ON THE TOTAL ABSOLUTE CURVATURE OF MANIFOLDS IMMERSED IN RIEMANNIAN MANIFOLD

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For the total curvature \( \int_C k(s) ds \) of a closed curve \( C \) of class \( C'' \) in an \( n \)-dimensional Euclidean space \( E^n \), as is well known, we have the inequality:

\[
\int_C k(s) ds \geq 2\pi
\]

where \( s \) denotes the arc length of \( C \) and \( k(s) \) the curvature of \( C \). The equality holds only when \( C \) is a plane convex curve. This was proved by Fenchel [6] in 1929, in the case \( n=3 \) and by Borsuk [1], in 1949, in the case \( n>3 \).

Chern and Lashof [4], in 1959, extended this result to the case of a compact orientable \( n \)-dimensional manifold \( M^n \) immersed in Euclidean space \( E^{n+N} \) and obtained the inequality

\[
\int_{M^n} K^*(p) dV \geq 2c_{n+N-1}
\]

where \( K^*(p) \) denotes the total curvature of \( M^n \) at \( p \), \( dV \) is the volume element of \( M^n \) and \( c_{n+N-1} \) denotes the volume of the \( n+N-1 \) dimensional unit sphere. The equality holds only when \( M^n \) belongs to a linear subspace \( E^{n+1} \) of dimensional \( n+1 \), and is imbedded as a convex hypersurface in \( E^{n+1} \).

The left hand sides of (1) and (2) represent the total curvature of \( C \) and \( M^n \), respectively. \( K^*(p) \) is defined by

\[
K^*(p) = \int_{S^{n-1}_p} |G(p, e)| d\sigma_{N-1}
\]

where \( S^{n-1}_p \) denotes the \( N-1 \) dimensional sphere of all unit normal vectors at \( p \) of \( M^n \), \( G(p, e) \) is the Lipschitz-Killing curvature at \( e \in S^{n-1}_p \) and \( d\sigma_{N-1} \) is the \((N-1)\)-differential form on the normal bundle of \( M^n \) in \( E^{n+N} \) that becomes its volume element on each fibre \( S^{n-1}_p \).

Willmore and Saleemi [12] had generalized Chern-Lashof's results by defining the total curvature of an orientable manifold immersed in a Riemannian manifold, but unfortunately, the results contained mistakes, and hence they are false.

The object of this paper is to generalize the Lipschitz-Killing curvature to the manifolds immersed in a complete, simply-connected Riemannian manifold with non-positive sectional curvature, and then to define the total absolute curvature as

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in the left hand side of (2) and (3), and to prove that many results in [3], [4], [5], [6], [8], [9] and [11] can hold too.

1. Preliminaries.

We shall assume throughout that all manifolds, maps, metric, etc. are differentiable of class $C^\infty$. We shall assume that $X$ is compact and orientable, and $Y$ is complete, simply-connected Riemannian manifold with non-positive sectional curvature. Then a well-known theorem of Cartan [2] states that any two points of $Y$ can be joined by a unique geodesic. Equivalently, at each point $m \in Y$ the exponential map $\text{exp}_m: T_m(Y) \to Y$ is a global diffeomorphism.

Now, let $f$ is an immersion from $X$ to $Y$: 

(4) \hspace{1cm} f: X \to Y

that is, we assume that the map $df_p$ induced on the tangent space $T_p(X)$ to $X$ at $p$ is of rank $n$ for all $p$ in $X$, where $\dim X = n$, and $\dim Y = n+N$. In the following, let $\bar{e}$ always denote the parallel translation of the tangent vector $e \in T_p(Y)$, $p \in X$, along the unique geodesic joining $f(p)$ and a fixed point.

Now, let $q$ be a point in $f(X)$, and set

(5) \hspace{1cm} B_v = \{ (p, e) : p \in X, \bar{e} \text{ the parallel translation of the unit tangent vector } e \text{ along the unique geodesic joining } f(p) \text{ and } q \text{ and perpendicular to } \exp_q^{-1}(f(X)) \text{ at } \exp_q^{-1}f(p), e \in T_{f(p)}(Y) \}.

