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ON THE EXISTENCE OF ANALYTIC MAPPINGS, II

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§1. Let R and S be Riemann surfaces which are the proper existence domains of n- and m-valued entire algebroid functions $f(z)$ and $F(w)$, respectively, where f and F are defined by irreducible equations

$$f^n+A_1(z)f^{n-1}+\cdots+A_{n-1}(z)f+A_n(z)=0, \tag{1}$$

$$F^m+B_1(w)F^{m-1}+\cdots+B_{m-1}(w)F+B_m(w)=0, \tag{2}$$

where $A_1, \cdots, A_n, B_1, \cdots, B_{m-1}$ and B_m are entire functions.

Let φ be an analytic mapping of R into S. Let $\mathfrak{P}_R$ and $\mathfrak{P}_S$ be the projection maps: $(z, f(z))\rightarrow z$ and $(w, F(w))\rightarrow w$, respectively. If φ preserves the projection maps, then we say that φ is a rigid analytic mapping. This means that every n-tuple of points on R having the same projection is carried to an m-tuple of points on S having the same projection. In this paper we study the analytic mappings of R into S. In the case of $n=m=2$, Ozawa obtained several interesting results [5], [6], [7], [8]. Here an analytic mapping means a non-trivial one.

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§2. In this section we assume that R and S have an infinite number of branch points. Put $h(z)=\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)$. Let E be the projection of all the branch points of R. Let $z_0\notin E$ be an arbitrary but fixed point in the z-plane. Let $U(z_0)$, $U(z_0)\cap E=\phi$ be a disk whose center is z_0. In $U(z_0)$ there exist n analytic branches of $\mathfrak{P}_R^{-1}(z)$: $\mathfrak{P}_R^{-1}(z)_1, \cdots, \mathfrak{P}_R^{-1}(z)_n$. Put $h_1(z)=\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)_1, \cdots, h_n(z)=\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)_n$. For these functions we define the fundamental symmetric polynomials:

$$H_1(z)=h_1(z)+h_2(z)+\cdots+h_n(z),$$

$$H_2(z)=h_1(z)h_2(z)+h_1(z)h_3(z)+\cdots+h_{n-1}(z)h_n(z),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$H_n(z)=h_1(z)h_2(z)\cdots h_n(z).$$

We can extend these functions over the z-plane except E. The resulting functions denoting with the same symbols are single-valued regular functions except E. Hence $h(z)$ satisfies the equation

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(3) $$h(z)^n - H_1(z)h(z)^{n-1} + \cdots + (-1)^n H_n(z) = 0.$$

It is easily seen that every point of the set E is a removable singularity of $H_j(z)$ $(j=1, \cdots, n)$. Thus $h(z)$ is an entire algebroid (or algebraic) function of z. We shall prove the following

THEOREM 1. *Assume that there exists an analytic mapping φ of R into S. If n is a prime number, then φ is rigid. If n is not a prime number, then the corresponding function $h(z)$ of φ is k-valued where k is a proper divisor of n and φ may or may not be rigid.*

Proof. We shall prove this along the same manner as in [9] pp. 29–34. In the first place we assume that $h(z)$ is n-valued. Then R is the proper existence domain of $h(z)$. Let p_0, $\mathfrak{P}_R(p_0) = z_0$ be a point on R whose order of ramification is $\lambda_{p_0} - 1$. Let q_0, $\mathfrak{P}_S(q_0) = w_0$ be the φ-image of p_0 on S. Then we have

(4) $$h(z) = w_0 + a_\tau (\sqrt[\lambda_{p_0}]{z - z_0})^\tau + \cdots, \qquad a_\tau \neq 0.$$

Put

(5) $$N(r; q_0, S) = \frac{1}{n\lambda_{q_0}} \int_0^r \{n(t; q_0, S) - n(0; q_0, S)\} \frac{dt}{t} + \frac{n(0; q_0, S)}{n\lambda_{q_0}} \log r,$$

where

$$n(r; q_0, S) = \sum_{\varphi(p) = q_0,\, |\mathfrak{P}_R(p)| \leq r} \tau.$$

Let q_1 and q_2 be distinct points on S. Then there exists a function $u(q; q_1, q_2)$ which is harmonic in q on S save at q_1 and q_2, has a positive normalized logarithmic singularity at q_1 and a negative normalized logarithmic singularity at q_2 and is bounded in the complement of some compact neighborhood of $\{q_1, q_2\}$ [1];

