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ON THE EXISTENCE OF ANALYTIC MAPPINGS, I

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§1. Introduction. Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions having no zero other than an infinite number of simple zeros. Let φ be a non-trivial analytic mapping of R into S. Let $\mathfrak{P}_R$ and $\mathfrak{P}_S$ be the projection maps $(z, y)\rightarrow z$ and $(w, u)\rightarrow w$, respectively. Then the composed function $h(z)=\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)$ reduces to an entire function of z [6]. Further when the order ρ_G of G is finite, let G_c be a canonical product having the same zeros with the same multiplicities as those of G. Similarly we use ρ_g and g_c with respect to g.

In this paper we shall prove the following two theorems:

THEOREM 1. *Assume that $\rho_{G_c}<\infty$ and $0<\rho_{g_c}<\infty$ and that there exists a non-trivial analytic mapping φ of R into S. Then ρ_{G_c} is an integral multiple of ρ_{g_c}.*

THEOREM 2. *Assume that there exists a non-trivial analytic mapping φ of R into itself. Then φ is a univalent conformal mapping of R onto itself and the corresponding function $h(z)$ is a linear function of the form $e^{2\pi ip/q}z+b$ with a suitable rational number p/q.*

Theorem 1 was proved in [7]. However there were some overlooked parts in the estimation of counting functions. Further when the order of G is finite and is not zero, theorem 2 can be derived from theorem 1. Here we shall show that theorem 2 holds good without any condition on the order of G.

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§2. Preliminaries. We need to quote some theorems in order to prove our theorems. In [6], Ozawa proved the following theorem:

THEOREM A. *If there exists a non-trivial analytic mapping φ of R into S, then there exists a pair of two entire functions $h(z)$ and $f(z)$ of z satisfying an equation of the form*

$$f(z)^2G(z)=g\circ h(z)$$

and vice-versa.

Here $h(z)$ is the composed function introduced in §1. Clearly an equation

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$f(z)^2G(z)=g\circ h(z)$ may be replaced by an equation of the form

$$f^*(z)^2G_c(z)=g_c\circ h(z)$$

with a suitable entire function $f^*(z)$.

In [1], Edrei and Fuchs proved the following theorem:

THEOREM B. *Let $E(z)$ and $F(z)$ be transcendental entire functions. Assume that the zeros of $E(z)$ have a positive exponent of convergence. Then the zeros of $E\circ F(z)$ cannot have a finite exponent of convergence.*

The proof of theorem B depends on a result of Valilon [9].

In [2], Fatou proved the following theorem:

THEOREM C. *Let $E(z)$ be a transcendental entire function. Then if $E(z)=z$ has at most a finite number of roots, $E\circ E(z)=z$ has an infinite number of roots.*

In the following $F(z)$ is a non-constant meromorphic function and the notations T, m, N, N_1 and $\bar{N}$ are used in the sense of Nevanlinna [3], [5]. The notation $N_2(r; a, F)$ is the N-function of simple a-points of F. Then Nevanlinna's first fundamental theorem is expressible in the following form:

THEOREM D. (Theorem 1.2 in [3]) *Suppose that $F(0)\neq a, \infty$ for a given complex number a. Then we have*

$$m(r; a, F)+N(r; a, F)=T(r, F)-\log|F(0)-a|+\varepsilon(a, r),$$

where $|\varepsilon(a, r)|\leqq\log^+|a|+\log 2$.

Next we quote some theorems fundamental to derive Nevanlinna's second fundamental theorem.

THEOREM E. (Theorem 2.1 in [3]) *Let $a_1, a_2, \cdots, a_q$ $(q>2)$ be distinct finite complex numbers and suppose that $|a_\mu-a_\nu|\geqq\delta$ with a fixed $\delta>0$ for $1\leqq\mu<\nu\leqq q$. Then we have*

$$m(r; \infty, F)+\sum_{\nu=1}^{q} m(r; a_\nu, F)\leqq 2T(r, F)-N_1(r, F)+S(r, F),$$

where

$$N_1(r, F)=N(r; 0, F')+2N(r; \infty, F)-N(r; \infty, F')$$

and

$$S(r, F)=m\left(r; \infty, \frac{F'}{F}\right)+m\left(r, \infty, \sum_{\nu=1}^{q}\frac{F'}{F-a_\nu}\right)+q\log^+\frac{3q}{\delta}+\log 2-\log|c|$$

with $F(z)-F(0)=cz^\lambda+\cdots$, $c\neq 0$.