It is clear that if the Riemannian manifold $Y$ is the $(n+N)$-dimensional Euclidean space $\mathbb{E}^{n+N}$, then $B_v$ is just the normal bundle of $X$, and so we call that $B_v$ is the pseudo-normal bundle of $X$ with respect to the point $q$, and each element of $B_v$ is called a pseudonormal vector of $X$ with respect to the point $q$. It is easy to see that $B_v$ is a bundle of $(N-1)$-dimensional spheres over $X$ by the properties of parallel translation, and is a $C^\infty$-manifold of dimension $n+N-1$. The mapping

(6) \hspace{1cm} \tilde{v}: B_v \to S_q^{n+N-1}

of $B_v$ into the unit sphere $S_q^{n+N-1}$ of $T_q(Y)$ defined by $\tilde{v}(p, e) = \bar{e}$ is the mapping with which we will be concerned in this paper.

Let $dV$ be the volume element of $X$. There is a differential form $d\sigma_{N-1}$ of degree $N-1$ on $B_v$ such that its restriction to a fibre is the volume element of the sphere of pseudo-normal vectors at $p \in X$; then $d\sigma_{N-1} \wedge dV$ is the volume element of $B_v$. Let $d\Sigma_{n+N-1}$ be the volume element of $S_q^{n+N-1}$. The function $G(p, q, e)$ defined by

(7) \hspace{1cm} \tilde{v}^*d\Sigma_{n+N-1} = G(p, q, e)dV \wedge d\sigma_{N-1}

where $\tilde{v}^*$ is the dual mapping on differential forms induced by $\tilde{v}$ is a function in $B_v$. It generalizes the Lipschitz-Killing curvature and we call it the $G$-Lipschitz-Killing curvature at $e$. We call the integral
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(8) \( K^*(p, q) = \int |G(p, q, e)| d\sigma_{n-1} \geq 0 \)

over the sphere of unit pseudo-normal vectors at \( f(p) \) the total absolute curvature at \( p \) with respect to \( q \), and define as the total absolute curvature of \( X \) with respect to \( q \) the integral

(9) \( \int_X K^*(p, q) dV \)

and the total absolute curvature of \( X \) itself as the following

(10) \( T(f) = \inf_{q \in f(X)} \int_X K^*(p, q) dV/c_{n+N-1}. \)

Since \( X \) is compact by the assumption, (9) is a continuous function of \( q \) over \( f(X) \). Hence, there exists at least one point in \( f(X) \), say \( o \) that

(11) \( T(f) = \int_X K^*(p, o) dV/c_{n+N-1}. \)

In the special case, \( Y = E^{n+N} \), we have \( \bar{e} = e \), and the exponential map can be identified with the identity map, hence it is clear that this definition is consistent with Chern-Lashof’s definition in the case where \( Y \) is Euclidean.

2. Generalization of Chern-Lashof’s theorems in [4].

THEOREM 1. Let \( X, Y \) and \( f \) be given as in section 1, then, the total absolute curvature \( T(f) \) satisfies the following inequality

(12) \( T(f) \geq \sum_{i=0}^{n} \beta_i \equiv \beta \)

where \( \beta_i \) is the \( i \)-th Betti number of \( X \).

Proof. For any \( e \in S^{n+N} \), where \( o \) is any one of the points in \( f(X) \) satisfying the equality (11), we define the following function \( g \):

(13) \( g: X \rightarrow R \)

by \( g(p) = \exp_{(p)}^{-1}(f(p)) \cdot e^* \), and let \( m_i(e^*) \) denote the number of critical points of index \( i \) for the function \( g \).