$$u(q; q_1, q_2) + \frac{1}{\lambda_{q_1}} \log \frac{1}{|w - w_1|},$$

$$u(q; q_1, q_2) - \frac{1}{\lambda_{q_2}} \log \frac{1}{|w - w_2|}$$

are harmonic at q_1 and q_2, respectively, where w_1 and w_2 are the projections of q_1 and q_2, respectively. Let $R(r)$ be the part of R whose projection lies on $|z| \leq r$ and $\Gamma(r)$ be the boundary of $R(r)$. We take a small neighborhood whose projection is a disk for every q_1- and q_2-points of φ and branch points of R. Then we have a subset $R'(r)$ of $R(r)$ with boundary $\Gamma'(r)$. We assume that there are no q_1- and q_2-points of φ and no branch points of R on $\Gamma(r)$. Then we have

$$\int_{\Gamma(r)} \frac{\partial v}{\partial n} ds = 0$$

with $v(p)=u(\varphi(p); q_1, q_2)$, and

$$r\frac{d\mu(r)}{dr}+\frac{n(r; q_1, S)}{n\lambda_{q_1}}-\frac{n(r; q_2, S)}{n\lambda_{q_2}}=0,$$

where

$$\mu(r)=\frac{1}{2n\pi}\int_{\gamma(r)} u(\varphi(\mathfrak{P}_R^{-1}(re^{i\theta})); q_1, q_2)d\theta,\ \gamma(r)=\mathfrak{P}_R(\Gamma(r)).$$

Using the continuity property of $\mu(r)$ we have

$$\mu(r)+N(r; q_1, S)-N(r; q_2, S)=A \text{ (const.)} \tag{6}$$

for every r.

Let K_{q_1} and K_{q_2}, $K_{q_1}\cap K_{q_2}=\phi$ be neighborhoods of q_1 and q_2 whose projections are disks with finite radii δ_{q_1} and δ_{q_2}, respectively. We define a function $u_{q_1}(q)$ as follows:

$$u_{q_1}(q)=\frac{1}{\lambda_{q_1}}\log\frac{\delta_{q_1}}{|w-w_1|},\qquad q\in K_{q_1},$$

$$=0,\qquad q\notin K_{q_1}.$$

We also define a function $u_{q_2}(q)$ analogously. Put

$$m(r; q_1, S)=\frac{1}{2n\pi}\int_{\gamma(r)} u_{q_1}(\varphi(\mathfrak{P}_R^{-1}(re^{i\theta})))d\theta.$$

Using $u_{q_2}(q)$ we also define $m(r; q_2, S)$ analogously. By (6) we have

$$m(r; q_1, S)+N(r; q_1, S)=m(r; q_2, S)+N(r; q_2, S)+Q(r), \tag{7}$$

where $A-B\leqq Q(r)\leqq A+B$, $B=\sup|u(q; q_1, q_2)-u_{q_1}(q)+u_{q_2}(q)|$. Let $\varphi(p)\neq q_1, q_2$ for every p with $\mathfrak{P}_R(p)=0$. Then we have

$$A=\lim_{r\to 0} u(r)=\frac{1}{n}\sum u(\varphi(\mathfrak{P}_R^{-1}(0); q_1, q_2)),$$

where the summation is taken for all choices of $\mathfrak{P}_R^{-1}(0)$.

From (7) we can derive a simple relation between the sum $m(r; q, S)+N(r; q, S)$ and $T(r, h)$. In the following $m(r; w_0)$, $N(r; w_0)$ and $T(r; h)$ are the Nevanlinna-Selberg corresponding functions for $h(z)$. Let $q_1, \cdots, q_j$ $(j\leqq m)$ be the points on S having the same projection w_0. Let $j=1$. Then we have

$$m(r; q_1, S)+N(r; q_1, S)=\frac{1}{m}\{m(r; w_0)+N(r; w_0)\}+O(1)=\frac{1}{m}T(r; h)+O(1). \tag{8}$$

Let $j>1$. Then we have

$$(9)\qquad \begin{aligned} &m\{m(r;\, q_\nu,\, S)+N(r;\, q_\nu,\, S)\}=\sum_{l=1}^{j}\lambda_{q_l}\{m(r;\, q_l,\, S)+N(r;\, q_l,\, S)\}+O(1)\\ &=m(r;\, w_0)+N(r;\, w_0)+O(1)=T(r;\, h)+O(1). \end{aligned}$$

The above relation (9) holds for all q_ν ($\nu=1, \cdots, j\leqq m$).

