X\,\omega)^0(Y^{\mathrm{II}}),\\ (\mathcal{L}_{X^{\mathrm{II}}}\,\omega^{\mathrm{II}})(Y^{\mathrm{II}})&=\mathcal{L}_{X^{\mathrm{II}}}\,(\omega^{\mathrm{II}}(Y^{\mathrm{II}}))-\omega^{\mathrm{II}}(\mathcal{L}_{X^{\mathrm{II}}}\,Y^{\mathrm{II}})=(\mathcal{L}_X(\omega(Y))-\omega(\mathcal{L}_X\,Y))^{\mathrm{II}}\\ &=((\mathcal{L}_X\,\omega)(Y))^{\mathrm{II}}=(\mathcal{L}_X\,\omega)^{\mathrm{II}}(Y^{\mathrm{II}})\end{aligned}$$

by virtue of (3. 4), (3. 8), (4. 7) and (4. 8). Consequently, Y being arbitrary, we find $\mathcal{L}_{X^0}\,\omega^0=0$, $\mathcal{L}_{X^{\mathrm{I}}}\,\omega^{\mathrm{I}}=(1/2)(\mathcal{L}_X\,\omega)^0$, $\mathcal{L}_{X^{\mathrm{II}}}\,\omega^{\mathrm{II}}=(\mathcal{L}_X\,\omega)^{\mathrm{II}}$. Similarly, we obtain the other formulas given in (4. 9). We have here

PROPOSITION 4. 2. *For any element K of $\mathcal{T}(M)$ the formulas*

$$\begin{aligned}&\mathcal{L}_{X^0}\,K^0=0, && \mathcal{L}_{X^0}\,K^{\mathrm{I}}=0 && \mathcal{L}_{X^0}\,K^{\mathrm{II}}=(\mathcal{L}_X\,K)^0,\\ &\mathcal{L}_{X^{\mathrm{I}}}\,K^0=0, && \mathcal{L}_{X^{\mathrm{I}}}\,K^{\mathrm{I}}=\frac{1}{2}(\mathcal{L}_X\,K)^0, && \mathcal{L}_{X^{\mathrm{I}}}\,K^{\mathrm{II}}=(\mathcal{L}_X\,K)^{\mathrm{I}},\\ &\mathcal{L}_{X^{\mathrm{II}}}\,K^0=(\mathcal{L}_X\,K)^0, && \mathcal{L}_{X^{\mathrm{II}}}\,K^{\mathrm{I}}=(\mathcal{L}_X\,K)^{\mathrm{I}}, && \mathcal{L}_{X^{\mathrm{II}}}\,K^{\mathrm{II}}=(\mathcal{L}_X\,K)^{\mathrm{II}}\end{aligned}$$

hold, X being an arbitrary element of $\mathcal{T}_0^1(M)$.

Proof. These formulas have been already proved in (4. 7), (4. 8) and (4. 9) respectively for K belonging to $\mathcal{T}_0^0(M)$, $\mathcal{T}_0^1(M)$ or $\mathcal{T}_1^0(M)$. Then we assume that these formulas are established for K belonging to $\mathcal{T}_s^r(M)$, where $r\leqq p$, $s\leqq q$. Taking an arbitrary element S of $\mathcal{T}_m^l(M)$ and an element T of $\mathcal{T}_{q-m}^{p-l}(M)$, we have

$$\begin{aligned}\mathcal{L}_{X^0}\,(S\otimes T)^0&=\mathcal{L}_{X^0}\,(S^0\otimes T^0)=(\mathcal{L}_{X^0}\,S^0)\otimes T^0+S^0\otimes(\mathcal{L}_{X^0}\,T^0)=0,\\ \mathcal{L}_{X^{\mathrm{I}}}\,(S\otimes T)^{\mathrm{I}}&=\mathcal{L}_{X^{\mathrm{I}}}\,(S^{\mathrm{I}}\otimes T^0+S^0\otimes T^{\mathrm{I}})\\ &=(\mathcal{L}_{X^{\mathrm{I}}}\,S^{\mathrm{I}})\otimes T^0+S^{\mathrm{I}}\otimes(\mathcal{L}_{X^{\mathrm{I}}}\,T^0)+(\mathcal{L}_{X^{\mathrm{I}}}\,S^0)\otimes T^{\mathrm{I}}+S^0\otimes(\mathcal{L}_{X^{\mathrm{I}}}\,T^{\mathrm{I}})\\ &=\frac{1}{2}((\mathcal{L}_X\,S)^0\otimes T^0+S^0\otimes(\mathcal{L}_X\,T)^0)\\ &=\frac{1}{2}((\mathcal{L}_X\,S)\otimes T+S\otimes(\mathcal{L}_X\,T))^0=\frac{1}{2}(\mathcal{L}_X\,S\otimes T)^0,\end{aligned}$$

$$\mathcal{L}_{X^{\rm II}}(S\otimes T)^{\rm II}=\mathcal{L}_{X^{\rm II}}(S^{\rm II}\otimes T^{0}+2S^{\rm I}\otimes T^{\rm I}+S^{0}\otimes T^{\rm II})$$

$$=(\mathcal{L}_{X^{\rm II}}S^{\rm II})\otimes T^{0}+S^{\rm II}\otimes(\mathcal{L}_{X^{\rm II}}T^{0})+2(\mathcal{L}_{X^{\rm II}}S^{\rm I})\otimes T^{\rm I}$$

$$+2S^{\rm I}\otimes(\mathcal{L}_{X^{\rm II}}T^{\rm I})+(\mathcal{L}_{X^{\rm II}}S^{0})\otimes T^{\rm II}+S^{0}\otimes(\mathcal{L}_{X^{\rm II}}S^{\rm II})$$

$$=((\mathcal{L}_{X}S)\otimes T+S\otimes(\mathcal{L}_{X}T))^{\rm II}=(\mathcal{L}_{X}S\otimes T)^{\rm II}$$

by virtue of (4. 1). Similarly, we can prove the other formulas given in Proposition 4. 2 for $K=S\otimes T$. Consequently, we have proved Proposition 4. 2 as consequences of $\mathcal{L}_{X}(S+T)=\mathcal{L}_{X}S+\mathcal{L}_{X}T$ for $S, T\in\mathcal{T}_{s}^{r}(M)$.

Linear mappings α *and* β. We shall define a linear mapping α: $\mathcal{T}_{s}^{r}(M)\rightarrow\mathcal{T}_{s-1}^{r}(T_{2}(M))$ ($s\geqq 1$). Let T be an element of $\mathcal{T}_{s}^{r}(M)$. Then $T^{\rm II}$ is a correspondence

$$T^{\rm II}:\quad(\tilde{X}_{1},\cdots,\tilde{X}_{s})\rightarrow T^{\rm II}(\tilde{X}_{1},\cdots,\tilde{X}_{s})\in\mathcal{T}_{0}^{r}(T_{2}(M)),$$

$\tilde{X}_{1},\cdots,\tilde{X}_{s}$ being arbitrary elements of $\mathcal{T}_{0}^{1}(T_{2}(M))$. If we consider a correspondence αT such that

$$\alpha T:\quad(\tilde{X}_{2},\cdots,\tilde{X}_{s})\rightarrow T^{\rm II}(A,\tilde{X}_{2},\cdots,\tilde{X}_{s})\in\mathcal{T}_{0}^{r}(T_{2}(M)),\tag{4. 10}$$

$\tilde{X}_{2},\cdots,\tilde{X}_{s}$ being arbitrary elements of $\mathcal{T}_{0}^{1}(T_{2}(M))$ and A the vector field defined by (2. 5). Then αT is an element of $\mathcal{T}_{s-1}^{r}(T_{2}(M))$. Then the correspondence α: $T\rightarrow\alpha T$ determines a linear mapping α: $\mathcal{T}_{s}^{r}(M)\rightarrow\mathcal{T}_{s-1}^{r}(T_{2}(M))$. Thus we have from (4. 10)

$$\begin{aligned}\alpha\omega&=\omega^{\rm II}(A)\qquad\text{for}\quad\omega\in\mathcal{T}_{1}^{0}(M),\\ \alpha df&=f^{\rm I}\qquad\text{for}\quad f\in\mathcal{T}_{0}^{0}(M).\end{aligned}\tag{4. 11}$$

When T has the form $T=\omega\otimes S$, $\omega\in\mathcal{T}_{1}^{0}(M)$, $S\in\mathcal{T}(M)$, taking account of (4. 1), we find

$$\alpha T=(\alpha\omega)S^{0}\qquad(T=\omega\otimes S)\tag{4. 12}$$

because of the formulas

$$\omega^{0}(A)=0,\qquad\omega^{\rm I}(A)=0,\qquad\omega^{\rm II}(A)=\alpha\omega\tag{4. 13}$$

for $\omega\in\mathcal{T}_{1}^{0}(M)$, which are direct consequences of (2. 5) and (3. 5).