Now, for every \( (p, e) \in B \) with \( \bar{e}(e) = e^* \), we have

(14) \( dg(p) = d(\exp_{(p)}^{-1}f)(p) \cdot e^* = 0 \)

hence, \( p \) is a critical point of the function \( g \). Conversely, if \( p \) is a critical point of the function \( g \), then, by the definition of the critical point, we have

\( dg(p)(\exp_{(p)}^{-1}f)(p) \cdot e^* = 0, \)

so that, \( (p, e) \) belongs to \( B \), where \( e \) is the unit vector at \( f(p) \) which is the inverse of the parallel translation of \( e^* \) along the unique geodesic joining \( f(p) \) and \( o \) in the
Riemannian manifold $Y$. Therefore, the number of all critical points of $g$ which is denoted by $m(e^*) \ (= \sum_{i=0}^{n} m_i(e^*))$ is equal to the number of points $(p, e) \in B_\theta$ which is transformed onto $e^*$ by $\tilde{v}$, hence, by (7) and (8), we have

$$\int_{X} K^*(p, o) dV = \int_{S_\theta^{n+N-1}} m(e^*) d\Sigma^{n+N-1}.$$  

But according to Sard's theorem, except a set of measure 0, $m(e^*)$ is well defined and finite, when $X$ is compact. So that, with respect to any coefficient field, we have the Morse's inequalities:

$$m_i(e^*)-m_{i-1}(e^*)+\cdots+(-1)^i m_0(e^*)$$

$$\geq \beta_i-\beta_{i-1}+\cdots+(-1)^i \beta_0, \quad i=0, 1, \ldots, n-1,$$

$$m_0(e^*)-m_{n-1}(e^*)+\cdots+(-1)^n m_0(e^*)=\beta_n-\beta_{n-1}+\cdots+(-1)^n \beta_0.$$  

Hence, we have

$$m_i(e^*) \geq \beta_i, \quad i=1, 2, \ldots, n,$$

$$m(e^*) \equiv \sum \beta_i = \beta$$

substitute these into (15), we have

$$T(f) = \int_{X} K^*(p, o) dV/c_{n+N-1} \geq \sum \beta_i = \beta$$

so that, we get the inequality (12).

**Theorem 2.** Under the same hypothesis of Theorem 1, if

$$T(f) < 3$$

then, $X$ is homeomorphic to a sphere of dimension $n$.

**Proof.** Let $o$ be one of the points in $f(X)$ such that it satisfies the equality (11), then, our hypothesis implies that there exists a set of positive measure on $S_\theta^{n+N-1}$ such that $e^*$ is a unit vector in the set, $\exp_\theta^{-1}(f(p)) \cdot e^*$ has just two critical points. For if not we would have $T(f) \geq 3$. Now, a theorem of Milnor [7] asserts that if a compact differentiable manifold $X$ has a real-valued differentiable function on it with only two critical points, then, $X$ is homeomorphic to a sphere. It follows from this that $X$ is homeomorphic to a sphere, and this theorem is proved.

**Theorem 3.** Under the same hypothesis of Theorem 1, if

$$2 < T(f) < 4,$$

then, $X$ is either homeomorphic to a sphere or is an even-dimensional manifold such that it has the homotopy type of an $n/2$-sphere with an $n$-cell attached, and in this case, the Betti numbers of $X$ are given by

$$\beta_0 = \beta_{n+2} = \beta_n = 1 \quad \text{and} \quad \beta_i = 0 \quad \text{for otherwise}.$$  

**Proof.** If there exists an $e^* \in S_\theta^{n+N-1}$ such that $m(e^*)=2$, then $X$ is homeomor-
phic to a sphere. If there exists no such $e^*$ in $S^{n+k-1}$, then by the hypothesis, there exists a set of positive measure on $S^{n+k-1}$ such that if $e^*$ is in the set, $\exp_{e^*}(f(p)) \cdot e^*$ has just three critical points. For this vector $e^*$, by (16), (17), (18), and $\beta_0=\beta_n=1$, it must be

$$m_i(e^*)=m_n(e^*)=1, \quad m_j(e^*)=0 \quad \text{for some } i \quad (1<i<n)$$

and

$$m_j(e^*)=1 \quad \text{for } j \quad (1<j<n, j \neq i).$$

Hence

$$\beta_j=0.$$

These relations and (17) imply $\beta_i=1$. Furthermore, by Poincaré duality theorem, we have $i=n-i$, that is $n$ is even and $i=n/2$. In this case, by a principle of the Morse theorem, $X$ is homotopic to an $n/2$-sphere with an $n$-cell attached.