Let $\{w_\nu\}_{\nu=1}^{k-1}$ be the projections of all the branch points $\{q_\nu\}$ of S. Put

$$n(r;\, S_h)=\sum_{R(r)}(\tau-1),$$

$$N(r;\, S_h)=\frac{1}{n}\int_0^r\{n(r;\, S_h)-n(0;\, S_h)\}\frac{dt}{t}+\frac{n(0;\, S_h)}{n}\log r,$$

where τ is the quantity given in (4). By the Nevanlinna-Selberg second fundamental theorem applied to $h(z)$ we have

$$(k-2n)T(r;\, h)\leqq\sum_{\nu=1}^{k-1}N(r;\, w_\nu)-N(r;\, S_h)+O(\log r\, T(r;\, h))$$

outside a set of finite measure. Using (8) and (9) we have

$$\left(k-1-\frac{1}{m}\sum_{\nu=1}^{k'}\lambda_{q_\nu}\right)T(r;\, h)+O(1)>\sum_{\nu=1}^{k-1}N(r;\, w_\nu)-\sum_{\nu=1}^{k'}\lambda_{q_\nu}N(r;\, q_\nu,\, S),$$

where k' is the number of branch points which lie over $\{w_\nu\}_{\nu=1}^{k-1}$. Hence we have

$$\begin{aligned} &\left(\frac{1}{m}\sum_{\nu=1}^{k'}\lambda_{q_\nu}-2n+1\right)T(r;\, h)\\ &\leqq\sum_{\nu=1}^{k'}\lambda_{q_\nu}N(r;\, q_\nu,\, S)-N(r;\, S_h)+O(\log r\, T(r;\, h)) \end{aligned}$$

outside a set of finite measure. Since φ is an analytic mapping of R into S we have

$$h(z)-w_0=\left(\sum_{n=1}^{\infty}a_n\sqrt[\lambda_{p_0}]{z-z_0}\right)^{\lambda_{q_0}},$$

where $h(z_0)=w_0$, $\mathfrak{P}_R(p_0)=z_0$ and $\mathfrak{P}_S(q_0)=w_0$. Hence we have

$$\sum_{\nu=1}^{k'}\lambda_{q_\nu}N(r;\, q_\nu,\, S)-N(r;\, S_h)\leqq\sum_{\nu=1}^{k'}N(r;\, q_\nu,\, S)=\frac{k'}{m}T(r;\, h)+O(1).$$

Consequently we have the following inequality:

$$\left(\frac{1}{m}\sum_{\nu=1}^{k}(\lambda_{q_\nu}-1)-2n+1\right)T(r;\, h)<O(\log r\, T(r;\, h))$$

outside a set of finite measure. On the other hand S has an infinite number of branch points whose order of ramification $\geqq 1$. This is a contradiction. Hence $h(z)=\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)$ cannot be an n-valued function of z for every analytic mapping

φ of R into S whenever it exists. Thus the equation (3) is reducible. Therefore the assertions of theorem 1 is proved by means of the same reason as in the theory of algebraic functions [2]. This completes the proof of theorem 1.

There exists a pair of Riemann surfaces for which there is a non-rigid analytic mapping. Let R and S be the proper existence domains of $f(z)$ and $F(w)$ defined by

$$f^{mn}=z\prod_{j=1}^{\infty}\left(1-\frac{z}{j^n}\right)^n, \qquad F^m=w\prod_{j=1}^{\infty}\left(1-\frac{w^n}{j^n}\right),$$

respectively. Then there exists an analytic mapping φ of R into S induced by $h(z)=\sqrt[n]{z}$, that is, $\varphi=\mathfrak{P}_S^{-1}\circ h\circ\mathfrak{P}_R$.

§ **3.** In this section we study the analytic mappings of R_n into S_m where R_n and S_m are the proper existence domains of n- and m-valued entire algebroid functions $f(z)$ and $F(w)$, respectively, defined by irreducible equations

$$f^n=G(z), \qquad F^m=g(w),$$

where G and g are entire functions having an infinite number of zeros whose orders are less than n and m and are coprime to n and m, respectively. Then R_n and S_m are regularly branched n- and m-sheeted covering Riemann surfaces, respectively.

