REMARKS ON UNRAMIFIED ABELIAN COVERING SURFACES OF A CLOSED RIEMANN SURFACE

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1. Introduction. Let R be a closed Riemann surface of genus p and W be an unramified unbounded regular covering surface of R whose covering transformation group $\Gamma(W)$ is abelian. Let C_{2i-1}, C_{2i} $(i=1, \cdots, p)$ be $2p$ canonical homology basis of R. Then $\Gamma(W)$ may be considered as an abelian group generated by C_i $(i=1, \cdots, 2p)$ with a number of defining relations among them:

$$\sum_{i=1}^{2p} r_{ki}C_i=0, \qquad k=1, \cdots, q \ (0\leqq q\leqq 2p)$$

with integral coefficients r_{ki}, whose $q\times 2p$ matrix

$$(r_{ki})$$

is of rank q. The rank r of $\Gamma(W)$ is defined by $2p-q$.

Mori [1] proved the following theorem for this surface W:

(1). $W\in O_G$ if and only if $r\leqq 2$.

(2). $W\in O_{AD}$.

(3). $W\in O_{AB}$ if there exists, for each $i=1, \cdots, p$, a relation of the form

$$r_{2i-1}C_{2i-1}+r_{2i}C_{2i}=0$$

with not both vanishing integral coefficients r_{2i-1} and r_{2i}. Especially this is the case, when W consists of a (finite or infinite) number of replicas of a planar surface obtained from R by cutting along p disjoint non-dividing loop cuts.

Let O_{MD} denote the class of Riemann surfaces not tolerating non-constant single-valued analytic function with a finite spherical area. Let O_{AB}^0 denote the class of Riemann surfaces any subregion of which tolerates no non-constant bounded analytic function whose real part vanishes continuously on its relative boundary.

In the present paper we shall prove the following theorem:

Theorem 1. *Let W be an unramified unbounded regular abelian covering surface of R. If W satisfies the condition in* (3) *and* $r\geqq 3$, *then*

$$W\in O_{AB}^0\frown O_{MD}.$$

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This theorem infers the truthness of a part of Tsuji's results. In this point we should make mention of some historical background. Tsuji [7], [8] stated the following fact: $W \in O_{MD}$ if $r \geqq 3$. Unfortunately there was a serious gap in his proof, which was pointed out by Sario [5]. Then the matter returned to the start.

In a way we shall prove the existence of meromorphic functions on W satisfying some growth conditions. It is not known whether there is an analytic function of the slowest growth on a given surface under some reasonable scale or not. This is one of the most important problems in the value-distribution theory. In the tendency of Picard's small theorem this has a decisive effect. This was shown in [4] by a somewhat artificial example of Riemann surface. Of course there also happens an effect in Picard's great theorem on a given surface [3]. Behnke-Stein's existence theorem on regular functions on every given open surface and its consequences would play the roles, if boldly says.

2. Proof of Theorem 1. We need some preparatory considerations. Kuroda [2] introduced the notion of O^0_{AB} and proved the Stoilow principle on Iversen's property and established a modular criterion for O^0_{AB}. His criterion may be stated in the following manner:

If a Riemann surface F admits a sequence of ring domains $R_{n,k}$ $(n=1, 2, \cdots; k=1, 2, \cdots, \nu(n))$ *such that, for each* n, *all of* $R_{n+1,k}$ $(k=1, 2, \cdots, \nu(n+1))$ *together separate the ideal boundary of R from all of* $R_{n,k}$ $(k=1, 2, \cdots, \nu(n))$ *and*

$$\varlimsup_{N\to\infty}\left\{\sum_1^N \log \mu_n - \log \nu(N)\right\} = \infty,$$

then F belongs to the class O^0_{AB}. *Here* μ_n *denotes the minimum harmonic modulus of ring domains* $R_{n,k}$ $(k=1, \cdots, \nu(n))$.

Toda-Matsumoto [6] proved the following two theorems making use of Kuroda's formulation of the Stoilow principle on Iversen's property.

Let F be an open Riemann surface of the class O^0_{AB}. *Let* Ω *be an end of F. If* $f(p)$ *be a single-valued meromorphic function on* Ω *which is bounded or has a finite spherical area, then* $f(p)$ *has a limit at each ideal boundary point of* $F \cap \bar{\Omega}$.

Let F be an open Riemann surface of the class O^0_{AB} *and have an ideal boundary point of positive harmonic measure. Then, for any compact subset K of F such that* $F-K$ *is connected,* $F-K$ *belongs to the class* O^0_{AB}.

Under the assumptions of Theorem 1 Mori [1] proved the following facts: $\mu_n \geqq \text{const.} > 1$ and $\nu(n) = O(n^{r-1})$. Thus we can say immediately that the assumptions of Kuroda's modular criterion for O^0_{AB} are satisfied and hence $W \in O^0_{AB}$.