**Theorem 4.** Let $X$ be an even-dimensional homology sphere with an immersion as in Theorem 1. If $\exp_{e^*}(f(X))$ is not imbedded as a convex hypersurface in some $(n+1)$-dimensional linear subspace of the tangent space to $Y$ at $o$ for all $o$ in $f(X)$. Then, the measure of the set $e^*$ in $S^{n+k-1}$ such that

$$m(e^*)=0$$

is positive.

For the proof, we need the following theorems:

**Theorem 5.** Under the same hypothesis of Theorem 1, let $q \in f(X)$, and set

$$h_t=(\exp_{t})^* f,$$

and let $\tau(h_t)$ denote the total curvature in the sense of Chern-Lashof, then we have

$$T(f)=\inf_{q \in f(X)} \tau(h_q).$$

**Proof.** Let us denote by $B_t^*$ the (usual) normal bundle of the immersion

$$h_t=\exp_{t} \circ f: X \rightarrow T_q(Y)$$

that is

$$B_t^*=(\{p, e': e' \text{ is the unit normal vectors to } q(X) \text{ at } h_q(p)\}.$$

We define

$$\tilde{\theta}': B_t^* \rightarrow S^{n+k-1}$$

by $\tilde{\theta}'(p, e')=e'$. Then, we have

$$\nu^* d\sigma_{n+k-1}=G'(p, q, e') dV \wedge d\sigma_{n+k-1}$$

where $d\sigma_{n+k-1}$ denotes a differential $(N-1)$-form on $B_t^*$ such that its restriction to
a fibre is the volume element of the sphere of the unit normal vectors.

Now, for a fixed $e'$ in $S_q^{n+N-1}$, let us define a function $w$ given by

$$w(p) = h_q(p) \cdot e',$$

then it is clear that the number of all critical points of the function $w$ is equal to the elements of $B'_t$ which is transformed into $e'$ by $v'$. Hence, if we denote by $m'(e')$ the number of all critical points of $w$, then we have the following equation:

$$\tau(h_q) = \int_{S_q^{n+N-1}} m'(e') d\Sigma_{n+N-1}/c_{n+N-1}.$$  

But it is easy to see that the number of all critical points of the function $g$ in (13) is equal to the number of all critical points of the function $w$ in (31), hence

$$\tau(h_q) = \int_{S_q^{n+N-1}} m(e') d\Sigma_{n+N-1}/c_{n+N-1}$$

(33)

$$= \int_{S_q^{n+N-1}} K^*(p, q) dV/c_{n+N-1}$$

so that

$$T(f) = \inf_{q \in f(X)} \tau(h_q).$$

**Theorem 6.** Under the same hypothesis of Theorem 1, we have $T(f) = 2$ if and only if, there exists some points $q$ in $f(X)$ such that (A): $((\exp_q^{-1})_* f)(X)$ is imbedded as a convex hypersurface in a $(n+1)$-dimensional linear subspace of the tangent space to $Y$ at $q$.

**Proof.** Since $X$ is compact, there exists at least one point, say $q$ such that equality (11) holds, hence by (33), we have

$$T(f) = \tau(h_q).$$

But by the assumption, we have $T(f) = 2$ and so that

$$\tau(h_q) = 2.$$ 

(35)

Therefore, by Theorem 3 in [4], we see that $h_q(X)$ is imbedded as a convex hypersurface in a $(n+1)$-dimensional linear subspace of the tangent space $T_q(Y)$ by $h_q$, that is $((\exp_q^{-1})_* f)(X)$ is imbedded as a convex hypersurface in an $(n+1)$-dimensional linear subspace of $T_q(Y)$.

Now, we turn to prove theorem 4.