We shall give a representation of an entire function $\mathfrak{f}$ on R_n. Let $p_1=(z, \sqrt[n]{G(z)})$, $p_2=(z, \omega\sqrt[n]{G(z)}), \cdots, p_{n-1}=(z, \omega^{n-2}\sqrt[n]{G(z)})$ and $P_n=(z, \omega^{n-1}\sqrt[n]{G(z)})$, where $\omega=\exp(2\pi i/n)$. Put

$$f_0=\{\mathfrak{f}(p_1)+\mathfrak{f}(p_2)+\cdots+\mathfrak{f}(p_n)\}/n,$$
$$f_1=\{\mathfrak{f}(p_1)+\omega^{n-1}\mathfrak{f}(p_2)+\cdots+\omega\mathfrak{f}(p_n)\}/n\sqrt[n]{G(z)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$f_{n-1}=\{\mathfrak{f}(p_1)+\omega\mathfrak{f}(p_2)+\cdots+\omega^{n-1}\mathfrak{f}(p_n)\}/n\sqrt[n]{G(z)}^{n-1}.$$

Now we introduce a local parameter around a branch point of R_n and expand $\mathfrak{f}$ in terms of the local parameter. Then the single-valuedness of f_j $(j=0, \cdots, n-1)$ are easily seen. Hence we have a representation of the form:

$$\mathfrak{f}(p)\{=\mathfrak{f}((z, \sqrt[n]{G(z)}))\}=f_0(z)+f_1(z)\sqrt[n]{G(z)}+\cdots+f_{n-1}(z)\sqrt[n]{G(z)}^{n-1},$$

where f_0 and f_1 are entire functions of z and f_j $(j=2, \cdots, n-1)$ are meromorphic functions of z which might have poles only at the zeros of G and the order of pole at a zero of G with order k is at most $[kj/n]$, where [ ] denotes the notation of Gauss. In the subsequent we say that a system of n functions (i.e. $L_0, \cdots, L_{n-1}$) satisfies the property (A) when L_0 and L_1 are entire functions and L_j $(j=2, \cdots, n-1)$ are meromorphic functions which might have poles only at the zeros of G and the order of pole at a zero of G with order k is at most $[kj/n]$. We shall prove the following

THEOREM 2. *Assume that n is a prime number. Then there is no analytic mapping of R_n into S_m, when $n\neq m$.*

Proof. By theorem 1 every analytic mapping φ of R_n into S_m is rigid whenever it exists. Hence the corresponding function $h(z)=\mathfrak{P}_{S_m}\circ\varphi\circ\mathfrak{P}_{R_n}^{-1}(z)$ is an entire function of z. Let g^* be the function of S_m defined by $g^*=\sqrt[m]{g\circ\mathfrak{P}_{S_m}}$. Then we have

$$g^*\circ\varphi\circ\mathfrak{P}_{R_n}^{-1}(z)=\lambda_0(z)+\lambda_1(z)\sqrt[n]{G(z)}+\cdots+\lambda_{n-1}(z)\sqrt[n]{G(z)}^{n-1}, \tag{10}$$

where $(\lambda_0, \cdots, \lambda_{n-1})$ satisfies the property (A). On the other hand we have

$$g^*\circ\varphi\circ\mathfrak{P}_{R_n}^{-1}(z)=\sqrt[m]{g\circ\mathfrak{P}_{S_m}}\circ\mathfrak{P}_{S_m}^{-1}\circ h\circ\mathfrak{P}_{R_n}\circ\mathfrak{P}_{R_n}^{-1}(z)=\sqrt[m]{g\circ h(z)}. \tag{11}$$

By (10) and (11) we have

$$g\circ h(z)=\{\lambda_0(z)+\lambda_1(z)\sqrt[n]{G(z)}+\cdots+\lambda_{n-1}(z)\sqrt[n]{G(z)}^{n-1}\}^m.$$

Since h is an entire function we have

$$\{\lambda_0+\lambda_1\omega\sqrt[n]{G}+\cdots+\lambda_{n-1}\omega^{n-1}\sqrt[n]{G^{n-1}}\}^m=\{\lambda_0+\lambda_1\sqrt[n]{G}+\cdots+\lambda_{n-1}\sqrt[n]{G^{n-1}}\}^m,$$

where $\omega=\exp(2\pi i/n)$. Hence at most one of $\lambda_0, \cdots, \lambda_{n-2}$ and λ_{n-1} does not vanish identically. We have one of the following functional equations:

$$g\circ h(z)=\lambda_j(z)^m G(z)^{jm/n} \qquad (j=0, \cdots, n-1).$$

Using the Nevanlinna-Selberg ramification relation we can easily see that $g\circ h(z)=\lambda_0(z)^m$ cannot hold in our case. Let $(n, m)=1$. Then $G^{jm/n}$ cannot reduce to a single-valued function of z. This is a contradiction. Hence there is no analytic mapping of R_n into S_m when $(n, m)=1$. Let $cn=m$ with a integer $c\geqq 2$. Then we have

$$g\circ h(z)=\lambda_j(z)^m G(z)^{jc} \qquad (j=0, \cdots, n-1).$$

However, since the orders of all the zeros of g are not divisors of m, by the Nevanlinna-Selberg ramification relation we can see that such functional equations cannot hold in our case. Consequently there is no analytic mapping of R_n into S_m when n is a prime number and $n\neq m$.

By the quite same method we can prove the following theorem.

THEOREM 3. *Suppose that there exists a rigid analytic mapping φ of R_n into S_m. Then n is an integral multiple of m by an integer c and the corresponding entire function $h(z)$ satisfy one of the following functional equations:*

$$f_j(z)^m G(z)^k=g\circ h(z),$$

where $k=cj$ is coprime to m and $(f_0, \cdots, f_{n-1})$ satisfies the property (A).

§4. Let R_n and S_m be the regularly branched surfaces defined in §3. We give the following theorems which can be proved by means of the same method as in [3], [4], [6]. Thus the proofs may be omitted here.

THEOREM 4 (cf. Theorem 3 in [4], Theorem 3 in [6]). *Suppose that there exists a rigid analytic mapping φ of R_n into S_m. Then the corresponding entire function $h(z)$ satisfies*

$$\overline{\lim_{r\to\infty}} \frac{N(r; 0, G)}{T(r; h)} = \infty.$$

Let G_c and g_c be canonical products having the same zeros with the same orders as those of G and g, respectively. Let ρ_{G_c} and ρ_{g_c} be the orders of G_c and g_c, respectively. Then we have the following

THEOREM 5 (cf. Theorem 1 in [3]). *Suppose that $\rho_{G_c}<\infty$ and $0<\rho_{g_c}<\infty$ and that there exists a rigid analytic mapping φ of R_n into S_m. Then ρ_{G_c} is an integral multiple of ρ_{g_c}.*

THEOREM 6 (cf. Theorem 2 in [3]). *Suppose that there exists a rigid analytic mapping φ of R_n into itself. Then φ is a univalent conformal mapping of R_n onto itself and the corresponding entire function $h(z)$ is a linear function of the form $e^{2\pi ip/q}z+b$ with a suitable rational number p/q.*

Recently Ozawa [8] introduced the notion of a finite modification of an ultrahyperelliptic surface and proved two interesting theorems. According to his definition we say that S_n is a finite modification of R_n when $G(z)$ and $g(z)$ have the same zeros for $|z|\geqq R_0$ for a suitable R_0.

THEOREM 7 (cf. Theorem 1 in [8]). *Suppose that there exists a rigid analytic mapping φ of R_n into S_n, which is a finite modification of R_n, then the corresponding entire function $h(z)$ reduces to the form $az+b$, that is, φ is a univalent conformal mapping of R_n onto S_n.*

THEOREM 8 (cf. Theorem 2 in [8]). *Suppose that there exists a rigid analytic mapping φ of R_n into S_n, which is a finite modification of R_n and that G and g have the same number of zeros in $|z|<R_0$, then φ is a univalent conformal mapping of R_n onto S_n and the corresponding entire function $h(z)$ reduces to the form $e^{2\pi ip/q}z+b$ with a suitable rational number p/q.*

§5. Let R be a Riemann surface defined in §1. Let S^* be a Riemann surface which is the proper existence domain of m-valued algebraic function $F^*(w)$ defined by an irreducible equation

$$F^{*m}+P_1(w)F^{*m-1}+\cdots+P_{m-1}(w)F^*+P_m(w)=0,$$

where P_j $(j=1, \cdots, m)$ are polynomials.

Let φ be an analytic mapping of R into S^*. Let $\mathfrak{P}_{S^*}$ be the projection map: $(w, F^*(w))\to w$. As before we define the corresponding function $h(z)=\mathfrak{P}_{S^*}\circ\varphi\circ\mathfrak{P}_R^{-1}(z)$.