We shall next define a linear mapping β: $\mathcal{T}_{s}^{r}(M)\rightarrow\mathcal{T}_{s-1}^{r}(T_{2}(M))$ ($s\geqq 1$). Let T be an element of $\mathcal{T}_{s}^{r}(M)$. If we consider a correspondence

$$\beta T:\quad(\tilde{X}_{2},\cdots,\tilde{X}_{s})\rightarrow T^{\rm II}(B,\tilde{X}_{2},\cdots,\tilde{X}_{s})\in\mathcal{T}_{s-1}^{r}(T_{2}(M)),\tag{4. 14}$$

$\tilde{X}_{2},\cdots,X_{s}$ being arbitrary elements of $\mathcal{T}_{0}^{1}(T_{2}(M))$ and B the vector field defined by (2. 5), then βT is an element of $\mathcal{T}_{s-1}^{r}(T_{2}(M))$. Thus the correspondence β: $T\rightarrow\beta T$ defines a linear mapping β: $\mathcal{T}_{s}^{r}(M)\rightarrow\mathcal{T}_{s-1}^{r}(T_{2}(M))$. We have now

$$\beta\omega=\omega^{\mathrm{II}}(B) \qquad \text{for} \quad \omega\in\mathcal{T}_1^0(M), \tag{4.15}$$
$$\beta(df)=f^{\mathrm{II}} \qquad \text{for} \quad f\in\mathcal{T}_0^0(M).$$

When T has the form $T=\omega\otimes S$, $\omega\in\mathcal{T}_1^0(M)$, $S\in\mathcal{T}(M)$, taking account of (4. 1), we obtain

$$\beta T=(\beta\omega)S^0+(\alpha\omega)S^{\mathrm{I}} \tag{4.16}$$

by virtue of the formulas

$$\omega^0(B)=0, \qquad \omega^{\mathrm{I}}(B)=\frac{1}{2}\alpha\omega, \qquad \omega^{\mathrm{II}}(B)=\beta\omega \tag{4.17}$$

for $\omega\in\mathcal{T}_1^0(M)$, which are direct consequences of (2. 5) and (3. 5).

§ 5. Local expressions.

In this section, we would like to find local expressions of the lifts of tensor fields in M. By components of a tensor field T in M we always mean those of T in coordinate neighborhood $(U,(x^h))$ of M and by components of a tensor field $\tilde{T}$ in $T_2(M)$ those of $\tilde{T}$ with respect to the induced coordinates $(\xi^A)=(x^h, y^h, z^h)$ in $\pi_2^{-1}(U)$. The local expression of a function f, a vector field X and a 1-form ω have been already given by (2. 2), (3. 1) and (3. 5) respectively.

Tensor fields of type (1, 1). Let F be an element of $\mathcal{T}_1^1(M)$, which is expressed by

$$F=F_i{}^h dx^i\otimes\frac{\partial}{\partial x^h}$$

in $(U, x^h))$. Taking the 0-th lift, we find

$$F^0=\left(F_i{}^h dx^i\otimes\frac{\partial}{\partial x^h}\right)^0$$
$$=(F_i{}^h)^0 dx^i\otimes\frac{\partial}{\partial y^h}$$

by virtue of (3. 10) and (4. 1). Taking the 1st lift, we have

$$F^{\mathrm{I}}=\left(F_i{}^h dx^i\otimes\frac{\partial}{\partial x^h}\right)^{\mathrm{I}}$$
$$=(F_i{}^h)^0\left(dy^i\otimes\frac{\partial}{\partial x^h}+\frac{1}{2}dx^i\otimes\frac{\partial}{\partial y^h}\right)+(F_i{}^h)^{\mathrm{I}}dy^i\otimes\frac{\partial}{\partial z^h}$$

by virtue of (3. 10) and (4. 1). Taking the 2nd lift, we obtain

$$F^{\mathrm{II}}=\left(F_i{}^h dx^i\otimes\frac{\partial}{\partial x^h}\right)^{\mathrm{II}}$$
$$=(F_i{}^h)^0\left(dz^i\otimes\frac{\partial}{\partial y^h}+dy^i\otimes\frac{\partial}{\partial y^h}+dx^i\otimes\frac{\partial}{\partial x^h}\right)$$
$$+(F_i{}^h)^{\mathrm{I}}\left(2dy^i\otimes\frac{\partial}{\partial z^h}+dx^i\otimes\frac{\partial}{\partial y^h}\right)+(F_i{}^h)^{\mathrm{II}}\left(dx^i\otimes\frac{\partial}{\partial z^h}\right)$$

by virtue of (3. 10) and (4. 1). Therefore we see that the lifts F^0, F^{I} and F^{II} of F have respectively the components of the form

(5. 1)
$$F^0:\ \begin{pmatrix}0&0&0\\0&0&0\\F_i{}^h&0&0\end{pmatrix},\qquad F^{\mathrm{I}}:\ \begin{pmatrix}0&0&0\\\frac{1}{2}F_i{}^h&0&0\\y^s\partial_sF_i{}^h&F_i{}^h&0\end{pmatrix},$$
$$F^{\mathrm{II}}:\ \begin{pmatrix}F_i{}^h&0&0\\y^s\partial_sF_i{}^h&F_i{}^h&0\\z^s\partial_sF_i{}^h+y^ty^s\partial_t\partial_sF_i{}^h&2y^s\partial_sF_i{}^h&F_i{}^h\end{pmatrix},$$

where $F_i{}^h$ denote the components of F. We have from (5. 1)

PROPOSITION 5. 1. *A tensor field F of type* (1, 1) *is of rank r, if and only if F^0 is of rank r, if and only if F^{I} is of rank $2r$, or, if and only if F^{II} is of rank $3r$.*

Let I be the identity tensor field of type (1, 1). Then, substituting $F_i{}^h=\delta_i^h$ in (5. 1), we find

(5. 2)
$$I^0:\ \begin{pmatrix}0&0&0\\0&0&0\\I&0&0\end{pmatrix},\qquad I^{\mathrm{I}}:\ \begin{pmatrix}0&0&0\\\frac{1}{2}I&0&0\\0&I&0\end{pmatrix},\qquad I^{\mathrm{II}}:\ \begin{pmatrix}I&0&0\\0&I&0\\0&0&I\end{pmatrix}.$$

Therefore *the 2nd lift I^{II} of the identity tensor field I of type* (1, 1) *is the identity tensor field of type* (1, 1) *in* $T_2(M)$. We have from (3. 1) and (5. 2)

(5. 3)
$$I^0X^0=0,\qquad I^0X^{\mathrm{I}}=0,\qquad I^0X^{\mathrm{II}}=X^0,$$
$$I^{\mathrm{I}}X^0=0,\qquad I^{\mathrm{I}}X^{\mathrm{I}}=\frac{1}{2}X^0,\qquad I^{\mathrm{I}}X^{\mathrm{II}}=X^{\mathrm{II}}.$$

for $X\in\mathcal{T}_0^1(M)$.

Tensor fields of type (0, 2). Let g be an element of $\mathcal{T}_2^0(M)$. Then we can easily verify that its lifts g^0, g^{I} and g^{II} have respectively components of the form

(5. 4)
$$g^0:\ \begin{pmatrix} g_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\qquad g^{\mathrm{I}}:\ \begin{pmatrix} y^s\partial_s g_{ji} & g_{ji} & 0 \\ g_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$g^{\mathrm{II}}:\ \begin{pmatrix} z^s\partial_s g_{ji}+y^t y^s\partial_t\partial_s g_{ji} & 2y^s\partial_s g_{ji} & g_{ji} \\ 2y^s\partial_s g_{ji} & 2y^s\partial_s g_{ji} & 0 \\ g_{ji} & 0 & 0 \end{pmatrix},$$

where g_{ji} denote the components of g.

Given an element $\tilde{h}$ of $\mathcal{T}^0_2(T_2(M))$, we denote by

$$\tilde{h}(d\xi, d\xi)=\tilde{h}_{CB}d\xi^C d\xi^B$$

the quadratic differential form corresponding to $\tilde{h}$, if $\tilde{h}$ is symmetric, $\tilde{h}_{CB}$ being the components of $\tilde{h}$. Let g be a *pseudo-Riemannian metric* in M. Then, taking account of (5. 4), we obtain

(5. 5)
$$g^0(d\xi, d\xi)=g_{ji}dx^j dx^i,$$
$$g^{\mathrm{I}}(d\xi, d\xi)=2g_{ji}dx^j\delta y^i,$$
$$g^{\mathrm{II}}(d\xi, d\xi)=2g_{ji}dx^j\delta v^i+2g_{ji}\delta y^j\delta y^i,$$

the differential forms δy^h and δz^h being defined respectively by

(5. 6)
$$\delta y^h=dy^h+\left\{ {h \atop s\ i} \right\}y^s dx^i,$$
$$\delta v^h=d\left(z^h+y^t y^s\left\{ {h \atop t\ s} \right\}\right)+\left\{ {h \atop i\ l} \right\}\left(z^l+y^t y^s\left\{ {l \atop t\ s} \right\}\right)dx^i$$
$$=dz^h+2\left\{ {h \atop i\ t} \right\}y^t dy^i+\left[y^t y^s\left(\partial_i\left\{ {h \atop t\ s} \right\}+\left\{ {h \atop i\ l} \right\}\left\{ {l \atop t\ s} \right\}\right)+\left\{ {h \atop i\ s} \right\}z^s\right]dx^i,$$

where $\left\{ {h \atop j\ i} \right\}$ denote the Christoffel's symbols constructed from g_{ji} and v^h are defined by

$$v^h=z^h+\left\{ {h \atop j\ i} \right\}y^i y^j.$$

We have, from (5. 5),

PROPOSITION 5. 2. *Let g be a pseudo-Riemannian metric in M (with r positive and $n-r$ negative signs). Then g^{II} is a pseudo-Riemannian metric in $T_2(M)$ (with $n+r$ negative and $2n-r$ positive signs).*

Let φ be a 2-form of the maximum rank in M. Then φ^{II} is also a 2-form of the maximum rank in $T_2(M)$ because of (5. 4). When $\varphi=d\eta$, η being a 1-form, then $\varphi^{\mathrm{II}}=d(\eta^{\mathrm{II}})$ as an immediate consequence of Proposition 4. 1. Thus we have

PROPOSITION 5. 3. *If φ is a 2-form defining an (almost) symplectic structure in M, then φ^{II} defines an (almost) symplectic structure in $T_2(M)$.*

Tensor fields of type (2, 0). Let G be a tensor field of type (2, 0) in M. Then we can easily verify that its lifts G^0, G^{I} and G^{I} have respectively components of the form

$$G^0:\begin{pmatrix}0&0&0\\0&0&0\\0&0&G^{ji}\end{pmatrix},\qquad G^{\mathrm{I}}:\begin{pmatrix}0&0&0\\0&0&\frac{1}{2}G^{ji}\\0&\frac{1}{2}G^{ji}&y^s\partial_s G^{ji}\end{pmatrix},$$

$$(5.7)\qquad G^{\mathrm{II}}:\begin{pmatrix}0&0&G^{ji}\\0&\frac{1}{2}G^{ji}&y^s\partial_s G^{ji}\\G^{ji}&y^s\partial_s G^{ji}&z^s\partial_s G^{ji}+y^t y^s\partial_t\partial_s G^{ji}\end{pmatrix},$$

where G^{ji} denote the components of G.

