ON A CERTAIN FUNCTIONAL-DIFFERENTIAL INEQUALITY

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Introduction.

Recently, the method using the relations described by some inequalities has been applied to the uniquenes problem for certain functional equations. For example, Nickel [5] has considered a functional equation including an operator T such that

$$F(t, x', x, Tx)=0, \tag{1}$$

and obtained various criteria for the uniqueness of solutions of (1). If the operator T will be defined suitably, (1) will yield various types of equations. For example, if $F(t, x, y, z)$ is of the form such that

$$F(t, x, y, z)=y-g(t, x)-z,$$

and if T is defined by

$$Tx=\int_0^t K(t, s, x(s))ds,$$

(1) is reduced to an integro-differential equation

$$x'=g(t, x)+\int_0^t K(t, s, x(s))ds.$$

Hence, the results in [5] will be applicable to the uniqueness problem of a very wider class of equations.

On the other hand, it has been shown in [2] that the Lyapunov function is applicable to the uniqueness problem for differential equations and also shown in [1] that some estimations for solutions of differential inequalities yield the uniqueness theorem for differential equations.

In this paper, a functional-differential inequality including an operator T such that

$$|x'-f(t, x, Tx)|\leqq\varepsilon(t),$$

in which a functional-differential equation corresponds to the case $\varepsilon(t)\equiv 0$, will be considered as well as the existence problem for

$$x'=f(t, x, Tx). \tag{2}$$

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§ 1. Existence theorem.

1. Existence theorem. As a preparation for § 2, we first consider briefly the existence problem of solutions for (2) in the introduction.

Let I be an interval $0 \leqq t \leqq t_1$, D_1 and D_2 the domains $|x-x_0| \leqq a$ and $|y| \leqq b$ in R^n respectively.[1] Let $f(t, x, y)$ be a continuous function of t, x, y defined on $I \times D_1 \times D_2$, and $|f(t, x, y)| \leqq M$ be satisfied on $I \times D_1 \times D_2$.

Next, we introduce a family of functions $x(t)$ continuous on the interval I_0: $0 \leqq t \leqq t_0 = \min(t_1, a/M)$ and contained in D_1. If we denote the family by $\mathfrak{M}$, it is clear that $\mathfrak{M}$ is a convex set. With this definition of $\mathfrak{M}$, we define an operator T which satisfies the following conditions:

(i) for any x in $\mathfrak{M}$, Tx is a continuous vector function on I_0 contained in D_2;

(ii) for any sequence $\{x_m(t)\}$ in $\mathfrak{M}$ uniformly convergent to $x(t)$ $(t \in I_0)$ in $\mathfrak{M}$, $(Tx_m)(t)$ also uniformly converges to $(Tx)(t)$ $(t \in I_0)$.[2]

Then, if we introduce a second operator U such that

$$Ux = x_0 + \int_0^t f(s, x(s), (Tx)(s))ds, \qquad t \in I_0$$

for any x in $\mathfrak{M}$, it is easily observed that the family $U\mathfrak{M}$ is a convex subset of $\mathfrak{M}$. From the hypotheses on f and I_0, it is easily shown that the inequality $|Ux| \leqq |x_0| + |a|$ is satisfied for any element in $U\mathfrak{M}$, that is, every element in $U\mathfrak{M}$ is uniformly bounded. Furthermore, for any points t', t'' in I_0, we have

$$|(Ux)(t') - (Ux)(t'')| \leqq \left| \int_{t'}^{t''} f(s, x(s), (Tx)(s))ds \right| \leqq M|t''-t'|,$$

which implies the equi-continuity of Ux. Since $\mathfrak{M}$ and $U\mathfrak{M}$ are convex sets, U is the contraction operator, and every element in $U\mathfrak{M}$ is uniformly bounded and equi-continuous, then it follows that there exists at least a fixed point in $\mathfrak{M}$ such that $Ux = x$. It is easily observed that this fixed point corresponds to a continuous solution of the equation

$$x(t) = x_0 + \int_0^t f(s, x(s), (Tx)(s))ds,$$

or equivalently, $x(t)$ is a solution of a functional-differential equation

$$x' = f(t, x, Tx), \qquad x(0) = x_0, \qquad t \in I_0. \tag{1. 1}$$

Thus, we have the following

THEOREM 1. *Let $f(t, x, y)$ be a continuous function of t, x, y and $|f| \leqq M$ on $I \times D_1 \times D_2$, and T a continuous operator defined above. Then, there exists at least a*

1) In this paper, it is supposed that, for any scalar function x, Tx is also scalar and the norm of x is as usual the sum of the absolute values of each element.