Then by the same method as in § 2 we can prove the following

THEOREM 9. *Assume that there exists an analytic mapping φ of R into S^*, when the genus of S^* is greater than $m(n-1)+1$. If n is a prime number, then φ is rigid. If n is not a prime number, then the corresponding function $h(z)$ is k-valued where k is a proper divisor of n and φ may or may not be rigid.*

§ 6. Let R_n be a regularly branched Riemann surface defined in § 3. Let S_m^* be a Riemann surface which is the proper existence domain of an m-valued algebraic function $F^*(w)$ defined by

$$F^{*m}=\prod_{j=1}^{p}(w-w_j)^{c_j} \tag{12}$$

where $c_j<m$ are positive integers which are coprime to m and $w_i\neq w_j$ for $i\neq j$. Then S_m^* is a closed and regularly branched m-sheeted covering Riemann surface.

Let φ be an analytic mapping of R_n into S_m^*. Let n be a prime number. By theorem 9 the corresponding function $h(z)=\mathfrak{P}_{S_m^*}\circ\varphi\circ\mathfrak{P}_{R_n}^{-1}(z)$ reduces to a single-valued function of z, when the genus of S_m^* is greater than $m(n-1)+1$. Using the same method as in the proof of theorem 2 we can prove the following

THEOREM 10. *Assume that n is a prime number and $n\neq m$. Then there is no analytic mapping of R_n into S_m^*, when the genus of S_m^* is greater than $m(n-1)+1$. Furthermore there is no rigid analytic mapping of R_n into S_m^*, when the genus of S_m^* is greater than 1.*

There exists a pair of Riemann surfaces for which there is a non-rigid analytic mapping.

EXAMPLE 1 (cf. [4], [6]). Let R_n be the proper existence domain of $\sqrt[n]{\operatorname{sn} z}$ with Jacobi's sn-function. Let S_2^* be the hyperelliptic surface which is the proper existence domain of $\sqrt{(1-w^{2n})(1-k^2w^{2n})}$. It is well-known that

$$\begin{aligned}(\operatorname{sn}'z)^2&=(1-\operatorname{sn}^2z)(1-k^2\operatorname{sn}^2z)\\&=(1-\sqrt[n]{\operatorname{sn} z})(\omega-\sqrt[n]{\operatorname{sn} z})\cdots(\omega^{n-1}-\sqrt[n]{\operatorname{sn} z})\\&\quad\cdot(-1-\sqrt[n]{\operatorname{sn} z})(-\omega-\sqrt[n]{\operatorname{sn} z})\cdots(-\omega^{n-1}-\sqrt[n]{\operatorname{sn} z})\\&\quad\cdot(1-\sqrt[n]{k\operatorname{sn} z})(\omega-\sqrt[n]{k\operatorname{sn} z})\cdots(\omega^{n-1}-\sqrt[n]{k\operatorname{sn} z})\\&\quad\cdot(-1-\sqrt[n]{k\operatorname{sn} z})(-\omega-\sqrt[n]{k\operatorname{sn} z})\cdots(-\omega^{n-1}-\sqrt[n]{k\operatorname{sn} z}),\end{aligned}$$

where $\omega=\exp(2\pi i/n)$. This shows that there exists an analytic mapping φ of R_n into S_2^*, which is induced by $\sqrt[n]{\operatorname{sn} z}$, that is, $h(z)=\sqrt[n]{\operatorname{sn} z}$, $\varphi=\mathfrak{P}_{S_2^*}^{-1}\circ h\circ\mathfrak{P}_{R_n}$, when the genus of S_2^* is $2n-1$. This mapping is not rigid.

EXAMPLE 2 (cf. [4], [6]). Let R_n be the proper existence domain of $\sqrt[n]{\wp(z)}$ with

Weierstrass' $\wp$-function with the primitive periods $2\omega_1$ and $2\omega_2$ where $n\geqq 5$ is an odd integer. Let $S_2{}^*$ be the hyperelliptic surface which is the proper existence of $2\sqrt{(w^n-e_1)(w^n-e_2)(w^n-e_3)}$, where $e_1=\wp(\omega_1)$, $e_2=\wp(\omega_2)$ and $e_3=\wp(\omega_1+\omega_2)$. Evidently the genus of $S_2{}^*$ is $3t-1$, when $n=2t+1$. It is well-known that