Tensor fields αT and βT. We shall give the local expressions of αT and βT defined in § 4. Taking account of (2. 5) and (3. 5), we have from (4. 11) and (4. 15)

$$(5.8)\qquad \alpha\omega=y^i\omega_i,\qquad \beta\omega=z^k\partial_k\omega_i+y^j y^i\partial_j\omega_i\qquad \text{for}\quad \omega\in\mathcal{T}_1^0(M)$$

with respect to the induced coordinates (x^h, y^h, z^h) in $\pi_2^{-1}(U)$, where ω_i denote the components of ω. Especially, we have from (5. 8)

$$(5.9)\qquad \alpha(dx^h)=y^h,\qquad \beta(dx^h)=z^h$$

in $\pi_2^{-1}(U)$.

Let T be an element of $\mathcal{T}_s^r(M)$ $(s\geqq 1)$ and assume that T has the expression

$$T=T_{i_1 i_2\cdots i_s}{}^{h_1\cdots h_r}dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_s}\otimes\frac{\partial}{\partial x^{h_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{h_r}}$$

in $(U,(x^h))$. Then, taking account of (4. 12), we have

$$\alpha T=\alpha\left(dx^j\otimes\left(T_{ji_2\cdots i_s}{}^{h_1\cdots h_r}dx^{i_2}\otimes\cdots\otimes dx^{i_s}\otimes\frac{\partial}{\partial x^{h_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{h_r}}\right)\right)$$
$$=\alpha(dx^j)(T_{ji_2\cdots i_s}{}^{h_1\cdots h_r})^0 dx^{i_2}\otimes\cdots\otimes dx^{i_s}\otimes\frac{\partial}{\partial z^{h_1}}\otimes\cdots\otimes\frac{\partial}{\partial z^{h_r}}$$

by virtue of (3. 10), since $\alpha(T+S)=\alpha T+\alpha S$ for $T, S\in\mathcal{T}_s^r(M)$. Thus, according to (5. 9), we obtain

$$(5.10)\qquad \alpha T=(y^j T_{ji_2\cdots i_s}{}^{h_1\cdots h_r})dx^{i_2}\otimes\cdots\otimes dx^{i_s}\otimes\frac{\partial}{\partial z^{h_1}}\otimes\cdots\otimes\frac{\partial}{\partial z^{h_r}}$$

with respect to the induced coordinates (ξ^A) in $\pi_2^{-1}(U)$. Especially, for any element F of $\mathcal{T}_1^1(M)$, αT has components of the form

$$\alpha F:\quad \begin{pmatrix} 0 \\ 0 \\ y^i F_i{}^h \end{pmatrix}, \tag{5.11}$$

$F_i{}^h$ denoting the components of F. For an element S of $\mathcal{T}_2^1(M)$, αS has components of the form

$$\alpha S:\quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y^j S_{ji}{}^h & 0 & 0 \end{pmatrix} \tag{5.12}$$

where $S_{ji}{}^h$ denote the components of S.

Let F be an element of $\mathcal{T}_1^1(M)$ with local expression

$$F = F_i{}^h dx^i \otimes \frac{\partial}{\partial x^h}.$$

Then, taking account of (4.16), we have

$$\begin{aligned}
\beta F &= \beta\left(dx^i \otimes \left(F_i{}^h \frac{\partial}{\partial x^h}\right)\right) \\
&= \beta(dx^i)\left(F_i{}^h \frac{\partial}{\partial x^h}\right)^0 + \alpha(dx^i)\left(F_i{}^h \frac{\partial}{\partial x^h}\right)^{\mathrm{I}} \\
&= \beta(dx^i)(F_i{}^h)^0 \frac{\partial}{\partial z^h} + (dx^i)\left((F_i{}^h)^{\mathrm{I}} \frac{\partial}{\partial z^h} + \frac{1}{2}(F_i{}^h)^0 \frac{\partial}{\partial y^h}\right) \\
&= \frac{1}{2} y^i (F_i{}^h)^0 \frac{\partial}{\partial y^h} + (z^i (F_i{}^h)^0 + y^i (F_i{}^h)^{\mathrm{I}}) \frac{\partial}{\partial z^h}
\end{aligned}$$

by virtue of (3.10) and (5.9), since $\beta(T+S) = \beta T + \beta S$ for $T, S \in \mathcal{T}_s^r(M)$. This means that βF has components of the form

$$\beta F:\quad \begin{pmatrix} 0 \\ \frac{1}{2} y^i F_i{}^h \\ z^i F_i{}^h + y^j y^i \partial_j F_i{}^h \end{pmatrix} \tag{5.13}$$

for any element $F \in \mathcal{T}_1^1(M)$, where $F_i{}^h$ denote the components of F.

By similar devices, we see that, for an element S of $\mathcal{T}_2^1(M)$, βS has components of the form

$$(5.14)\qquad \beta S:\ \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2}y^k S_{ki}{}^h & 0 & 0 \\ z^k S_{ki}{}^h + y^t y^s \partial_t S_{si}{}^h & y^k S_{ki}{}^h & 0 \end{pmatrix},$$

where $S_{ji}{}^h$ denote the components of S.

If we take account of (3.1) and (5.8), we obtain the following formulas:

$$(5.15)\qquad \begin{aligned} &X^0(\alpha\omega)=0, && X^0(\beta\omega)=(\omega(X))^0, \\ &X^{\mathrm{I}}(\alpha\omega)=\frac{1}{2}(\omega(X))^0, && X^{\mathrm{I}}(\beta\omega)=\alpha(\mathcal{L}_X\,\omega)+(\alpha(d\omega))(X), \\ &X^{\mathrm{II}}(\alpha\omega)=\alpha(\mathcal{L}_X\,\omega), && X^{\mathrm{II}}(\beta\omega)=\beta(\mathcal{L}_X\,\omega) \end{aligned}$$

for $\omega\in\mathcal{T}_1^0(M)$ and $X\in\mathcal{T}_0^1(M)$.

§6. Lifts of tensor fields of type (1, 1).

Formulas. Let F be an element of $\mathcal{T}_1^1(M)$. Then, taking account of (3.1) and (5.1), we find easily the following

$$(6.1)\qquad \begin{aligned} &F^0X^0=0, && F^0X^{\mathrm{I}}=0 && F^0X^{\mathrm{II}}=(FX)^0, \\ &F^{\mathrm{I}}X^0=0, && F^{\mathrm{I}}X^{\mathrm{I}}=\frac{1}{2}(FX)^0, && F^{\mathrm{I}}X^{\mathrm{II}}=(FX)^{\mathrm{I}}, \\ &F^{\mathrm{II}}X^0=(FX)^0, && F^{\mathrm{II}}X^{\mathrm{I}}=(FX)^{\mathrm{I}}, && F^{\mathrm{II}}X^{\mathrm{II}}=(FX)^{\mathrm{II}} \end{aligned}$$

for $F\in\mathcal{T}_1^1(M)$, X being an arbitrary element of $\mathcal{T}_0^1(M)$.

For any two elements F and G of $\mathcal{T}_1^1(M)$, we defined an element FG of $\mathcal{T}_1^1(M)$ by $(FG)X=F(GX)$, X being an arbitrary element of $\mathcal{T}_0^1(M)$. Then we find the following formulas:

$$(6.2)\qquad \begin{aligned} &G^0F^0=0, && G^0F^{\mathrm{I}}=0 && G^0F^{\mathrm{II}}=(GF)^0, \\ &G^{\mathrm{I}}F^0=0, && G^{\mathrm{I}}F^{\mathrm{I}}=\frac{1}{2}(GF)^0, && G^{\mathrm{I}}F^{\mathrm{II}}=(GF)^{\mathrm{I}}, \\ &G^{\mathrm{II}}F^0=(GF)^0, && G^{\mathrm{II}}F^{\mathrm{I}}=(GF)^{\mathrm{I}}, && G^{\mathrm{II}}F^{\mathrm{II}}=(GF)^{\mathrm{II}} \end{aligned}$$

for $G, F\in\mathcal{T}_2^1(M)$. In fact, taking account of (6.1), we have

$$(G^0F^0)X^{\mathrm{II}}=G^0(F^0X^{\mathrm{II}})=0,$$

$$(G^{\mathrm{I}}F^{\mathrm{I}})X^{\mathrm{II}}=G^{\mathrm{I}}(F^{\mathrm{I}}X^{\mathrm{II}})=G^{\mathrm{I}}(FX)^{\mathrm{I}}=\frac{1}{2}(G(FX))^{\mathrm{I}}=\frac{1}{2}(GF)^0X^{\mathrm{II}}$$

$$(G^{\mathrm{II}}F^{\mathrm{II}})X^{\mathrm{II}}=G^{\mathrm{II}}(F^{\mathrm{II}}X^{\mathrm{II}})=G^{\mathrm{II}}(FX)^{\mathrm{II}}=(G(FX))^{\mathrm{II}}=(GF)^{\mathrm{II}}X^{\mathrm{II}}$$

for any element X of $\mathcal{T}_0^1(M)$. Thus we have $G^0F^0=(GF)^0$, $G^{\mathrm{I}}F^{\mathrm{I}}=(1/2)(GF)^0$ and

$G^{II}F^{II}=(GF)^{II}$. The other formulas given in (6. 2) are proved in a similar way.