2) Since the operator T is always supposed to be continuous, some class of equations, for example, difference-differential equations of neutral type will be excluded.

continuous solution of the functional-differential equation (1. 1) *on* $0 \leqq t \leqq \min(t_1, a/M)$.

2. Maximal and minimal solutions. It may not be expected that the uniqueness of solutions is established, even if $f(t, x, y)$ is continuous on $I_0 \times D_1 \times D_2$. From this reason, it is useful to introduce the maximal and minimal solutions of (1. 1) as in the theory of differential equations.

In this paragraph, all variables are supposed to be scalar, and we first prepare a following

LEMMA 1. *In the two equations*

$$x' = f(t, x, Tx), \qquad x(0) = x_0, \tag{1. 2}$$

$$y' = g(t, y, Ty), \qquad y(0) = y_0, \tag{1. 3}$$

suppose that $f(t, x, y)$ *and* $g(t, x, y)$ *are continuous on* $I_0 \times D_1 \times D_2$ *and the existence of continuous solutions of* (1. 2) *and* (1. 3) *on* I_0 *is already established.*

Then, if $x_0 \leqq y_0$, *and if* $f(t, u, v) < g(t, u, v)$ *is satisfied on* $I_0 \times D_1 \times D_2$, *every solution of* (1. 2) *is not greater than any solution of* (1. 3) *on* I_0.

The proof of this lemma is so similar to that in the theory of differential equations that it is omitted.

Corresponding to the equation (1. 1), for any constant $\varepsilon > 0$ we consider an equation

$$x' = f(t, x, Tx) + \varepsilon, \qquad x(0) = x_0. \tag{1. 4}$$

Then, it follows from Theorem 1 that there exists at least a continuous solution of (1. 4) on an interval I_ε: $0 \leqq t \leqq t_\varepsilon = \min(t_0, a/(M+\varepsilon))$. It is apparent that I_ε tends to I_0 as $\varepsilon \to +0$. Since the solution may depend on ε, we denote it by $x(t, \varepsilon)$. From the above Lemma 1, we obtain that any continuous solution $x(t)$ of (1. 1) is not greater than $x(t, \varepsilon)$ on I_ε, that is, we have an inequality $x(t) \leqq x(t, \varepsilon)$ on I_ε. By a wellknown theorem of Dini, as $\varepsilon \to +0$, the function $x(t, \varepsilon)$ uniformly converges to a function $\bar{\varphi}(t)$ which is a continuous solution of (1. 1) on I_0. Hence, the inequality $x(t) \leqq \bar{\varphi}(t)$ remains valid on I_0 for any solution $x(t)$ of (1. 1).

Similarly, if we consider an equation

$$x' = f(t, x, Tx) - \varepsilon, \qquad x(0) = x_0$$

for $\varepsilon > 0$, there exists a continuous solution $\underline{\varphi}(t)$ of (1. 1), for which the inequality $\underline{\varphi}(t) \leqq x(t) \leqq \bar{\varphi}(t)$ is fulfilled on I_0 for any solution $x(t)$ of (1. 1). Thus, we obtain two continuous solutions $\bar{\varphi}(t)$ and $\underline{\varphi}(t)$ which are called the maximal and minimal solutions respectively.

§ 2. Functional-differential inequalities.

In order to derive the uniqueness theorem for (1. 1), we first deal with the functional-differential inequalities such that

$$|x' - f(t, x, Tx)| \leqq \varepsilon_i(t), \qquad x(0) = x_i \ (i = 1, 2) \tag{2. 1}$$

on I_0, which is reduced to the equation (1. 1), if $\varepsilon_i(t)\equiv 0$. Since every solution of (2. 1) may depend on x_i, we denote it by $x(t, x_i)$, or sometimes it will be abbreviated by $x^i(t)$.