$$\begin{aligned}\wp'(z)^2&=4\{\wp(z)-e_1\}\{\wp(z)-e_2\}\{\wp(z)-e_3\}\\&=4\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_1}\}\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_1}\,\omega\}\cdots\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_1}\,\omega^{n-1}\}\\&\quad\cdot\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_2}\}\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_2}\,\omega\}\cdots\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_2}\,\omega^{n-1}\}\\&\quad\cdot\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_3}\}\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_3}\,\omega\}\cdots\{\sqrt[n]{\wp(z)}-\sqrt[n]{e_3}\,\omega^{n-1}\},\end{aligned}$$

where $\omega=\exp(2\pi i/n)$. This shows that there exists an analytic mapping φ of R_n into $S_2{}^*$, which is induced by $\sqrt[n]{\wp(z)}$, that is, $h(z)=\sqrt[n]{\wp(z)}$, $\varphi=\mathfrak{P}_{S_2^*}^{-1}\circ h\circ\mathfrak{P}_{R_n}$. This analytic mapping is not rigid.

EXAMPLE 3 (cf. [4]). Put

$$z=\Psi(w)=\int_0^w\frac{dt}{\sqrt[3]{\{(t-a_1)(t-a_2)(t-a_3)\}^2}},$$

where a_1, a_2 and a_3 are non-zero distinct complex numbers. Let $w=f(z)$ be the inverse function, then it can be continued over the whole plane as a single-valued meromorphic function. Let $S_3{}^*$ be the regularly branched three-sheeted covering Riemann surface which is the proper existence domain of $\sqrt[3]{\{(w^n-a_1)(w^n-a_2)(w^n-a_3)\}^2}$. Let R_n be the proper existence domain of $\sqrt[n]{f(z)}$, then it is a regularly branched n-sheeted covering Riemann surface. $f(z)$ satisfies

$$\begin{aligned}f'(z)^3&=\{(f(z)-a_1)(f(z)-a_2)(f(z)-a_3)\}^2\\&=\{(\sqrt[n]{f(z)}-\sqrt[n]{a_1})(\sqrt[n]{f(z)}-\sqrt[n]{a_1}\,\omega)\cdots(\sqrt[n]{f(z)}-\sqrt[n]{a_1}\,\omega^{n-1})\\&\quad\cdot(\sqrt[n]{f(z)}-\sqrt[n]{a_2})(\sqrt[n]{f(z)}-\sqrt[n]{a_2}\,\omega)\cdots(\sqrt[n]{f(z)}-\sqrt[n]{a_2}\,\omega^{n-1})\\&\quad\cdot(\sqrt[n]{f(z)}-\sqrt[n]{a_3})(\sqrt[n]{f(z)}-\sqrt[n]{a_3}\,\omega)\cdots(\sqrt[n]{f(z)}-\sqrt[n]{a_3}\,\omega^{n-1})\}^2,\end{aligned}$$

where $\omega=\exp(2\pi i/n)$. This shows that there exists an analytic mapping φ of R_n into $S_3{}^*$, which is induced by $\sqrt[n]{f(z)}$, that is, $h(z)=\sqrt[n]{f(z)}$, $\varphi=\mathfrak{P}_{S_2^*}^{-1}\circ h\circ\mathfrak{P}_{R_n}$, when the genus of $S_3{}^*$ is $3n-2$. This analytic mapping is not rigid.

We can prove the following theorem by the same method as in theorem 2.

THEOREM 11. *Suppose that there exists a rigid analytic mapping φ of R_n into $S_m{}^*$, when the genus of $S_m{}^*$ is greater than* 1. *Then n is an integral multiple of m and the corresponding meromorphic function $h(z)$ satisfies one of the following functional equations*:

$$f(z)^m G(z)^k=\prod_{j=1}^{p}(h(z)-w_j)^{c_j}, \tag{13}$$

where $k<m$ is an integer being coprime to m and f is a meromorphic function.

By this theorem we can prove the following theorem:

THEOREM 12. *Suppose that there exists a rigid analytic mapping φ of R_n into S_m^*, when the genus of S_m^* is greater than 1. Then the corresponding meromorphic function $h(z)$ satisfies*

$$\left(p-\frac{2m}{m-1}\right)\Big/(m-1)\leqq\overline{\lim_{r\to\infty}}\frac{N(r;0,G)}{T(r;h)}\leqq\sum_{j=1}^{p}c_j.$$