LEMMA F. (Lemma 2.3 in [3]) *Suppose that* $0<r<R$. *Then we have*

$$m\left(r; \infty, \frac{F'}{F}\right)<4 \log^+ T(R, F)+4 \log^+ \log^+ \frac{1}{|F(0)|}+5 \log^+ R$$

$$+6 \log^+ \frac{1}{R-r}+\log^+ \frac{1}{r}+14,$$

with $F(0)\neq 0, \infty$.

LEMMA G. (Lemma 2.4 in [3]) *Suppose that* $T(r)$ *is continuous, increasing and* $T(r)\geqq 1$ *for* $r_0\leqq r<\infty$. *Then we have*

$$T\left(r+\frac{1}{T(r)}\right)<2T(r)$$

outside a set of r *which has linear measure at most* 2.

We shall use the following precise form of Nevanlinna's second fundamental theorem which is easily obtained by combining theorem E, lemma F and lemma G.

THEOREM H. *Let* $a_1, a_2 \cdots, a_q (q>2)$ *be distinct finite complex numbers. Suppose that* $F(z)$ *is a non-constant meromorphic function with* $F(z)-F(0)=cz^\lambda+\cdots, c\neq 0$, *and that* $F(0)\neq 0, \infty, a_1, a_2, \cdots, a_q$. *Further suppose that* $T(r_0, F)\geqq 1$. *Then we have*

$$(q-1)T(r, F)\leqq N(r; \infty, F)+\sum_{\nu=1}^{q} N(r; a_\nu, F)-N_1(r, F)-\log |c|$$

$$+K_1 \log T(r, F)+K_2 \log r+K_3 \sum_{\nu=1}^{q} \{\log^+|F(0)-a_\nu|+\log^+ \log^+|F(0)-a_\nu|\}$$

$$+K_4 \log^+ \log^+ \frac{1}{|F(0)|}+K_5$$

outside a set E_F *of* r *which has linear measure at most* $2+r_0$, *where* $K_i\,(i=1, \cdots, 5)$ *are absolute constants depending on given complex numbers* $a_1, a_2, \cdots, a_q$, *but independent of the function* $F(z)$.

Finally we quote a lemma concerning the characteristic of a composed function:

LEMMA I. *If* $F(z)$ *and* $E(z)$ *are two transcendental entire functions, then for any given positive number* K *there exists a number* r_0 *such that*

$$T(r, F\circ E)\geqq \frac{1}{3} T(r^K, F)$$

for all $r\geqq r_0$, *and* r_0 *depends on* K *and* E *but not on* F.

The proof of lemma I is contained in the proof of lemma 2.6 in [3].

§ 3. **Proof of theorem 1.** By theorem A we may consider the possibility of a functional equation $f(z)^2G_c(z)=g_c\circ h(z)$ with two suitable entire functions $f(z)$ and $h(z)$.

The first aim confronting us is to prove that $h(z)$ must be of finite order in our case. Assume contrarily that $h(z)$ is of infinite order. Since g_c has no zero other than an infinite number of simple zeros, we have

$$N(r;0,g_c\circ h)=N_2(r;0,g_c\circ h)+N_1(r;0,g_c\circ h)+\bar{N}_1(r;0,g_c\circ h),$$

$$\begin{aligned}(3.1)\qquad \bar{N}_1(r;0,g_c\circ h)&\leqq N_1(r;0,g_c\circ h)\leqq N(r;0,h')\leqq T(r,h')+O(1)\\ &\leqq T(r,h)+O(\log(rT(r,h)))\leqq 2T(r,h)\end{aligned}$$

outside a set of finite measure. Further we have

$$N(r;0,g_c\circ h)\geqq\sum_{\nu=1}^{p}N(r;w_\nu,h)$$

for an arbitrary but fixed number p of zeros $\{w_\nu\}$ of g_c and for all r. Since $h(z)$ is transcendental, by Nevanlinna's second fundamental theorem applied to $h(z)$ we have

$$\begin{aligned}\sum_{\nu=1}^{p}N(r;w_\nu,h)&\geqq(p-1)T(r,h)-O(\log(rT(r,h)))\\ &\geqq(p-2)T(r,h)\end{aligned}$$

outside a set of finite measure. Thus we have

$$N(r;0,g_c\circ h)\geqq KT(r,h),$$

and by (3.1)