In the sequel, it is supposed that f and T be the same as defined in § 1, and $\varepsilon_i(t)$ $(i=1, 2)$ continuous on I_0. Then, we introduce a V-function as follows.

Let $V(t, x)$ be a function of t and x satisfying the following conditions:

(i) $V(t, x)$ is continuous and non-negative for $t\in I_0$ and $|x|<\infty$;

(ii) $V(t, x)=0$ implies $x=0$ uniformly in t;

(iii) $V(t, x)$ satisfies the Lipschitz condition such that

$$|V(t, x)-V(t, y)|\leqq k(t)|x-y|,$$

where $k(t)$ is continuous on I_0.

Corresponding to such a function $V(t, x)$, we define two quantities

$$\mathfrak{d}V(t, x_1, y_1, x_2, y_2), \qquad DV(t, x(t)-y(t))$$

by setting

$$\mathfrak{d}V(t, x_1 y_1, x_2, y_2)$$
$$=\overline{\lim_{h\to 0}}\frac{1}{h}(V(t, x_1-y_1+h(f(t, x_1, x_2)-f(t, y_1, y_2))-V(t, x_1-y_1)) \tag{2. 2}$$

$$DV(t, x(t)-y(t))$$
$$=\overline{\lim_{h\to 0}}\frac{1}{h}(V(t+h, x(t+h)-y(t+h))-V(t, x(t)-y(t)) \tag{2. 3}$$

for any x_i, y_i, t and any continuous functions $x(t), y(t)$ $(t\in I_0)$.

LEMMA 2. *For any solutions x, y of* (2. 1), *the inequality*

$$|\mathfrak{d}V(t, x, y, Tx, Ty)-DV(t, x-y)|\leqq k(t)(\varepsilon_1(t)+\varepsilon_2(t)) \tag{2. 4}$$

remains valid on I_0.

Proof. By the definitions of (2. 2) and (2. 3), it follows that

$$|\mathfrak{d}V(t, x, y, Tx, Ty)-DV(t, x-y)|$$
$$\leqq\overline{\lim_{h\to 0}}\frac{1}{|h|}k(t+h)(|x(t+h)-x(t)-hf(t, x(t), (Tx)(t))|+|y(t+h)-y(t)-hf(t, y(t), (Ty)(t))|)$$
$$\leqq k(t)(\varepsilon_1(t)+\varepsilon_2(t)).$$

Now, we choose a function $\omega(t, x, y)$ such that it is a continuous and non-negative function of t, x, y for $t\in I_0$, $0\leqq x<\infty$, $|y|<\infty$. Furthermore, it is supposed that $\omega(t, x, y)$ is monotone increasing with respect to y for any fixed t and x. With this choice of the function $\omega(t, x, y)$, we consider a functional-differential equation

$$r'=\omega(t, r, Tr)+k(t)(\varepsilon_1(t)+\varepsilon_2(t)). \tag{2. 5}$$

Here, it is necessary to re-define the operator T with some additional conditions. Suppose that the operator T satisfies the following conditions:

(i) for any continuous function x on I_0, Tx is also a continuous function on I_0;

(ii) T is a continuous operator;

(iii) for any continuous functions x and y on $0 \leqq t < s$, where s is an arbitrary constant not greater than t_0, if $x \leqq y$ for $0 \leqq t < s$, then $Tx \leqq Ty$ holds good for $t = s$.

From the above definition, it is observed that there exists at least a continuous solution of (2.5) on a certain interval ($0 \leqq t \leqq t_1$ ($\leqq t_0$)). Hence, in the following, it is supposed that I_0 is the existence interval of continuous solutions of (2.5).