**Proof of Theorem 4.** Since by the assumption, $X$ is an even-dimensional, orientable, compact manifold, and is a homology sphere, we have $\beta_0 = \beta_n = 1$, and $\beta_i = 0$ for otherwise. So that we have

$$\sum_{i=0}^{n} \beta_i = 2 \quad \text{and} \quad \sum_{i=0}^{n} (-1)^i \beta_i = 2,$$ 

(36)

and hence, (17) becomes
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\[ m_n(e^*) - m_{n-1}(e^*) + \cdots + m_2(e^*) - m_1(e^*) + m_0(e^*) = 2 \]

that is

\[ m_1(e^*) + m_2(e^*) + \cdots + m_{n-1}(e^*) = m_0(e^*) + m_2(e^*) + \cdots + m_n(e^*) - 2, \]

so that

\[ m(e^*) = 2(m_0(e^*) + m_2(e^*)) \]

if the equality holds almost everywhere in \( S_q^{n+N-1} \) for every \( q \) in \( f(X) \), then we get \( T(f) = 2 \), but by theorem 6, this implies that (A) of theorem 6 holds. This is a contradiction, so that we have proved the theorem completely.

4. Manifold immersed in a total geodesic submanifold.

**THEOREM 7.** Under the same hypothesis of Theorem 1, if \( f(X) \) lies in an \((n+N')\)-dimensional totally geodesic submanifold \( \bar{Y} \) of \( Y \). Then

\[ T(f) = \bar{T}(f) \]

where \( \bar{T}(f) \) denotes the total absolute curvature of the mapping

\[ \bar{f} : X \longrightarrow \bar{Y} \subset Y \]

where \( \bar{f}(p) = f(p) \) for every \( p \in X \).

Before to prove this theorem, we first prove that if \( N' = N-1 \), then this theorem is valid, and we state it as a lemma:

**LEMMA 1.** If \( N' = N-1 \) in the theorem 7, then this theorem holds.

**Proof.** We consider the bundle of all frames

\[ (p, e_1, e_2, \cdots, e_{n+N-1}) \]

such that \( e_1, e_2, \cdots, e_{n+N-1} \in T_{\bar{f}(p)}(\bar{Y}) \), \( o \in f(X) \) and \( \tilde{e}_i, \cdots, \tilde{e}_n \) are tangent to \( ((\text{exp}_o^{-1}) \circ f)(X) \) at \( ((\text{exp}_o^{-1}) \circ f)(p) \) and \( \tilde{e}_{n+1}, \cdots, \tilde{e}_{n+N-1} \) are normal to \( ((\text{exp}_o^{-1}) \circ f)(X) \) at \( ((\text{exp}_o^{-1}) \circ f)(p) \).

Now, let us put

\[ \bar{w}_{n+N-1,A} = d\tilde{e}_{n+N-1,A} \]

then

\[ d\tilde{e}_{n+N-1} = \sum_{A=1}^{n+N-2} \bar{w}_{n+N-1,A} \tilde{e}_A. \]

Now, let \( d\Sigma_{n+N-2} \) denote the volume element of the unit \((n+N-2)\)-sphere in \( T_o(\bar{Y}) \), then we have

\[ d\Sigma_{n+N-2} = \bar{w}_{n+N-1,1} \wedge \cdots \wedge \bar{w}_{n+N-1,n+N-2} \]

hence we have
where $\tilde{B}_n$ denotes the pseudo-normal bundle of $X$ with respect to $\bar{Y}$, and $\tilde{\varphi}$ the corresponding mapping from $\tilde{B}_n$ to $S_o^{n+N-2}$.