$$N_2(r;0,g_c\circ h)\geqq KT(r,h)$$

for an arbitrary but fixed number K and for all r outside a set of finite measure. Using the functional equation $f(z)^2G_c(z)=g_c\circ h(z)$, we have

$$\begin{aligned}KT(r,h)&\leqq N_2(r;0,g_c\circ h)\leqq N(r;0,G_c)\\ &\leqq T(r,G_c)+O(1)\end{aligned}$$

outside a set of finite measure. Hence

$$(3.2)\qquad (K-1)T(r,h)\leqq T(r,G_c)$$

for an arbitrary but fixed number K and for all r outside a set of finite measure. Since G_c is of finite order, there exists a constant C_1 such that

$$(3.3)\qquad T(r,G_c)\leqq r^{C_1}$$

holds for all sufficiently large r. On the other hand, since h is of infinite order, there exists a sequence $\{r_n\}\uparrow\infty$ $(r_1\geqq 2)$ such that

(3. 4) $$T(r_n, h)\geqq r_n{}^{C_2}, \qquad C_2=2C_1$$

remains true. Further we have $T(r, h)\geqq T(r_n, h)$ for $r\geqq r_n$ and $(r_n+1)^{C_1}\leqq r_n{}^{C_2}$ for all n. Therefore by (3. 3) and (3. 4)

$$T(r, h)\geqq T(r, G_c)$$

holds on the set $S=\cup_{n=1}^{\infty} S_n$, $S_n=[r_n, r_n+1]$. Since S is of infinite measure, this contradicts (3. 2) with $K>2$. Therefore $h(z)$ must be of finite order.

Next we shall prove that $h(z)$ must be a polynomial. Assume that $h(z)$ is transcendental and of finite order. Then by the same estimations as above we have

$$N_1(r; 0, g_c\circ h)\leqq 2T(r, h),$$

$$N(r; 0, g_c\circ h)\geqq N_2(r; 0, g_c\circ h)\geqq KT(r, h)$$

for an arbitrary but fixed number K and for all sufficiently large r. Therefore we have

(3. 5) $$\lim_{r\to\infty}\frac{N(r; 0, g_c\circ h)}{T(r, h)}=\infty, \qquad \lim_{r\to\infty}\frac{N(r; 0, g_c\circ h)}{N_2(r; 0, g_c\circ h)}=1.$$

Using the functional equation $f(z)^2G_c(z)=g_c\circ h(z)$, we have

$$N_2(r; 0, g_c\circ h)\leqq N(r; 0, G_c),$$

$$N(r; 0, f)\leqq N(r; 0, h')\leqq T(r, h)+O(\log(rT(r, h)))$$

$$\leqq 2T(r, h)$$

for all sufficiently large r. Hence we have

$$N(r; 0, f)=o(N(r; 0, g_c\circ h))=o(N_2(r; 0, g_c\circ h))=o(N(r; 0, G_c))$$

for all sufficiently large r. Thus we have

(3. 6) $$N(r; 0, f)\leqq N(r; 0, G_c)$$

for all sufficiently large r. The exponent of convergence of zeros of G_c, which is equal to the order of G_c [5], is finite by assumption. Further by (3. 6) the exponent of convergence of zeros of f is finite. Therefore the exponent of convergence of zeros of f^2G_c is finite. On the other hand, since $h(z)$ is not polynomial and ρ_g is positive, by theorem B the exponent of convergence of zeros of $g_c\circ h(z)$ cannot be finite. Consequently we have a contradiction and $h(z)$ must be a polynomial.

Let $h(z)$ be a polynomial of the form $a_0z^\nu+a_1z^{\nu-1}+\cdots+a_\nu$. Then we have for any ε with $0<\varepsilon<1$

$$n(r; 0, g_c\circ h)\geqq n(|a_0|r^\nu(1-\varepsilon);\ 0, g_c)-O(1)$$

and hence

$$N(r; 0, g_c\circ h)\geqq N(|a_0|r^\nu(1-\varepsilon);\ 0, g_c)-O(\log r).$$

Therefore we have

$$\rho_{N(r;\ 0,\ g_c\circ h)}\geqq \nu\rho_{N(r;\ 0,\ g_c)}=\nu\rho_{g_c}.$$

Further we have by (3.5) and (3.6)

$$\rho_{N(r;\ 0,\ g_c\circ h)}\leqq\rho_{N(r;\ 0,\ G_c)}=\rho_{G_c}\leqq\rho_{N(r;\ 0,\ g_c\circ h)}\leqq\rho_{g_c\circ h}.$$

On the other hand we have by Pólya's method [8]

$$\rho_{g_c\circ h}\leqq\nu\rho_{g_c}.$$

Therefore we have the desired result:

(3.7) $$\rho_{G_c}=\nu\rho_{g_c}.$$

REMARK. *In the case of* $\rho_{G_c}=0$ *and* $\rho_{g_c}>0$ (3.7) *implies that there is no non-trivial analytic mapping of* R *into* S.