THEOREM 2. *Let $r_0(t)$ be the maximal solution of* (2.5) *under the initial condition $r(0) = V(0, x_1 - x_2)$. Then, if*

$$\mathfrak{d} V(t, x, y, Tx, Ty) \leqq \omega(t, V(t, x-y), (TV)(t, x-y)) \tag{2.6}$$

for any continuous functions x, y on I_0, we obtain the following estimation

$$V(t, x(t, x_1) - x(t, x_2)) \leqq r_0(t), \qquad t \in I_0. \tag{2.7}$$

Proof. Corresponding to the equation (2.5), we consider an equation

$$r' = \omega(t, r, Tr) + k(t)(\varepsilon_1(t) + \varepsilon_2(t)) + \rho, \qquad \rho > 0. \tag{2.8}$$

Let $r_\rho(t)$ be a continuous solution of (2.8) under the initial condition $r(0) = V(0, x_1 - x_2) + \rho$. Since $V(0, x_1 - x_2) < r_\rho(0)$, it follows from Lemma 1 and the continuity of V and r_ρ that there exists an interval $0 \leqq t \leqq t_2$, on which the inequality $V(t, x(t, x_1) - x(t, x_2)) \leqq r_\rho(t)$ remains valid. Then, if we denote by t_3 the supremum of t_2, and if $t_2 < t_0$, it turns out that

$$V(t_3, x(t_3, x_1) - x(t_3, x_2)) = r_\rho(t_3),$$

$$\begin{aligned} r'_\rho(t_3) &= \lim_{t \to t_3} \frac{r_\rho(t) - r_\rho(t_3)}{t - t_3} \\ &\leqq \overline{\lim_{t \to t_3}} \frac{V(t, x^1(t) - x^2(t)) - V(t_3, x^1(t_3) - x^2(t_3))}{t - t_3} \\ &= D V(t_3, x^1(t_3) - x^2(t_3)). \end{aligned}$$

Hence, from the above relations and the properties of ω and T, it follows that

$$\begin{aligned} &\omega(t_3, V(t_3, x^1(t_3) - x^2(t_3)), (TV)(t_3, x^1(t_3) - x^2(t_3))) + k(t_3)(\varepsilon_1(t_3) + \varepsilon_2(t_3)) + \rho \\ &\leqq \omega(t_3, r'_\rho(t_3), (Tr_\rho)(t_3)) + k(t_3)(\varepsilon_1(t_3) + \varepsilon_2(t_3)) + \rho \\ &= r'_\rho(t_3) \\ &\leqq D V(t_3, x^1(t_3) - x^2(t_3)) \\ &\leqq \mathfrak{d} V(t_3, x^1(t_3), x^2(t_3), (Tx^1)(t_3), (Tx^2)(t_3)) + k(t_3)(\varepsilon_1(t_3) + \varepsilon_2(t_3)) \\ &\leqq \omega(t_3, V(t_3, x^1(t_3) - x^2(t_3)), (TV)(t_3, x^1(t_3) - x^2(t_3))) + k(t_3)(\varepsilon_1(t_3) + \varepsilon_2(t_3)), \end{aligned}$$

which is a contradiction, since $\rho > 0$. Hence, the inequality

$$V(t, x(t, x_1) - x(t, x_2)) \leqq r_\rho(t)$$

remains valid on the whole interval I_0. Since $r_\rho(t)$ uniformly converges to the maximal solution $r_0(t)$ of (2.5) as $\rho \to +0$, we obtain the desired inequality (2.7), which completes our proof.

In the above result, if $\varepsilon_i(t)\equiv 0$ $(i=1, 2)$, we can apply Theorem 2 to the uniqueness problem for the functional-differential equation (1. 1).

THEOREM 3. *Under the hypotheses of Theorem* 2, *if the equation* (2. 5) *has only the zero solution, the uniqueness of solutions of* (1. 1) *is established.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of (1. 1). Then, it follows from Theorem 2 that

$$V(t, x_1(t)-x_2(t))\leqq r_0(t)\equiv 0, \qquad t\in I_0.$$

Since $V\geqq 0$ and $V(t, x)=0$ implies $x=0$ uniformly in t, we have $x_1(t)\equiv x_2(t)$, which proves the uniqueness of solutions.

If $V=|x|$, the inequality (2. 6) is replaced by

$$|f(t, x, Tx)-f(t, y, Ty)|\leqq\omega(t, \phi, T\phi), \qquad \phi\equiv V(t, x-y). \tag{2. 9}$$

Then, we have the following corollary which corresponds to a theorem of Perron in the theory of differential equations.