Now, let $e$ be one of the two unit vectors perpendicular to $\bar{Y}$, in the tangent space $T_{f(p)}(\bar{Y})$, a unit pseudo-normal vector at $f(p)$ can be written uniquely in the form:

$$e'_{n+N}=(\cos \theta)e_{n+N-1}+(\sin \theta)e, \quad -\pi/2 < \theta \leq \pi/2,$$

where $e_{n+N-1}$ is the unit vector in the direction of its projection in $T_{f(p)}(\bar{Y})$, let

$$e'_{n+1}=(\sin \theta)e_n+(\cos \theta)e_n$$

and

$$e'_{s-1}=e_s \quad \text{for} \quad s=1, \ldots, n+N-2$$

We also denote by $B'_n$ and $\tilde{\varphi}'$ the corresponding pseudo-normal bundle and the mapping to $S_o^{n+N-1}$. Then, the total absolute curvature with respect to the point $o$ of the immersion $f: X \rightarrow Y$ is given by

$$T(f, o)=\int_{B'_n} |\tilde{\varphi}'_1(\tilde{\varphi}'_{n+1}, \ldots, \tilde{\varphi}'_{n+N-1})|/|C_{n+N-1}|$$

Now, by (44) we have the following equality by the properties of parallel translation that

$$e'_{n+N}=(\cos \theta)e_{n+N-1}+(\sin \theta)e.$$

Therefore we get

$$d\tilde{e}'_{n+N}=(\cos \theta)d\tilde{e}_{n+N-1}+(-\sin \theta)e_{n+N-1}+(\cos \theta)e)d\theta$$

$$=(\cos \theta)d\tilde{e}_{n+N-1}-\tilde{e}'_{n+N-1}d\theta.$$

But we have

$$d\tilde{e}_{n+N-1} \cdot \tilde{e}'_{n+N-1}=d\tilde{e}_{n+N-1} \cdot ((\sin \theta)e_{n+N-1}-(\cos \theta)e)$$

$$=-(\cos \theta)(d\tilde{e}_{n+N-1} \cdot \tilde{e})$$

$$=-(\cos \theta)(\tilde{e}_{n+N-1} \cdot \tilde{e})$$

$$=0.$$
also

\[ \phi_{n+N,s} = \tilde{e}_{n+N}^* \cdot \tilde{e}^*_s = ((\cos \theta) \tilde{e}_{n+N-1} - \tilde{e}_{n+N}^* d\theta) \cdot \tilde{e}^*_s \]

\[ = (\cos \theta) \tilde{w}_{n+N-1,s}. \]

Therefore by (47) we have

\[ T(f, o) = \frac{1}{C_{n+N-1}} \int_{\partial o} |\nu^* ((\cos \theta)^{n+N-2} \tilde{w}_{n+N-1,1} \wedge \cdots \wedge \tilde{w}_{n+N-1,n+N-2} \wedge d\theta)| \]

\[ = \frac{C_{n+N-2}}{C_{n+N-1}} \left( \int_{-\pi/2}^{\pi/2} |\cos \theta|^{n+N-2} d\theta \right) \bar{T}(f, o) \]

hence by the fact

\[ c_h = \frac{2(\Gamma(1/2))^{k+1}}{\Gamma((k+1)/2)} \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} |\cos \theta|^k d\theta = \frac{\Gamma(1/2)\Gamma((k+1)/2)}{\Gamma((k+2)/2)} \]

we have

\[ (54) \quad T(f, o) = \bar{T}(f, o) \]

so that we get \( T(f) = \bar{T}(f) \). This completes the proof of the lemma.

**Lemma 2.** Let \( f: X \to E^{n+N} \) be an immersion of a compact, orientable differentiable manifold of dimension \( n \) in the Euclidean space \( E^{n+N} \), such that \( f(X) \) lies in an \( (n+N') \)-dimensional linear subspace \( E \) of \( E^{n+N} \). Let \( \bar{f} \) denote the induced mapping of \( f \) into the \( (n+N') \)-dimensional linear subspace \( E \). Then, they have the same total absolute curvature, that is

\[ (55) \quad T(f) = \bar{T}(f) = T(\bar{f}). \]

**Proof.** Since any linear subspace of the Euclidean space \( E^{n+N} \) can be regarded as a totally geodesic submanifold of \( E^{n+N} \), we can select a sequence of linear subvariety of \( E^{n+N} \):

\[ (56) \quad E = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{N-N'} = E^{n+N} \]

satisfies that

\[ (57) \quad \dim E_i - \dim E_{i-1} = 1, \quad i = 1, 2, \ldots, N-N'. \]

so that by the fact that each \( E_{i-1} \) can be regarded as totally geodesic submanifold of \( E_i \), hence we can apply Lemma 1, \( (N-N') \)-times and then we will get the desired result (55).