§4. **Proof of theorem 2.** By theorem A it is sufficient to consider the possibility of a functional equation $f(z)^2G(z)=G\circ h(z)$ with two suitable entire functions $f(z)$ and $h(z)$. And it is sufficient to prove that $h(z)$ is a linear function $az+b$. For we can prove the desired result by the same argument as in [7, p. 6].

At first we shall prove that $h(z)$ must be a polynomial. Assume that $h(z)$ is a transcendental entire function. Then its iteration $h_{n+1}(z)=h\circ h_n(z)=h_n\circ h(z)$ with $h_1(z)=h(z)$ is transcendental an satisfies an equation $f_n(z)^2G(z)=G\circ h_n(z)$ with a suitable entire function $f_n(z)$. By theorem C the equation $h(z)=z$ or $h\circ h(z)=z$ has an infinite number of roots. Therefore we may assume that the equation $h(z)=z$ has an infinite number of roots. Let z_0 be an arbitrary non-zero root of $h(z)=z$. And we choose distinct complex numbers $w_1, w_2, \cdots, w_{10}$ from the set of zeros of $G(z)$. Without loss of generality we may assume that $z_0\neq w_i\ (i=1, 2, \cdots, 10)$. In the following argument complex numbers $z_0, w_1, w_2, \cdots, w_{10}$ are fixed.

If we put $h(z)=z_0+c(z-z_0)^k+\cdots$, $c\neq 0$, then we have

(4.1) $$h_n(z)=z_0+c^{1+k+\cdots+k^{n-1}}(z-z_0)^{k^n}+\cdots.$$

Since $h(z)$ is transcendental, we have

$$\lim_{r\to\infty}\frac{T(r, h)}{\log r}=\infty,$$

and for an arbitrary given constant K there exists a number r_1 such that

$$T(r, h)>K \log r$$

holds for all $r>r_1$.

By lemma I there exists a number r_2 such that

$$T(r, h_n)=T(r, h_{n-1}\circ h)\geqq \frac{1}{3} T(r^{6k}, h_{n-1})$$

$$\geqq \frac{1}{3^2} T(r^{(6k)^2}, h_{n-2})\geqq \cdots \geqq \frac{1}{3^{n-1}} T(r^{(6k)^{n-1}}, h)$$

for all $r>r_2$ and for all n. If we put $r_3=\max(2, r_1, r_2)$, we have

$$T(r, h_n)>K(2k)^{n-1} \log r \tag{4.2}$$

for all $r>r_3$ and for all n.

Consider a function $H_n(z)=h_n(z+z_0)$. If we put $r_0=2(r_3+|z_0|)$, then we have

$$T(r_0, H_n(z))\geqq \frac{1}{3} \log^+ M\left(\frac{r_0}{2}, H_n(z)\right)\geqq \frac{1}{3} \log^+ M(r_3, h_n(z))$$

$$\geqq \frac{1}{3} T(r_3, h_n(z)).$$

Since we may assume that $K \log r_3>3$, we have by (4.2)

$$T(r_0, H_n(z))>(2k)^{n-1}. \tag{4.3}$$

Further $H_n(0)=z_0$. Therefore we can apply theorem H to $H_n(z)$ and by (4.1) we have

$$\begin{aligned} 9T(r, H_n)\leqq &\sum_{\nu=1}^{10} N(r; w_\nu, H_n)-N(r; 0, H_n')+\log^+\left(\frac{1}{|c|}\right)^{1+k+\cdots+k^{n-1}} \\ &+K_1 \log T(r, H_n)+K_2 \log r+K_3 \end{aligned} \tag{4.4}$$

outside a set E_n of r which has linear measure at most 2, where K_1, K_2, K_3 are constants which depend on $z_0, w_1, \cdots, w_{10}$ but not on n. On the other hand by an equation $f_n(z)^2G(z)=G\circ H_n(z-z_0)$ we have