We see from (6. 2) that, for any element F of $\mathcal{T}_1^1(M)$, F^0, F^I and F^{II} are commutative with each other and the identities

$$(F^0)^2=0, \qquad (F^I)^3=0 \qquad \text{for} \quad F\in\mathcal{T}_1^1(M) \tag{6.3}$$

hold.

Let $P(t)$ be a polynomial of t and $F\in\mathcal{T}_1^1(M)$. Then, taking account of (6. 2), we obtain

$$(P(F))^{II}=P(F^{II}) \tag{6.4}$$

and hence, for example,

$$(F^2+I)^{II}=(F^{II})^2+I, \qquad (F^3+F)^{II}=(F^{II})^3+F^{II} \tag{6.5}$$

for any element F of $\mathcal{T}_1^1(M)$.

A tensor field F of type (1, 1) is called an *almost complex structure* if $F^2+I=0$. A tensor field F is called an *f-structure of rank r* if $F^3+F=0$ and F is of rank r everywhere. Thus, taking account of Proposition 5. 1, we have from (6. 5)

PROPOSITION 6. 1. *Let F be an element of $\mathcal{T}_1^1(M)$. Then F^{II} is an almost complex structure in $T_2(M)$ if and only if F is so in M. F^{II} is an f-structure of rank $3r$ in $T_2(M)$ if and only if F is an f-structure of rank r in M.*

Contraction in Lifts. Let F be an element of $\mathcal{T}_1^1(M)$. We denote by $c(F)$ the element of $\mathcal{T}_0^0(M)$ obtained by contraction, i.e., $c(F)=F_i^i$ if F has components F_i^h. Then we have from (5. 1)

$$c(F^0)=0, \qquad c(F^I)=0, \qquad c(F^{II})=3(c(F))^0 \tag{6.6}$$

for $F\in\mathcal{T}_1^1(M)$. For example, we have

$$c((\omega\otimes X)^0)=0, \qquad c((\omega\otimes X)^I)=0, \qquad c((\omega\otimes X)^{II})=3(\omega(X))^0, \tag{6.7}$$

X and ω being respectively elements of $\mathcal{T}_0^1(M)$ and $\mathcal{T}_1^0(M)$.

Torsion tensors and Nijenhuis tensors. Let S be an element of $\mathcal{T}_2^1(M)$ such that $S=Z\otimes\omega\otimes\pi$, $Z\in\mathcal{T}_0^1(M)$, $\omega,\pi\in\mathcal{T}_1^0(M)$. Then, taking account of (3. 8) and (4. 1), we have the following formulas:

$$\begin{gathered} S^0(X^{II}, Y^{II})=(S(X, Y))^0, \qquad S^I(X^{II}, Y^{II})=(S(X, Y))^I, \\ S^{II}(X^{II}, Y^{II})=(S(X, Y))^{II} \end{gathered} \tag{6.8}$$

for $S\in\mathcal{T}_2^1(M)$, X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$.

Let there be given two elements G and F of $\mathcal{T}_1^1(M)$. Then their *torsion tensor* $N_{F,G}$ is by definition a tensor field of type (1, 2) given by

$$\begin{aligned} N_{F,G}(X, Y)=&[FX, GY]+[GX, FY]+FG[X, Y]+GF[X, Y] \\ &-F[X, GY]-F[GX, Y]-G[X, FY]-G[FX, Y], \end{aligned} \tag{6.9}$$

X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$. Thus, taking account of (3. 9), (6. 1) and (6. 2), we obtain

$$
\begin{aligned}
&N_{F^0, G^0}(X^{II}, Y^{II})=0, \qquad N_{F^0, G^I}(X^{II}, Y^{II})=0,\\
&F_{F^0, G^{II}}(X^{II}, Y^{II})=(N_{F,G}(X, Y))^0,\\
&N_{F^I, G^I}(X^{II}, Y^{II})=\frac{1}{2}(N_{F,G}(X, Y))^0,\\
&N_{F^I, G^{II}}(X^{II}, Y^{II})=(N_{F,G}(X, Y))^I,\\
&N_{F^{II}, G^{II}}(X^{II}, Y^{II})=(N_{F,G}(X, Y))^{II},
\end{aligned}
\tag{6. 10}
$$

X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$. Thus, we have from (6. 10)

$$
\begin{aligned}
&N_{F^0, G^0}=0, && N_{F^I, G^I}=\frac{1}{2}(N_{F,G})^0,\\
&N_{F^0, G^I}=0, && N_{F^I, G^{II}}=(N_{F,G})^I,\\
&N_{F^0, G^{II}}=(N_{F,G})^0, && N_{F^{II}, G^{II}}=(N_{F,G})^{II}
\end{aligned}
\tag{6. 11}
$$

for $F, G\in\mathcal{T}_1^1(M)$ by virtue of (6. 8).

The *Nijenhuis tensor* N_F of an element F of $\mathcal{T}_1^1(M)$ is defined by $N_F=(1/2)N_{F,F}$. Thus we have from (6. 11)

PROPOSITION 6. 2. *For any element F of $\mathcal{T}_1^1(M)$*

$$N_{F^0}=0, \qquad N_{F^I}=\frac{1}{2}(N_F)^0, \qquad N_{F^{II}}=(N_F)^{II}$$

hold.

PROPOSITION 6. 3. *Let F be an almost complex structure in M. Then the almost complex structure F^{II} is a complex structure in $T_2(M)$ if and only if F is so in M.*

§ 7. Lifts of affine connections.

Lifts of affine connections. Let ∇ be an affine connection in M, which has coefficients $\Gamma_j{}^h{}_i$ in $(U, (x^h))$. We now introduce in $\pi_2{}^{-1}(U)$ an affine connection ∇^{II} having coefficients $\tilde{\Gamma}_C{}^A{}_B$ with respect to the induced coordinates $(\xi^A)=(x^h, y^h, z^h)$ such that

$$
(\tilde{\Gamma}_C{}^h{}_B)=\begin{pmatrix} (\Gamma_j{}^h{}_i)^0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}
\tag{7. 1}
$$

for each fixed index h,

$$(\tilde{\Gamma}_C{}^{\bar{h}}{}_B)=\begin{pmatrix}(\Gamma_j{}^h{}_i)^{\mathrm{I}} & (\Gamma_j{}^h{}_i)^0 & 0\\ (\Gamma_j{}^h{}_i)^0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{7. 2}$$

for each fixed $\bar{h}$ and

$$(\tilde{\Gamma}_C{}^{\bar{\bar{h}}}{}_B)=\begin{pmatrix}(\Gamma_j{}^h{}_i)^{\mathrm{II}} & 2(\Gamma_j{}^h{}_i)^{\mathrm{I}} & (\Gamma_j{}^h{}_i)^0\\ 2(\Gamma_j{}^h{}_i)^{\mathrm{I}} & 2(\Gamma_j{}^h{}_i)^0 & 0\\ (\Gamma_j{}^h{}_i)^0 & 0 & 0\end{pmatrix} \tag{7. 3}$$

for each fixed index $\bar{\bar{h}}$, where $(\Gamma_j{}^h{}_i)^0$, $(\Gamma_j{}^h{}_i)^{\mathrm{I}}$ and $(\Gamma_j{}^h{}_i)^{\mathrm{II}}$ denote respectively the 0-th, the 1st and the 2nd lifts of the functions $\Gamma_j{}^h{}_i$ given in $(U, (x^h))$. We note here that the transformation law of coefficients $\Gamma_j{}^h{}_i$ of an affine connection is given by

$$\Gamma_{j'}{}^{h'}{}_{i'}=\frac{\partial x^{h'}}{\partial x^h}\left(\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^i}{\partial x^{i'}}\Gamma_j{}^h{}_i+\frac{\partial^2 x^h}{\partial x^{j'}\partial x^{i'}}\right) \tag{7. 4}$$

in $U\cap U'$. Thus, taking account of (1. 5), (1. 6), (7. 1), (7. 2) and (7. 3), we know by virtue of (7. 4) that the affine connection ∇^{II} introduced above in each $\pi_2^{-1}(U)$ determines globally in $T_2(M)$ an affine connection, which is denoted also by ∇^{II}. The affine connection ∇^{II} constructed thus in $T_2(M)$ is called the *lift of the affine connection* ∇ given in M.