COROLLARY 1. *If the inequality* (2. 9) *is satisfied, and if the equation* (2. 5) *has only the zero solution, the uniqueness of solutions of* (1. 1) *is established.*

On the other hand, let $M(r)$ be a function satisfying the following conditions:

(i) $M(r)$ is defined and continuous for $0\leqq r<\infty$;

(ii) $M(0)=0$ and $M(r)$ is non-decreasing for $0\leqq r<\infty$, and $M(r)=0$ if and only if $r=0$;

(iii) $$\lim_{\varepsilon\to+0}\int_\varepsilon^r\frac{d\rho}{M(\rho)}=\infty.$$

Then, we obtain the following

COROLLARY 2. *Suppose that* $M(r)$ *is the same function defined as above and the inequality*

$$|f(t, x, Tx)-f(t, y, Ty)|\leqq\varphi(t)(M(|x-y|)+M(T|x-y|))$$

is fulfilled for any t *and any continuous functions* x, y *on* I_0. *Then, if* T *is a bounded operator, the uniqueness of solutions is established.*

§ 3. Applications.

1. Integro-differential equations. In (2. 1), if $f(t, x, y)$ is of the form

$$f(t, x, y)=g(t, x)+y,$$

and if the operator T is defined by

$$Tx=\int_0^t K(t, s, x(s))ds,$$

the inequality (2. 1) is considered to be an integro-differential inequality such that

$$\left|x'-g(t,x)-\int_0^t K(t,s,x(s))ds\right| \leqq \varepsilon_i(t), \qquad x(0)=x_i, \tag{3.1}$$

and the equation (1. 1) becomes an integro-differential equation such that

$$x'=g(t,x)+\int_0^t K(t,s,x(s))ds, \qquad x(0)=x_0, \tag{3.2}$$

which is called the integro-differential equation of Volterra type. For these inequality and equation, we can apply every result as already shown in the preceding sections. As an example, we consider a particular case with strong conditions that g and K satisfy the Lipschitz conditions such that

$$|g(t,x_1)-g(t,x_2)| \leqq L|x_1-x_2|,$$

$$|K(t,s,y_1)-K(t,s,y_2)| \leqq M|y_1-y_2|,$$

where L and M are positive constants. Then, if we choose a function

$$\omega=Lr(t)+M\int_0^t r(s)ds$$

as an ω-function, the result in Theorem 2 yields the estimation

$$|x(t,x_1)-x(t,x_2)| \leqq \frac{M}{\sqrt{L^2+M}}\left(|x_1-x_2|(e^{\lambda_1 t}-e^{\lambda_2 t})+\int_0^t k(s)\varepsilon(s)(e^{-\lambda_1 s}-e^{-\lambda_2 s})ds\right),$$

where

$$\lambda_1=\frac{1}{2}(L+\sqrt{L^2+M}), \qquad \lambda_2=\frac{1}{2}(L-\sqrt{L^2+M}), \qquad \varepsilon(t)=\varepsilon_1(t)+\varepsilon_2(t).$$

From the above estimation, it follows that, if $\varepsilon_i(t)\equiv 0$ $(i=1,2)$, the uniqueness of solutions of (3. 2) is established, and furthermore, every solution of (3. 2) is a continuous function of initial values.

2. Difference-differential equations. In (2. 1), if T is defined by

$$Tx=x(s),$$

where s ranges over $\alpha \leqq s \leqq t$, we obtain the functional-differential inequality and equation respectively such that

$$|x'-f(t,x,x(s))| \leqq \varepsilon(t)$$

and

$$x'=f(t,x,x(s)).$$

On the other hand, if T is defined by

$$Tx=\begin{cases} x(t-h) & (h \leqq t), \\ \varphi(t) & (-h \leqq t < 0), \end{cases}$$

where h is a positive constant, we obtain the difference-differential inequality and equation such that

$$|x'(t)-f(t, x(t), x(t-h))| \leqq \varepsilon(t)$$

and

$$x'(t)=f(t, x(t), x(t-h)).$$

These functional-differential inequalities and equations have been already investigated in detail, for example, Cf. [3, 4, 6, 7]

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