Suppose that $W \notin O_{MD}$, then there exists a non-constant single-valued meromorphic function $f(p)$ with a finite spherical area. Since W has only one ideal boundary point for $r \geqq 2$ and $W \in O^0_{AB}$, $f(p)$ has a limit at the ideal boundary point. Thus $f(p)$ may be considered as a bounded function on some end Ω of W. Further $W \notin O_G$ for $r \geqq 3$. Thus we have that any end Ω of W belongs to the class O_{AB}, which shows that $f(p)$ must reduce to a constant in Ω and on W. This is a contradiction.

As a corollary we have the following theorem.

THEOREM 2. *Let Φ be a surface obtained from R by cutting along s $(3 \leqq s \leqq p)$ non-dividing independent cycles $C_1, \cdots, C_s$ such that the intersection number $C_i \times C_j = 0$ for any i, j, $1 \leqq i, j \leqq s$. Let W be an unramified abelian covering surface of R and Φ be a fundamental domain of covering transformation group $\Gamma(W)$ of rank $r \geqq 3$. Then $W \in O_{MD} \frown O^0_{AB}$.*

Proof. Let Φ' be a surface obtained from Φ by cutting along $p-s$ non-dividing cycles $C_{s+1}, \cdots, C_p$ of original surface R such that $C_i \times C_j = 0$ for any i, j $(1 \leqq i, j \leqq p)$. Let W' be a surface constructed from a number of replicas Φ_n' of Φ' subjecting to the group $\Gamma(W)$. Let $\{W_k'\}$ be an infinite number of replicas of W'. W_k' and W_{k+1}' are connected along any C_{jnk}^+ and $C_{jn,k+1}^-$, $j = s+1, \cdots, p$; $n = 0, \pm 1, \cdots$; $k = 0, \pm 1, \cdots$, where C_{jnk}^+ and C_{jnk}^- show two boundary curves of Φ_n' in W_k' which correspond to a cycle C_j on Φ or on R. The resulting surface W'' is also an unramified abelian covering surface of R whose covering transformation group $\Gamma(W'')$ is isomorphic to $\Gamma(W)$ added a new generator α in a commutative manner. By Theorem 1 we have $W'' \in O_{MD} \frown O^0_{AB}$. Since $W \notin O_G$ and its boundary element exists only one, it is sufficient to prove $W \in O^0_{AB}$. If $W \notin O^0_{AB}$, then there exists a subregion Ω of W which tolerates a non-constant single-valued bounded regular function $f(p)$ whose real part vanishes continuously on $\partial\Omega$. By the element α of $\Gamma(W'')$ we extend Ω to W'' subjecting to the transformation α and then we obtain an open set Ω' in which $f(p)$ can be extended. Then the resulting function $F(p)$ is a non-constant single-valued bounded regular function in Ω' such that Re $F(p)$ vanishes continuously on $\partial\Omega'$. This contradicts $W'' \in O^0_{AB}$.

3. A growth problem. Let W be an unramified abelian covering surface of a closed Riemann surface R of genus p whose covering transformation group $\Gamma(W)$ is free abelian of rank r and is generated by r generators $C_1, \cdots, C_r$. Let W_A be a domain defined by

$$\sum \prod_{i=1}^{r} C_i \, m_i \, R^*,$$

where R^* is a fundamental domain of $\Gamma(W)$ and the summation is taken over all combinations $(m_1, \cdots, m_r)$ of integers m_i satisfying $|m_i| \leqq A$ for each $i = 1, \cdots, r$. Let $f(p)$ be a single-valued meromorphic function on R whose image is $p+1$ sheeted sphere. Let $F(p)$ be the extension of $f(p)$ by $\Gamma(W)$. Then $F(p)$ is a single-valued meromorphic function on W. Let $S(A, F)$ be the mean sheet number of $F(W_A)$, that is,

$$S(A, F) = \frac{1}{\pi} \iint_{W_A} \frac{|F'|^2}{(1+|F|^2)^2} \, d\sigma.$$

Then we have

$$S(A, F)=C\,(2A)^r,$$

hence we can infer that

$$\lim_{A\to\infty} \frac{\log S(A, F)}{\log A}=r.$$

THEOREM 3. *There exists a single-valued meromorphic function of order r with regard to the above scale on W.*

If $W\notin O_{MD}$ (this case occurring very rarely but surely in two cases $r=1$ and $r=2$), then there is a single-valued regular function with a finite spherical area on W (and hence of order zero). It is not known what value is the smallest possible one as the order of a single-valued meromorphic function on W, when either $r\geqq 3$ or $1\leqq r\leqq 2$ but $W\in O_{MD}$. The above argument suggests a possibility for establishment of a general value-distribution theory on W by making use of our scale of measurement.

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