We obtain here the following formulas:

$$\begin{aligned}
&\nabla^{\mathrm{II}}_{X^0} Y^0=0, && \nabla^{\mathrm{II}}_{X^0} Y^{\mathrm{I}}=0, && \nabla^{\mathrm{II}}_{X^0} Y^{\mathrm{II}}=(\nabla_X Y)^0,\\
&\nabla^{\mathrm{II}}_{X^{\mathrm{I}}} Y^0=0, && \nabla^{\mathrm{II}}_{X^{\mathrm{I}}} Y^{\mathrm{I}}=\frac{1}{2}(\nabla_X Y)^0, && \nabla^{\mathrm{II}}_{X^{\mathrm{I}}} Y^{\mathrm{II}}=(\nabla_X Y)^{\mathrm{I}},\\
&\nabla^{\mathrm{II}}_{X^{\mathrm{II}}} Y^0=(\nabla_X Y)^0, && \nabla^{\mathrm{II}}_{X^{\mathrm{II}}} Y^{\mathrm{I}}=(\nabla_X Y)^{\mathrm{I}} && \nabla^{\mathrm{II}}_{X^{\mathrm{II}}} Y^{\mathrm{II}}=(\nabla_X Y)^{\mathrm{II}}
\end{aligned} \tag{7. 5}$$

for $X, Y\in\mathcal{T}^1_0(M)$. In fact, taking account of (3. 1), (7. 1), (7. 2) and (7. 3), we see that $\nabla^{\mathrm{II}}_{X^{\mathrm{II}}} Y^{\mathrm{II}}$ has components of the form

$$(\nabla^{\mathrm{II}}_{X^{\mathrm{II}}} Y^{\mathrm{II}})^h=X^j\left(\frac{\partial Y^h}{\partial x^j}+\Gamma_j{}^h{}_i Y^i\right)=X^j\nabla_j Y^h,$$

$$\begin{aligned}
(\nabla^{\mathrm{II}}_{X^{\mathrm{II}}} Y^{\mathrm{II}})^{\bar{h}}=&X^j\left[\frac{\partial}{\partial x^j}(y^s\partial_s Y^h)+(y^s\partial_s\Gamma_j{}^h{}_i)Y^i+\Gamma_j{}^h{}_i(y^s\partial_s Y^i)\right]\\
&+(y^s\partial_s X^j)\left[\frac{\partial}{\partial y^j}(y^s\partial_s Y^h)+\Gamma_j{}^h{}_i Y^i\right]\\
=&y^s\partial_s\left(X^j\left(\frac{\partial Y^h}{\partial x^j}+\Gamma_j{}^h{}_i Y^i\right)\right)=y^s\partial_s(X^j\nabla_j Y^h),
\end{aligned}$$

$$(\nabla^{II}_{X^{II}} Y^{II})^{\bar{\bar{h}}} = X^j\Big[\frac{\partial}{\partial x^j}(z^s\partial_s Y^h + y^t y^s \partial_t\partial_s Y^h) + (z^s\partial_s \Gamma_j{}^h{}_i + y^t y^s\partial_t\partial_s \Gamma_j{}^h{}_i)Y^i$$
$$+2(y^s\partial_s\Gamma_j{}^h{}_i)(y^t\partial_t Y^i) + \Gamma_j{}^h{}_i(z^s\partial_s Y^i + y^t y^s\partial_t\partial_s Y^i)\Big]$$
$$+(y^k\partial_k X^j)\Big[\frac{\partial}{\partial y^j}(z^s\partial_s Y^h + y^t y^s\partial_t\partial_s Y^h) + 2(y^s\partial_s\Gamma_j{}^h{}_i)Y^i + 2\Gamma_j{}^h{}_i(y^s\partial_s Y^i)\Big]$$
$$+(z^k\partial_k X^j + y^m y^l\partial_m\partial_l X^j)\Big[\frac{\partial}{\partial z^j}(z^s\partial_s Y^h + y^t y^s\partial_t\partial_s Y^h) + \Gamma_j{}^h{}_i Y^i\Big]$$
$$= z^s\partial_s(X^j(\partial_j Y^h + \Gamma_j{}^h{}_i Y^i)) + y^t y^s\partial_t\partial_s(X^j(\partial_j Y^h + \Gamma_j{}^h{}_i Y^i))$$
$$= z^s\partial_s(X^j\nabla_j Y^h) + y^t y^s\partial_t\partial_s(X^j\nabla_j Y^h).$$

Therefore we find $\nabla^{II}_{X^{II}} Y^{II} = (\nabla_X Y)^{II}$. Similarly, we obtain the other formulas given in (7. 5).

Comparing (7. 5) with (4. 2) or (6. 1), we find easily the formulas

$$\nabla^{II} Y^0 = (\nabla Y)^0, \quad \nabla^{II} Y^I = (\nabla Y)^I, \quad \nabla^{II} Y^{II} = (\nabla Y)^{II} \tag{7. 6}$$

for $Y \in \mathcal{T}^1_0(M)$.

We also obtain the following formulas:

$$\begin{array}{lll} \nabla^{II}_{X^0}\,\omega^0 = 0, & \nabla^{II}_{X^0}\,\omega^I = 0, & \nabla^{II}_{X^0}\,\omega^{II} = (\nabla_X\,\omega)^0, \\ \nabla^{II}_{X^I}\,\omega^0 = 0, & \nabla^{II}_{X^I}\,\omega^I = \frac{1}{2}(\nabla_X\,\omega)^0, & \nabla^{II}_{X^I}\,\omega^{II} = (\nabla_X\,\omega)^I \\ \nabla^{II}_{X^{II}}\,\omega^0 = (\nabla_X\,\omega)^0, & \nabla^{II}_{X^{II}}\,\omega^I = (\nabla_X\,\omega)^I, & \nabla^{II}_{X^{II}}\,\omega^{II} = (\nabla_X\,\omega)^{II} \end{array} \tag{7. 7}$$

for $X \in \mathcal{T}^1_0(M)$, $\omega \in \mathcal{T}^0_1(M)$. In fact, taking an arbitrary element Y of $\mathcal{T}^1_0(M)$, we have

$$(\nabla^{II}_{X^{II}}\,\omega^{II})(Y^{II}) = \nabla^{II}_{X^{II}}(\omega^{II}(Y^{II})) - \omega^{II}(\nabla^{II}_{X^{II}}\,Y^{II})$$
$$= (\nabla_X(\omega(Y)) - \omega(\nabla_X Y))^{II}$$
$$= ((\nabla_X\,\omega)(Y))^{II} = (\nabla_X\,\omega)^{II}(Y^{II})$$

by virtue of (3. 8) and (7. 5). Thus we have $\nabla^{II}_{X^{II}}\,\omega^{II} = (\nabla_X\,\omega)^{II}$, because Y is arbitrary. The other formulas given in (7. 7) are proved in a similar way.

We have from (7. 7) the formulas

$$\nabla^{II}\omega^0 = (\nabla\omega)^0, \quad \nabla^{II}\omega^I = (\nabla\omega)^I, \quad \nabla^{II}\omega^{II} = (\nabla\omega)^{II} \tag{7. 8}$$

for $\omega \in \mathcal{T}^0_1(M)$. In fact, we have from (4. 2) and (7. 7)

$$\gamma_{X^{II}}(\nabla^{II}\omega^{II}) = \nabla^{II}_{X^{II}}\,\omega^{II} = (\nabla_X\omega)^{II} = (\gamma_X(\nabla\omega))^{II} = \gamma_{X^{II}}(\nabla\omega)^{II}$$

for any element X of $\mathcal{T}^1_0(M)$. Thus we have $\nabla^{II}\omega^{II} = (\nabla\omega)^{II}$. Similarly, we can prove the other formulas given in (7. 8).

We have here from (7. 6) and (7. 8)

PROPOSITION 7. 1. *For any element K of $\mathcal{T}(M)$*

$$\nabla^{II} K^{0}=(\nabla K)^{0}, \qquad \nabla^{II} K^{I}=(\nabla K)^{I}, \qquad \nabla^{II} K^{II}=(\nabla K)^{II}$$

hold.

We have directly from Proposition 7. 1 the formulas

$$\begin{array}{llll} & \nabla^{II}_{X^0} K^{0}=0, & \nabla^{II}_{X^0} K^{I}=0, & \nabla^{II}_{X^0} K^{II}=(\nabla_X K)^{0}, \\ (7.9) & \nabla^{II}_{X^I} K^{0}=0, & \nabla^{II}_{X^I} K^{I}=\frac{1}{2}(\nabla_X K)^{0}, & \nabla^{II}_{X^I} K^{II}=(\nabla_X K)^{I}, \\ & \nabla^{II}_{X^{II}} K^{0}=(\nabla_X K)^{0}, & \nabla^{II}_{X^{II}} K^{I}=(\nabla_X K)^{I}, & \nabla^{II}_{X^{II}} K^{II}=(\nabla_X K)^{II} \end{array}$$

for $X\in\mathcal{T}^1_0(M)$, $K\in\mathcal{T}^r_s(M)$ by virtue of (4. 2).

The Curvature and the torsion tensors. Denoting by T the torsion tensor of an affine connection ∇ in M, we have by definition

$$T(X, Y)=(\nabla_X Y-\nabla_Y X)-[X, Y] \qquad \text{for} \qquad X, Y\in\mathcal{T}^1_0(M).$$

Taking the second lift, we obtain

$$\begin{aligned}(T(X, Y))^{II}&=(\nabla^{II}_{X^{II}} Y^{II}-\nabla^{II}_{Y^{II}} X^{II})-[X^{II}, Y^{II}]\\ &=\tilde{T}(X^{II}, Y^{II})\end{aligned}$$

by virtue of Proposition 7. 1 and (3. 9), where $\tilde{T}$ denotes the torsion tensor of ∇^{II}. This equation implies, together with (6. 8), $T^{II}(X^{II}, Y^{II})=\tilde{T}(X^{II}, Y^{II})$. Thus, we have $T^{II}=\tilde{T}$, since X and Y are arbitrary. Therefore we have

PROPOSITION 7. 2. *The torsion tensor of the lift ∇^{II} of an affine connection ∇ given in M coincides with the 2nd lift T^{II} of the torsion tensor T of ∇.*

The curvature tensor R of an affine connection ∇ in M is a tensor field of type (1, 3) such that, for any two elements X and Y of $\mathcal{T}^1_0(M)$, $R(X, Y)$ is an element of $\mathcal{T}^1_1(M)$ satisfying the condition

$$R(X, Y)Z=(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z)-\nabla_{[X,Y]}Z$$

for any element Z of $\mathcal{T}^1_1(M)$. Taking the 2nd lift, we find

$$\begin{aligned}(R(X, Y)Z)^{II}&=(\nabla^{II}_{X^{II}}\nabla^{II}_{Y^{II}} Z^{II}-\nabla^{II}_{Y^{II}}\nabla^{II}_{X^{II}} Z^{II})-\nabla^{II}_{[X^{II},Y^{II}]} Z^{II}\\ &=\tilde{R}(X^{II}, Y^{II})Z^{II}\end{aligned}$$

by virtue of Proposition 7. 1 and (3. 9), where $\tilde{R}$ denotes the curvature tensor of ∇^{II}. On the other hand, we can verify $(R(X, Y)Z)^{II}=R^{II}(X^{II}, Y^{II})Z^{II}$ by virtue of $\nabla_{X^{II}} K^{II}=(\nabla_X K)^{II}$ given in (4. 2). Therefore we find $\tilde{R}(X^{II}, Y^{II})Z^{II}=R^{II}(X^{II}, Y^{II})Z^{II}$, which implies $\tilde{R}=R^{II}$ since X, Y and Z are arbitrarily taken. Thus we have

PROPOSITION 7. 3. *The curvature tensor of the lift* ∇^{II} *of an affine connection* ∇ *given in* M *coincides with the 2nd lift* R^{II} *of the curvature tensor* R *of* ∇.