**Proof of Theorem 7.** Since by (33) we have

\[ T_q(\exp_{q^{-1}} f) = T(q, f) \quad \text{for each} \ q \ \text{in} \ f(X). \]

we can identify the tangent space \( T_q(Y) \) with \( E^{n+N} \). Now, by the assumption, \( Y \)
is a totally geodesic submanifold of $Y$, hence if we denote the exponential map of $Y$ at $q \in f(X)$ by $\text{Exp}_q$, then we have

$$(59) \quad (\text{exp}_q)^{-1} \, Y = (\text{Exp}_q)^{-1}.$$ 

Hence by the fact $f(X)$ lies in $Y$, and the result in Lemma 2, we can easily get (38).

5. Immersions with minimal total absolute curvature.

**Definition.** An immersion $f: X \rightarrow Y$ is said to be minimal, if $T(f) = \beta(X)$, where $X$ and $Y$ are both under the same hypothesis as in Theorem 1.

**Theorem 8.** If (4) is a minimal immersion with the real field as coefficient field, then $X$ has no torsion.

**Proof.** Let $\beta_i(X, Z_p)$ be the $i$-th Betti number of $X$ with the coefficient field, $Z \mod p$ ($p$ is a prime), and $\beta_i(X, R)$ be the $i$-th Betti number of $X$ with the real coefficient field $R$, then

$$\beta_i(X, R) \leq \beta_i(X, Z_p), \quad i = 0, 1, \ldots, n. \quad (60)$$

But by the hypothesis, we have

$$T(f) = \beta(X, R) \quad (61)$$

and by Theorem 1, we have

$$T(f) \geq \beta(X, Z_p) \quad (62)$$

so that we have

$$\beta_i(X, R) = \beta_i(X, Z_p) \quad \text{for} \quad i = 0, 1, \ldots, n. \quad (63)$$

which means that $X$ has no torsion.

**Theorem 9.** Let

$$\rho(p) = \sup_{q \in g^{q+N-1}, e \in f(X)} |G(p, q, e)|. \quad (64)$$

Then, we have

$$\int_X \rho(p) dV \geq \frac{\beta c_n + N - 1}{c N - 1}. \quad (65)$$

The proof of this theorem is similar to the proof of Theorem 5 in [11], and the following corollary follows immediately.

**Corollary.** Under the same hypothesis and notations, we have

$$\nu(X) \geq \frac{\beta c_n + N - 1}{H c^{N - 1}}. \quad (65)$$
where $H = \sup_{p \in X} \rho(p)$ and $v(X)$ denotes the volume of $X$.

6. Total absolute curvature of product immersion.

The following theorem appears in Willmore–Saleemi’s paper.

**Theorem 10.** Let $f: X \to Y$ and $f': X' \to Y'$ be two immersions satisfying the hypothesis in Theorem 1. Let

$$f \times f': X \times X' \to Y \times Y'$$

be the product immersion, and let $T(f \times f')$ be the corresponding total absolute curvatures, then

$$T(f \times f') = T(f) \times T(f').$$

**Proof.** Suppose that $\dim X = n$, $\dim X' = n'$, $\dim Y = n+N$, and $\dim Y' = n'+N'$. Now, if $q \in f(X)$ and $q' \in f'(X')$, let $B_q$, $B'_q$ and $\tilde{v}$, $\tilde{v}'$ be the pseudo-normal bundles and the mappings of $f$ and $f'$, respectively. Then, with the notations in Lemma 1, we have

$$T(f, q) = \int_{B_q} |\tilde{v}^*(\bar{w}_{n+N,1} \wedge \cdots \wedge \bar{w}_{n+N,n+n+N-1})| e_{n+N-1}$$