$$\sum_{\nu=1}^{10} N(r; w_\nu, H_n)-N(r; 0, H_n')\leqq \sum_{\nu=1}^{10} \bar{N}(r; w_\nu, H_n)$$

$$\leqq \sum_{\nu=1}^{10} \bar{N}_1(r; w_\nu, H_n)+\sum_{\nu=1}^{10} N_2(r; w_\nu, H_n)$$

$$\leqq \frac{1}{2} \sum_{\nu=1}^{10} N(r; w_\nu, H_n)+N(r; 0, G^*)$$

$$\leqq 5T(r, H_n)+N(r; 0, G^*),$$

where $G^*(z)=G(z+z_0)$. Hence (4.4) reduces to

$$4T(r, H_n)\leqq N(r; 0, G^*)+\log^+\left(\frac{1}{|c|}\right)^{1+k+\cdots+k^{n-1}}+K_1 \log T(r, H_n)+K_2 \log r+K_3.$$

The exceptional set E_n has linear measure at most r_0+2. Therefore we have

$$0\leqq\{N(r'; 0, G^*)-T(r', H_n)\}+\left\{\log\left(\frac{1}{|c|}\right)^{1+k+\cdots+k^{n-1}}-T(r', H_n)\right\}$$
$$+\{K_1 \log T(r', H_n)-T(r', H_n)\}+\{K_2 \log r'+K_3-T(r', H_n)\}$$

for at least one number r'; $r_0\leqq r'\leqq 4r_0$. By (4.3) each term in the right hand side is negative for sufficiently large n. We have a contradiction. Consequently $h(z)$ must be a polynomial.

Next assume that $h(z)$ is a polynomial of degree at least 2. Then $h'(z)$ has a finite number of zeros. Further there exists a number K_0 such that $h(z)$ has at least two simple w-points in $|z|<|w|$ if $|w|>K_0$. Suppose that $G(z)$ has p simple zeros in $|z|\leqq K_0$ and q simple zeros in $K_0<|z|<K'$. We may assume that $p<q$. Then $G\circ h(z)$ has at least $2q$ simple zeros in $|z|<K'$. This is a contradiction since $G(z)$ and $G\circ h(z)$ have the same number of simple zeros. Consequently $h(z)$ must be a linear function.

§ 5. **Remarks.** Let R_3 and S_3 be two regularly branched three-sheeted covering surfaces defined by two equations $y^3=G_3(z)$ and $u^3=g_3(w)$, respectively, where G_3 and g_3 are two entire functions having no zero other than an infinite number of simple or double zeros. In [4] one of the authors proved the following:

If there exists a non-trivial analytic mapping φ of R_3 into S_3, then there exists an entire function $h(z)$ of z such that either $\nu(z)^3G_3(z)=g_3\circ h(z)$ or $\mu(z)^3G_3(z)^2=g_3\circ h(z)$ remains true where $\nu(z)$ is an entire function of z and $\mu(z)$ a single-valued regular function of z excepting possibly all the double zeros of $G_3(z)$ at which it might have simple poles. The converse holds good.

Using this we have the following theorems.

THEOREM 1′. *Assume that $\rho_{G_{3c}}<\infty$ and $0<\rho_{g_{3c}}<\infty$ and that there exists a non-trivial analytic mapping φ of R_3 into S_3. Then $\rho_{G_{3c}}$ is an integral multiple of $\rho_{g_{3c}}$.*

THEOREM 2′. *Assume that there exists a non-trivial analytic mapping φ of R_3 into itself, then φ is a univalent conformal mapping of R_3 onto itself and the corresponding entire function $h(z)$ is a linear function of the form $e^{2\pi i p/q}z+b$ with a suitable rational number p/q.*

Theorem 1′ was stated in [4] without proof. We can prove theorem 1′ and theorem 2′ by the same method as in § 3 and § 4. Further by the same method as in § 4 we can prove the following:

Let $g(z)$ be an entire function having no zero other than an infinite number of zeros with multiplicity at most $m-1$. Then if an equation

$$f(z)^m g(z)=g\circ h(z)$$

holds good with two suitable entire functions $f(z)$ and $h(z)$, $h(z)$ must be a linear function of the form $e^{2\pi i p/q}z+b$ with a suitable rational number p/q.

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