As a corollary to Propositions 7. 1 and 7. 3, we have

PROPOSITION 7. 4. *Let* T *and* R *be respectively the torsion and the curvature tensors of an affine connection* ∇ *given in* M. *According as* $T=0$, $\nabla T=0$, $R=0$ *or* $\nabla R=0$, *we have* $T^{II}=0$, $\nabla^{II}T^{II}=0$, $R^{II}=0$ *or* $\nabla^{II}R^{II}=0$. *In particular,* $T_2(M)$ *is locally symmetric with respect to the lift* ∇^{II} *of* ∇ *if and only if* M *is so with respect to* ∇.

Let g be a pseudo-Riemannian metric in M and ∇ the Riemannian connection determined by g. Then we have from Proposition 7. 1

$$\nabla^{II}g^{II}=(\nabla g)^{II}=0.$$

On the other hand, since ∇ is torsionless, so is ∇^{II} by virtue of Proposition 7. 2. Consequently, ∇^{II} should coincide with the Riemannian connection determined by g^{II}. Thus we have

PROPOSITION 7. 5. *Let* g *be a pseudo-Riemannian metric in* M *and* ∇ *its Riemannian connection. Then the lift* ∇^{II} *of* ∇ *is the Riemannian connection determined by the 2nd lift* g^{II} *of* g.

We have from Propositions 7. 4 and 7. 5

PROPOSITIONS 7. 6. *Let* g *be a pseudo-Riemannian metric in* M. *Then* $T_2(M)$ *is locally symmetric with respect to* g^{II} *if and only if* M *is so with respect to* g.

Let P be an element of $\mathcal{T}_3^1(M)$. Then we have from (4. 2)

$$\gamma_{X^{II}}\gamma_{Y^{II}}P^{II}=(\gamma_X\gamma_Y P)^{II},$$

in which both sides belong to $\mathcal{T}_1^1(T_2(M))$.

If we take account of (6. 6), we find

$$c(\gamma_{X^{II}}\gamma_{Y^{II}}P^{II})=c((\gamma_X\gamma_Y P)^{II})=3(c(\gamma_X\gamma_Y P))^0,$$

which implies

PROPOSITION 7. 7. *Let* g *be a pseudo-Riemannian metric in* M. *Then the Ricci tensor* $\tilde{K}$ *of* g^{II} *coincides with* $3K^0$, *where* K *denotes the Ricci tensor of* g.

If g^{II} is an Einstein metric in $T_2(M)$, we have $\tilde{K}=ag^{II}$ with a constant a, $\tilde{K}$ being the Ricci tensor of g^{II}. However, we have from proposition 7. 7 $\tilde{K}=3K^0$. Thus we have $ag^{II}=3K^0$, which, together with (5. 4), implies $a=0$. Therefore we have

PROPOSITION 7. 8. *Let* g *be a pseudo-Riemannian metric in* M. *If* g^{II} *is an*

Einstein metric in $T_2(M)$, then g^{II} is of zero Ricci tensor. If g^{II} is of constant curvature, then g^{II} is locally flat.

Let $\tilde{K}_{CB}$ denote the components of the Ricci tensor $\tilde{K}=3K^0$ of g^{II} and $\tilde{G}^{CB}$ the contravariant components of g^{II}. Then, taking account of (5. 4) and (5. 7), we have $\tilde{k}=\tilde{K}_{CB}\tilde{G}^{CB}=3(K_{ji}g^{ji})^0$, where K_{ji} denote the components of the Ricci tensor of g and g^{ji} the contravariant components of g. Thus we have

PROPOSITION 7. 9. *Let g be a pseudo-Riemannian metric in M. Let k and $\tilde{k}$ be the curvature scalars of g and g^{II} respectively. Then $\tilde{k}=3k^0$. If g is of constant curvature scalar, so is g^{II}.*

A pseudo-Riemannian metric g is of constant curvature k in M if

$$R(X, Y)Z=k(g(Z, Y)X-g(Z, X)Y) \qquad \text{for} \quad X, Y\in \mathcal{T}_0^1(M)$$

with a constant k, R denoting the curvature tensor of g. Taking the 2nd lift, we have

$$\begin{aligned} R^{\mathrm{II}}(X^{\mathrm{II}}, Y^{\mathrm{II}})Z^{\mathrm{II}}&=(R(X, Y)Z)^{\mathrm{II}}=(k(g(Z, Y)X-g(Z, X)Y))^{\mathrm{II}} \\ &=k[g^{\mathrm{II}}(Z^{\mathrm{II}}, Y^{\mathrm{II}})X^0+2g^{\mathrm{I}}(Z^{\mathrm{II}}, Y^{\mathrm{II}})X^{\mathrm{I}}+g^0(Z^{\mathrm{II}}, Y^{\mathrm{II}})X^{\mathrm{II}} \\ &\quad -g^{\mathrm{II}}(Z^{\mathrm{II}}, X^{\mathrm{II}})Y^0-2g^{\mathrm{I}}(Z^{\mathrm{II}}, X^{\mathrm{II}})Y^{\mathrm{I}}-g^0(Z^{\mathrm{II}}, X^{\mathrm{II}})Y^{\mathrm{II}}]. \end{aligned}$$

If we take account of $I^0X^{\mathrm{II}}=X^0$, $I^{\mathrm{I}}X^{\mathrm{II}}=X^{\mathrm{I}}$ given in (5. 3), we have from the equation above

$$\begin{aligned} &R^{\mathrm{II}}(X^{\mathrm{II}}, Y^{\mathrm{II}})Z^{\mathrm{II}} \\ &=k[g^{\mathrm{II}}(Z^{\mathrm{II}}, Y^{\mathrm{II}})I^0X^{\mathrm{II}}+2g^{\mathrm{I}}(Z^{\mathrm{II}}, Y^{\mathrm{II}})I^{\mathrm{I}}X^{\mathrm{II}}+g^0(Z^{\mathrm{II}}, Y^{\mathrm{II}})X^{\mathrm{II}} \\ &\quad -g^{\mathrm{II}}(Z^{\mathrm{II}}, X^{\mathrm{II}})I^0Y^{\mathrm{II}}-2g^{\mathrm{I}}(Z^{\mathrm{II}}, X^{\mathrm{II}})I^{\mathrm{I}}Y^{\mathrm{II}}-g^0(Z^{\mathrm{II}}, X^{\mathrm{II}})Y^{\mathrm{II}}] \end{aligned} \tag{7. 10}$$

which gives the curvature tensor R^{II} of *g^{II} in $T_2(M)$ when g is of constant curvature in M.*

§ 8. Lifts of infinitesimal transformations.

Let g be a pseudo-Riemannian metric in M. Then we have from Proposition 4. 2

$$\mathcal{L}_{X^0}\, g^{\mathrm{II}}=(\mathcal{L}_X\, g)^0, \qquad \mathcal{L}_{X^{\mathrm{I}}}\, g^{\mathrm{II}}=(\mathcal{L}_X\, g)^{\mathrm{I}}, \qquad \mathcal{L}_{X^{\mathrm{II}}}\, g^{\mathrm{II}}=(\mathcal{L}_X\, g)^{\mathrm{II}} \tag{8. 1}$$

for any element X of $\mathcal{T}_0^1(M)$. If X is a Killing vector field with respect to g, i.e., if $\mathcal{L}_X\, g=0$, then X^0, X^{I} and X^{II} are so with respect to g^{II}. Thus, taking account of (5. 2), we have

PROPOSITION 8. 1. *Let g be a pseudo-Riemannian metric in M. If X is a*

Killing vector field with respect to g in M, then X^0, X^I, X^{II} are all Killing vector fields with respect to the pseudo-Riemannian metric g^{II} in $T_2(M)$.