Proof. In this case every branch point of R_n has its φ-image on a branch point of S_m^*. This is proved as in [5]. Hence $h(z)$ should be a transcendental meromorphic function of z. In fact, assume that $h(z)$ is a polynomial. Then every branch point of S_m^* is covered only finitely often by $\varphi(R_n)$ and every branch point of R_n is carried to a branch point of S_m^*. There is an infinite number of branch points on R_n. This is a contradiction. Using (13) we have

$$k\,N(r;0,G)\leqq\sum_{j=1}^{p}c_jN(r;w_j,h)=\sum_{j=1}^{p}c_jT(r;h)+O(1).$$

Hence we have

$$\overline{\lim_{r\to\infty}}\frac{N(r;0,G)}{T(r;h)}\leqq\sum_{j=1}^{p}c_j.$$

Let $w_1,\cdots,w_{p-1}$ and w_p be the values defined in (12). Then by the second fundamental theorem applied to $h(z)$ we have

$$(p-2)T(r;h)\leqq\sum_{j=1}^{p}\bar{N}(r;w_j,h)+O(\log r\,T(r;h))$$

outside a set of finite measure. On the other hand we have

$$\sum_{j=1}^{p}\bar{N}(r;w_j,h)=\sum_{j=1}^{p}\bar{N}_{m-1}(r;w_j,h)+\sum_{j=1}^{p}\bar{N}_m(r;w_j,h),$$

$$(m-1)\sum_{j=1}^{p}\bar{N}_m(r;w_j,h)\leqq N(r;0,h')\leqq 2\,T(r;h)+O(\log r\,T(r;h))$$

outside a set of finite measure, where $\bar{N}_{m-1}$ denotes the counting function of w_j-points whose multiplicities are less than m and N_m that of other w_j-points which are counted only once, respectively. By (13) we have

$$\sum_{j=1}^{p}\bar{N}_{m-1}(r;w_j,h)\leqq k\,N(r;0,G).$$

Therefore we have

$$\left(p-\frac{2m}{m-1}\right)T(r;\, h)\leqq k\, N(r;\, 0,\, G)+O(\log r\, T(r;\, h))$$

outside a set of finite measure. Hence we have

$$\left(p-\frac{2m}{m-1}\right)\Big/(m-1)\leqq \overline{\lim_{r\to\infty}}\frac{N(r;\, 0,\, G)}{T(r;\, h)}.$$

This completes the proof.

In the case of $n=m=2$, Ozawa proved the assertion of theorem 12. Further he showed the sharpness of the left and right hand side inequalities of (14) [7].

We give the sharpness of the right hand side inequality of (14) in the case of $n=m$ and the sharpness of the left hand side inequality of (14) in the case of $n=m=3$ by the following examples:

EXAMPLE 4. Let $G(z)$ be $e^{pz}-1$ and let $P(w)$ be

$$\prod_{j=1}^{p}(w-w_j)=\prod_{j=1}^{p}(w-\omega^j),$$

where $\omega=\exp(2\pi i/p)$. Let R_n and $S_n{}^*$ be Riemann surfaces which are the proper existence domains of $\sqrt[n]{G(z)}$ and $\sqrt[n]{P(w)}$, respectively. Then there exists a rigid analytic mapping which is induced by $h(z)=e^z$, that is, $\varphi=\mathfrak{P}_{S_2^*}^{-1}\circ h\circ\mathfrak{P}_{R_n}$. In this case

$$N(r;\, 0,\, G)=p\; T(r;\, e^z)=p\; T(r;\, h).$$

Hence we have

$$\overline{\lim_{r\to\infty}}\frac{N(r;\, 0,\, G)}{T(r;\, h)}=p.$$

EXAMPLE 5. Let $G(z)$ be

$$\{(f(z)-a_4)\,(f(z)-a_5)\cdots(f(z)-a_p)\}^2$$

where $f(z)$ is the function defined in example 3 and $a_4, \cdots, a_p$ are $p-3$ different complex numbers and they are different with a_1, a_2 and a_3. Let R_3 be the proper existence domain of $\sqrt[3]{G(z)}$ and let S_3^* be the proper existence domain of $\sqrt[3]{\{(w-a_1)\,(w-a_2)\cdots(w-a_p)\}^2}$. Then there is an analytic mapping φ whose corresponding function is $f(z)$. In this case

$$2N(r;\, 0,\, G)=(p-3)\; T(r;\, f(z))=(p-3)\; T(r;\, h).$$

Hence

$$\overline{\lim_{r\to\infty}}\frac{N(r;\, 0,\, G)}{T(r;\, h)}=\frac{p-3}{2}.$$

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