(68)

$$T(f', q') = \int_{B'_q} |\tilde{v}'^*(\bar{w}_{n'+N',1} \wedge \cdots \wedge \bar{w}_{n'+N',n'+N'-1})| e_{n'+N'-1}$$

(67)

Let $B$ be the set of the forms

$$b = (p, p'; e_1, \ldots, e_{n+N}, e'_1, \ldots, e'_{n'+N'})$$

such that the parallel translation vectors $\bar{e}_1, \ldots, \bar{e}_{n+N}$ of $e_1, \ldots, e_{n+N}$ along the unique geodesic joining $f(p)$ and $q$ are unit normal vectors of $((\exp_q^1) \circ f(X))$ at $\exp_q^1 \circ f(p)$ and $(p', e'_1, \ldots, e'_{n'+N'})$ are the analogous element of the immersion $f'$.

Now, let $B^q$ denote the pseudo-normal bundle of the product immersion $f \times f'$ and we also denote the corresponding mapping from $B^q$ to $S^{n+N+n'+N'-1}_{(0,\pi)}$ by $\tilde{v}$, that is

$$\tilde{v}((p, p'), e) = \bar{v},$$

(70)

where $\bar{v}$ denotes the parallel translation vector of $e$ along the unique geodesic joining $(p, p')$ and $(q, q')$.

Let us consider the unit vector

$$e^* = \bar{e}_{n+N} \cos \theta + \bar{e}^*_{n+N} \sin \theta.$$

(71)

It is clear that $e^*$ is a unit normal vector to $\exp_{q', q''}^1(f(X) \times f'(X'))$ at $\exp_{q', q''}^1(f(p), f'(p'))$ and we have

$$de^* = \cos \theta \, d\bar{e}_{n+N} + \sin \theta \, d\bar{e}^*_{n+N} + (\cos \theta \, \bar{e}^*_{n+N} - \sin \theta \, \bar{e}_{n+N})d\theta$$

(72)

so that if we set $r = n+N+n'+N'-1$, we have
\[ T(f \times f'; (q, q')) = \left( \int_{B_q} |\hat{\beta}^*(\varpi_{n+1} \cdots \varpi_{n+N-1} \varpi') \cdots \right) \left( \int_{B_{q'}} |\hat{\beta}^*(\varpi'_{n'+N'} \cdots \varpi'_{n'+N'-1}) \right)^{\alpha/2} \left( \cos \theta \right)^{n+N-1} \left( \sin \theta \right)^{n'-N'} \, d\theta \right) / c_r \]

(73)

\[ = \left( \int_{B_q} |\hat{\beta}^*(\varpi_{n+1} \cdots \varpi_{n+N-1})| \int_{B_{q'}} |\hat{\beta}^*(\varpi'_{n'+N'} \cdots \varpi'_{n'+N'-1})| \left( \cos \theta \right)^{n+N-1} \left( \sin \theta \right)^{n'-N'} \, d\theta \right) / c_r \]

\[ = \left( c_{n+N-1} c_{n'+N'-1} T(f, q) T(f', q') B \left( \frac{1}{2} (n+N), \frac{1}{2} (n'+N') \right) \right) / 2c_r \]

So that we get

(74)

\[ \inf_{(q, q') \in (f(\mathbb{X}) \times f'(\mathbb{X}'))} T(f \times f'; (q, q')) = \inf_{q \in f(\mathbb{X})} T(f, q) \times \inf_{q' \in f'(\mathbb{X}')} T(f', q') \]

that is

(75)

\[ T(f \times f') = T(f) T(f'). \]

**Corollary 1.** If \( f: X \to Y \) and \( f': X' \to Y' \) are both minimal immersions, then the product immersion is also minimal immersion.

This follows immediately from theorem 10 and the definition of the minimal immersion.

**Corollary 2.** If \( f: X \to Y \) and \( f': X' \to Y' \) are both minimal immersion with the real field, as the coefficient field, then the product immersion has no torsion.

This follows from theorem 8 and 10.

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**References**


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