Similarly, taking account of Proposition 6. 1, we have

PROPOSITION 8. 2. *If X is an (almost) analytic vector field in M with respect to an (almost) complex structure F, i.e., if $\mathcal{L}_X F=0$, then X^0, X^I and X^{II} are so also in $T_2(M)$ with respect to the (almost) complex structure F^{II}.*

Let X be a *conformal Killing vector field* in M with respect to a pseudo-Riemannian metric g. Then we have $\mathcal{L}_X g=ag$, $a\in\mathcal{T}^0_0(M)$. Thus, taking account of (8. 1), we obtain

$$\mathcal{L}_{X^{II}}\, g^{II}=a^{II}g^0+2a^Ig^I+a^0g^{II},$$

which implies

PROPOSITION 8. 3. *Let X be a conformal Killing vector field in M with respect to a pseudo-Riemannian metric g. Then X^{II} is conformal in $T_2(M)$ with respect to g^{II} if and only if X is homothetic, i.e., if and only if $\mathcal{L}_X g=ag$ holds with a constant a. If this is the case, X^{II} is necessarily homothetic.*

Let ∇ be an affine connection in M. Then, for any element X of $\mathcal{T}^1_0(M)$, the Lie derivative of ∇ with respect to X is an element $\mathcal{L}_X \nabla$ of $\mathcal{T}^1_2(M)$ defined by

$$(8.\ 2)\qquad (\mathcal{L}_X \nabla)(Y, Z)=\mathcal{L}_X (\nabla_Y Z)-\mathcal{L}_Y (\nabla_X Z)-\mathcal{L}_{[X,Y]}Z,$$

X, Y and Z belonging to $\mathcal{T}^1_0(M)$. Thus, taking account of (3. 9), (8. 2) and Proposition 4. 2, we obtain

$$\begin{aligned}(\mathcal{L}_{X^{II}} \nabla^{II})(Y^{II}, Z^{II})&=\mathcal{L}_{X^{II}} (\nabla^{II}_{Y^{II}} Z^{II})-\nabla^{II}_{Y^{II}} (\mathcal{L}_{X^{II}} Z^{II})-\nabla^{II}_{[X^{II},Y^{II}]} Z^{II}\\ &=(\mathcal{L}_X (\nabla_Y Z)-\nabla_Y (\mathcal{L}_X Z)-\nabla_{[X,Y]}Z)^{II}\\ &=((\mathcal{L}_X \nabla)(Y, Z))^{II}=(\mathcal{L}_X \nabla)^{II}(Y^{II}, Z^{II})\end{aligned}$$

for any element Y and Z of $\mathcal{T}^1_0(M)$. Thus we find

$$(8.\ 2)\qquad \mathcal{L}_{X^{II}} \nabla^{II}=(\mathcal{L}_X \nabla)^{II}.$$

Similarly, we can prove the other formulas given in Proposition 8. 4. Thus we have

PROPOSITION 8. 4. *Let ∇ be an affine connection in M. Then, for any element X of $\mathcal{T}^1_0(M)$, the formulas*

$$\mathcal{L}_{X^0} \nabla^{II}=(\mathcal{L}_X \nabla)^0,\qquad \mathcal{L}_{X^I} \nabla^{II}=(\mathcal{L}_X \nabla)^I,\qquad \mathcal{L}_{X^{II}} \nabla^{II}=(\mathcal{L}_X \nabla)^{II}$$

hold in $T_2(M)$.

A vector field X is called an *infinitesimal affine transformation* with respect to an affine connection ∇ if $\mathcal{L}_X \nabla=0$. As a consequence of Proposition 8. 4, we have

PROPOSITION 8. 5. *Let ∇ be an affine connection in M. If X is an infinitesimal affine transformation in M with respect to ∇, then X^0, X^{I} and X^{II} are so also in $T_2(M)$ with respect to ∇^{II}.*

A vector field X in M is called an *infinitesimal projective transformation* with respect to an affine connection ∇ if

$$(\mathcal{L}_X \nabla)(Y, Z)=\eta(Z)Y+\eta(Y)Z,$$

η being a certain element of $\mathcal{T}_1^0(M)$. Taking the 2nd lift, we have

$$\begin{aligned}(\mathcal{L}_{X^{\mathrm{II}}} \nabla^{\mathrm{II}})(Y^{\mathrm{II}}, Z^{\mathrm{II}})=&\eta^{\mathrm{II}}(Z^{\mathrm{II}})Y^0+2\eta^{\mathrm{I}}(Z^{\mathrm{II}})Y^{\mathrm{I}}+\eta^0(Z^{\mathrm{II}})Y^{\mathrm{II}}\\ &+\eta^{\mathrm{II}}(Y^{\mathrm{II}})Z^0+2\eta^{\mathrm{I}}(Y^{\mathrm{II}})Z^{\mathrm{I}}+\eta^0(Y^{\mathrm{II}})Z^{\mathrm{II}}\end{aligned}$$

by virtue of Propositton 8. 4. Thus we have

PROPOSITION 8. 6. *Let X be an infinitesimal projective transformation in M with respect to an affine connection ∇. Then X^{II} is an infinitesimal projective transformation with respect to ∇^{II} if and onlf if X is affine. If this is the case, X^{II} is necessarily affine with respect to ∇^{II}.*

Let X be an element of $\mathcal{T}_0^1(M)$ and $\exp(tX)$ denote a local 1-parameter group of transformations of M generated by X. Then, according to (1. 10) and (3. 1), X^{II} generates a local 1-parameter group of $T_2(M)$ and

$$\exp(tX^{\mathrm{II}})=(\exp(tX))^*$$

holds. Hence we have

PROPOSITION 8. 7. *If a vector field X in M is complete in the sense that it generates a global 1-parameter group of transformations of M, then X^{II} is also complete in $T_2(M)$.*

REMARK. From the local expressions (3. 1) of X^0 and X^{I}, we see immediately that X^0 and X^{I} are complete in $T_2(M)$ whether X is complete in M or not.

Taking account of the Remark stated above, we have, from Propositions 8. 1 and 8. 7,

PROPOSITION 8. 8. *If M is homogeneous pseudo-Riemannian manifold with* *metric g, so is $T_2(M)$ with metric g^{II}.*

Similarly, we have from Proposition 8. 2

PROPOSITION 8. 9. *If M is homogeneous (almost) complex manifold with (almost) complex structure F, so is $T_2(M)$ with (almost) complex structure F^{II}.*

Similarly, we have from Proposition 8. 5

PROPOSITION 8. 10. *If a group G of affine transformations of M with respect to an affine connection ∇ is transitive in M, the group G^* of affine transformations of $T_2(M)$ with respect to ∇^{II} is transitive in $T_2(M)$, where G^* denotes the group of transformations generated by vector fields X^0, X^{I} and X^{II}, X in M being an arbitrary element belonging to the Lie algebra of vector fields generating G.*

Let M be a pseudo-Riemannian (resp. affine) symmetric space with metric g (resp. connection ∇). If we take an arbitrary point P in M, then there exists in M a symmetry S_{P} with center P, that is to say, S_{P} is in M an isometry of g (resp. an affine transformation of ∇) such that $S_{\mathrm{P}}(\mathrm{P})=\mathrm{P}$, $(S_{\mathrm{P}})^2=$identity. We note here that M is identified with the zero-cross section $\bar{M}$ of $T_2(M)$, which is defined by equations $y^h=0$, $z^h=0$ with respect to the induced coordinates $(\xi^A)=(x^h, y^h, z^h)$ in each $\pi_2^{-1}(U)$. For any point P of M we denote by $\bar{\mathrm{P}}$ the point of $\bar{M}$ corresponding to P. Then the transformation $(S_{\mathrm{P}})^*$ induced from S_{P} (Cf. § 1) is a symmetry with center $\bar{\mathrm{P}}$ with respect to g^{II} (resp. ∇^{II}). On the other hand, $T_2(M)$ is homogeneous with respect to g^{II} (resp. ∇^{II}), because M is so with respect to g (resp. ∇). Therefore, taking an arbitrary point σ in $T_2(M)$, we know that there exists an isometry (resp. an affine transformation) $\tilde{\varphi}$ such that $\tilde{\varphi}(\bar{\mathrm{P}})=\sigma$. Hence, the transformation $\tilde{\varphi}\circ(S_{\mathrm{P}})^*\circ\tilde{\varphi}^{-1}$ is a symmetry with center σ, i.e., $T_2(M)$ is symmetric with respect to g^{II} (resp. ∇^{II}). Thus we have

PROPOSITION 8. 11. *If M is symmetric with respect to a pseudo-Riemannian metric g (resp. an affine connection ∇), so is $T_2(M)$ with respect to g^{II} (resp. ∇^{II})*

§ 9. Geodesics.

Let ∇ be a *torsionless* affine connection in M. We denote by $\Gamma_j{}^h{}_i$ the coefficients of ∇ in a coordinate neighborhood $(U, (x^h))$ of M, where $\Gamma_j{}^h{}_i=\Gamma_i{}^h{}_j$. Let $\tilde{C}$ be a curve in $T_2(M)$ and suppose that $\tilde{C}$ is expressed locally by equations

$$\xi^A=\xi^A(t), \qquad \text{i.e.,}$$

$$x^h=x^h(t), \qquad y^h=y^h(t), \qquad z^h=z^h(t) \tag{9. 1}$$

with respect to the induced coordinates $(\xi^A)=(x^h, y^h, z^h)$ in $\pi_2^{-1}(U)$, t being a parameter. We now put along $\tilde{C}\cap\pi_2^{-1}(U)$

$$v^h=z^h+y^jy^i\Gamma_j{}^h{}_i \tag{9.2}$$

and

$$\frac{\delta y^h}{dt}=\frac{dy^h}{dt}+\Gamma_j{}^h{}_i\frac{dx^j}{dt}y^i,\qquad \frac{\delta^2y^h}{dt^2}=\frac{d}{dt}\left(\frac{\delta y^h}{dt}\right)+\Gamma_j{}^h{}_i\frac{dx^j}{dt}\frac{\delta y^i}{dt};$$

(9.3)

$$\frac{\delta v^h}{dt}=\frac{dv^h}{dt}+\Gamma_j{}^h{}_i\frac{dx^j}{dt}v^i,\qquad \frac{\delta^2v^h}{dt^2}=\frac{d}{dt}\left(\frac{\delta v^h}{dt}\right)+\Gamma_j{}^h{}_i\frac{dx^j}{dt}\frac{\delta v^i}{dt},$$

where $x^h(t)$, $y^h(t)$ and $z^h(t)$ are the functions appearing in (9.1). Denoting by C the projection $\pi_2(\tilde{C})$ of $\tilde{C}$ in M, we see that the curve C is expressed as $x^h=x^h(t)$ in $(U,(x_h))$, $x^h(t)$ being the functions appearing in (9.1). Then the quantities

$$y^h,\ v^h,\ \frac{\delta y^h}{dt},\ \frac{\delta v^h}{dt},\ \frac{\delta^2y^h}{dt^2},\ \frac{\delta^2v^h}{dt^2}$$

defined above are respectively global vector fields along C.

A curve $\tilde{C}$ in $T_2(M)$ is a geodesic with respect to ∇^{II}, t being an affine parameter, if and only if its local expression (9.1) satisfies the differential equations

$$\frac{d^2\xi^A}{dt^2}+\tilde{\Gamma}_C{}^A{}_B\frac{d\xi^C}{dt}\frac{d\xi^B}{dt}=0,\qquad \text{i.e.,}$$

(9.4)

$$\frac{d^2x^h}{dt^2}+\tilde{\Gamma}_C{}^h{}_B\frac{d\xi^C}{dt}\frac{d\xi^B}{dt}=0,$$

$$\frac{d^2y^h}{dt^2}+\tilde{\Gamma}_C{}^{\bar{h}}{}_B\frac{d\xi^C}{dt}\frac{d\xi^B}{dt}=0,$$

$$\frac{d^2z^h}{dt^2}+\tilde{\Gamma}_C{}^{\bar{\bar{h}}}{}_B\frac{d\xi^C}{dt}\frac{d\xi^B}{dt}=0,$$

where $\Gamma_C{}^h{}_B$, $\Gamma_C{}^{\bar{h}}{}_B$ and $\Gamma_C{}^{\bar{\bar{h}}}{}_B$ are the coefficients of ∇^{II} given by (7.1), (7.2) and (7.3). The equations (9.4) are equivalent to the equations

$$\frac{d^2x^h}{dt^2}+\Gamma_j{}^h{}_i\frac{dx^j}{dt}\frac{dx^i}{dt}=0, \tag{9.5}$$

$$\frac{d^2y^h}{dt^2}+(y^s\partial_s\Gamma_j{}^h{}_i)\frac{dx^j}{dt}\frac{dx^i}{dt}+2\Gamma_j{}^h{}_i\frac{dx^j}{dt}\frac{dy^i}{dt}=0, \tag{9.6}$$

$$\frac{d^2z^h}{dt^2}+(z^s\partial_s\Gamma_j{}^h{}_i+y^ty^s\partial_t\partial_s\Gamma_j{}^h{}_i)\frac{dx^j}{dt}\frac{dx^i}{dt}$$
$$+4(y^s\partial_s\Gamma_j{}^h{}_i)\frac{dx^j}{dt}\frac{dy^i}{dt}+2\Gamma_j{}^h{}_i\frac{dy^j}{dt}\frac{dy^i}{dt}+2\Gamma_j{}^h{}_i\frac{dx^j}{dt}\frac{dz^i}{dt}=0. \tag{9. 7}$$

Making use of (9. 2) and (9. 3), we see that the system of differential equations (9. 5), (9. 6) and (9. 7) is equivalent to the the system of differential equations

$$\frac{d^2x^h}{dt^2}+\Gamma_j{}^h{}_i\frac{dx^j}{dt}\frac{dx^i}{dt}=0, \tag{9. 8}$$

$$\frac{\delta^2y^h}{dt^2}+R_{kji}{}^hy^k\frac{dx^j}{dt}\frac{dx^i}{dt}=0, \tag{9. 9}$$

$$\frac{\delta^2v^h}{dt^2}+R_{kji}{}^hv^k\frac{dx^j}{dt}\frac{dx^i}{dt}+4R_{kji}{}^hy^k\frac{dx^j}{dt}\frac{\delta y^i}{dt}$$
$$+(\nabla_tR_{sji}{}^h-\nabla_jR_{its}{}^h)y^ty^s\frac{dx^j}{ds}\frac{dx^i}{ds}=0, \tag{9. 10}$$

where $R_{kji}{}^h$ denote the components of the curvature tensor of ∇. That is to say, the system of differential equations (9. 8), (9. 9) and (9. 10) determines in $T_2(M)$ geodesics with respect to the affine connection ∇^{II}. Thus we have

PROPOSITION 9. 1. *Let $\tilde{C}$ be a geodesic in $T_2(M)$ with respect to ∇^{II}, where ∇ is a torsionless affine connection in M, and suppose that $\tilde{C}$ has the local expression* (9. 1). *Then the projection $C=\pi_2(\tilde{C})$ is a geodesic in M with respect to ∇. The vector field $y^h(t)$ defined along C is a Jacobi field with respect to ∇. The vector field $v^h(t)$ defined by* (9. 2) *along C satisfies the differential equation* (9. 10). *The affine parameter of $\tilde{C}$ induces naturally an affine parameter along C.*

Conversely, if there exists in M a geodesic with respect to ∇, C having the local expression $x^h=x^h(t)$ with affine parameter t, if there is given a Jacobi vector field $y^h(t)$ along C, and, if there is given a vector field $v^h(t)$ satisfying along C the differential equation (9. 10), *then the curve $\tilde{C}$ defined in $T_2(M)$ by the local expression $x^h=x^h(t)$, $y^h=y^h(t)$, $z^h=v^h(t)-y^j(t)y^i(t)\Gamma_j{}^h{}_i(x^s(t))$ is a geodesic in $T_2(M)$ with respect to ∇^{II}.*

Taking account of (9. 8), (9. 9) and (9. 10) we see easily that, *if there is given in M a geodesic C with respect to a torsionless affine connection ∇, C having the local expression $x^h=x^h(t)$, and a Jacobi field $v^h(t)$ along C, then the curve $\tilde{C}$ defined in $T_2(M)$ by the local expression $x^h=x^h(t)$, $y^h=0$, $z^h=v^h(t)$ is a geodesic with respect to ∇^{II}.*

We say that M is *complete* with respect to an affine connection (resp. a pseudo-Riemannian metric g) if along any geodesic any affine parameter takes an arbitrarily given real value. Then, taking account of (9. 8), (9. 9) and (9. 10), we have

PROPOSITION 9. 2. *If M is complete with respect to a torsionless affine connection ∇ (resp. a pseudo-Riemannian metric g), so is $T_2(M)$ with respect to ∇^{II} (resp. g^{II}).*

According to [15], we have from (9. 8), (9. 9) and (9. 10)

PROPOSITION 9. 3. *Let $\tilde{C}$ be a geodesic in $T_2(M)$ with respect to ∇^{II}, ∇ being a torsionless affine connection in M. Then the projection $\pi_{12}(\tilde{C})$ of $\tilde{C}$ in the tangent bundle $T_1(M)$ is also a geodesic with respect to ∇^C, where ∇^C is the complete lift of the affine connection ∇ in the sense of* [15].

BIBLIOGRAPHY

[1] DOMBROWSKI, P., On the geometry of the tangent bundle. J. Reine und Angew. Math. **210** (1962), 73–88.

[2] LEDGER, A. J., AND K. YANO, The tangent bundle of a locally symmetric space. J. London Math. Soc. **40** (1965), 487–492.

[3] LEDGER, A. J., AND K. YANO, Almost complex structures on tensor bundles. Journal of Differential Geometry. **1** (1967), 355–368.

[4] MORIMOTO, A., Note on prolongations of affine connections to tangent bundles. To appear.

[5] MORIMOTO, A., Prolongations of G-structures to tangent bundles. To appear.

[6] SASAKI, S., On the differential geometry of tangent bundles of Riemannian manifolds. Tôhoku Math. J. **10** (1958), 338–354.

[7] SATÔ, I., Almost analytic vector fields in almost complex manifolds. Tôhoku Math. J. **17** (1965), 185–199.

[8] TACHIBANA, S., AND M. OKUMURA, On the almost complex structure of tangent bundles of Riemannian spaces. Tôhoku Math. J. **14** (1962), 156–161.

[9] TANNO, S., An almost complex structure of the tangent bundle of an almost contact manifold. Tôhoku Math. J. **17** (1965), 7–15.

[10] TONDEUR, P., Structure presque kählérienne sur la fibré des vecteurs covariants d'une variéte riemannienne. C. R. Acad. Sci. Paris **254** (1962), 407–408.

[11] YANO, K., Tensor fields and connections on cross-sections in the tangent bundle of a differentiable manifold. Proc. Royal Soc. Edinburgh, Ser. A, **67** (1967), 277–288.

[12] YANO, K., Tensor fields and connections on cross-section in the cotangent bundle. Tôhoku Math. J. **19** (1967), 32–48.

[13] YANO, K., AND E. T. DAVIES, On the tangent bundles of Finsler and Riemannian manifold. Rend. Cir. Math. Palermo, Series II, **12** (1963), 1–18.

[14] YANO, K., AND S. ISHIHARA, Almost complex structures induced in tangent bundles. Kōdai Math. Sem. Rep. **19** (1967), 1–27.

[15] YANO, K., AND S. KOBAYASHI, Prolongations of tensor fields and connections to

tangent bundle, I, II. J. Math. Soc. Japan **186** (1966), 194–210, 236–246.

[16] YANO, K., AND A. J. LEDGER, Linear connections on tangent bundles. J. London Math. Soc. **39** (1964), 495–500.

[17] YANO, K., AND E. M. PATTERSON, Vertical and complete lifts from a manifold to its cotangent bundle. J. Math. Soc. Japan **19** (1967), 91–113.

[18] YANO, K., AND E. M. PATTERSON, Horizontal lifts from a manifold to its tangent bundle. J. Math. Soc. Japan **19** (1967), 185–198.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.