ON EVANS' SOLUTION OF THE EQUATION $\Delta u = Pu$ ON RIEMANN SURFACES

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Introduction.

Let R be an open Riemann surface. By a density $P(z)$ on R we mean a non-negative continuously differentiable function of local parameters $z=x+\sqrt{-1}\,y$ such that the expression $P(z)dxdy$ is invariant under the change of local parameters z. Then we can consider the elliptic partial differential equation

$$(E) \qquad \Delta u(z)=P(z)u(z), \qquad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

which is invariantly defined on R. Throughout this paper, we always assume

$$P(z)dxdy \not\equiv 0$$

on R. By a solution u of (E) on an open subset D of R we mean that u is twice continuously differentiable function satisfying (E) on D.[1] An *Evans' solution* $e(z)$ of (E) on R is a solution of (E) on R satisfying

$$\lim_{R\ni z\to A_\infty} e(z)=\infty,$$

where A_∞ is the Alexandroff's ideal boundary point of R. The purpose of this paper is to give a sufficient condition for the existence of Evans' solution of (E) on R.

Let $(R_n)_{n=0}^\infty$ be a normal exhaustion of R and $\Omega_{0,n}$ $(n>0)$ be the continuous function on $\bar{R}_n-R_0$ such that $\Omega_{0,n}$ is a solution of (E) on $R_n-\bar{R}_0$ with $\Omega_{0,n}=1$ on ∂R_0 and $\Omega_{0,n}=0$ on ∂R_n. Then there exists a continuous function Ω_0 on $R-R_0$ such that Ω_0 is a solution of (E) on $R-\bar{R}_0$ with $\Omega_0=1$ on ∂R_0 and

$$\Omega_0(z)=\lim_{n\to\infty}\Omega_{0,n}(z)>0$$

on $R-R_0$. Clearly $\Omega_0(z)$ does not depend on the special choice of exhaustions $(R_n)_{n=1}^\infty$.

We consider the condition

$$(\Omega) \qquad \sigma=\inf_{z\in R-R_0}\Omega_0(z)>0.$$

It is easy to see that the condition (Ω) does not depend on the special choice of

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1) For fundamental properties of solutions of (E), refer to the list in pp. 152–153 in [4].

R_0. Hence the condition (Ω) depends only on the pair (R, P). The main result of this paper is the following

THEOREM 1. *Suppose that the condition (Ω) is satisfied. Then the equation (E) possesses an Evans' solution of (E) on R.*

For the proof of this theorem, first we construct the extension $G(p,q)$ of the Green function on R with respect to (E) to the Čech compactification R^* of R (§ 1). Using this extended Green kernel $G(p,q)$, we define for each subset K of R^* a "transfinite diameter" $D(K)$ and a modified "Tchebycheff's constant" $E(K)$ of K and then we prove $E(K) \geqq D(K)$ (Proposition 2, § 2). Usually in the potential theory, Fekete's relation $E(K)=D(K)$ is proved for compact sets K by using symmetricity and continuity of the kernel function. But our kernel $G(p,q)$ not necessarily satisfies symmetricity and continuity. In spite of this, we can prove the half of Fekete's relation: $E(K) \geqq D(K)$. This fact may have the independent interest. Next we prove $D(\Gamma)=\infty$, where Γ is the Čech boundary R^*-R of R (Proposition 3, § 3). After these preparations, we conclude the proof of Theorem 1 (§ 4).

Only in the proof of $D(\Gamma)=\infty$, we use the assumption (Ω). The converse of this is true. Namely, the following Theorem 2 holds which is proved in § 5. This shows that without (Ω) the standard method due to Evans to construct Evans' solution based on the transfinite diameter with respect to the Green kernel cannot be applied.

THEOREM 2. *The following conditions are mutually equivalent*:

(a) (Ω);

(b) $\inf_{z \in R} G(z,w) > 0$ *for each w in R*;

(c) $D(\Gamma)=\infty$.

Finally we give an application of Theorem 1 to the function theory. An Evans-Selberg's potential on R is a harmonic function $h(z)$ on R with one negative logarithmic singularity in R such that $\lim_{R \ni z \to A_\infty} h(z)=\infty$. Applying Theorem 1, we prove the following in § 6:

THEOREM 3 (*Evans-Selberg-Kuramochi*[2]). *There exists an Evans-Selberg's potential on R if and only if R is of null boundary.*

§ 1. Green kernel on Čech compactification.

LEMMA 1.1. *There exists a unique compact Hausdorff space R^*, called the Čech compactification of R, such that*

(C. 1) *R is an open subspace of R^*, or equivalently, $\Gamma=R^*-R$, which is called*

2) See Kuramochi [2].

the Čech boundary of R, is compact in R^*;

(C. 2) *R is dense in R^*;*

(C. 3) *any bounded continuous function*[3] *on R is uniquely extended to R^* so as to be continuous on R^*.*

In fact, since R is completely regular, by a theorem of Čech,[4] there exists a unique compact Hausdorff space R^* satisfying (C. 2) and (C. 3). As R is locally compact, so (C. 1) is satisfied for this R^*.[5]

LEMMA 1.2. *Any continuous function f on R is uniquely extended to R^* so as to be continuous on R^*.*

In fact, let $g(z)=\max(f(z), 0)$ and $h(z)=g(z)-f(z)$ on R. Then $(1+g(z))^{-1}$ and $(1+h(z))^{-1}$ are bounded continuous functions on R and so continuously extended to R^*. Then the same is true for $g(z)$ and $h(z)$ and since R is dense in R^*, these extensions are unique. We denote these extensions by the same notations. If $g(p)=\infty$ at some point p in R^*, then there exists a neighborhood V of p such that $g(p)>0$ on V. Hence $g(z)=f(z)$ on $V\cap R$ and so $h(z)=0$ on $V\cap R$. As $V\cap R$ is dense in V, so $h(p)=0$. Similarly, $h(p)=\infty$ implies $g(p)=0$. Hence the expression $g(p)-h(p)$ has a definite meaning and gives a continuous extension of $f(z)=g(z)-h(z)$. Again, since R is dense in R^*, the extension is unique. Q. E. D.

Let $g(z, w)$ be the Green function on R with respect to (E) with its pole w in R. It is positive, symmetric $g(z, w)=g(w, z)$ and continuous on $R\times R$.[6] If we fix z in R, then $g(z, w)$ is continuous on R with respect to w and so continuously extended to R^* in a unique way. We set

$$g(z, p)=\lim_{R\ni w\to p} g(z, w) \qquad (p\in R^*).$$

LEMMA 1.3. *For any point p in $\Gamma=R^*-R$, $g(z, p)$ is a solution of (E) on R and so extended continuously to R^* in a unique way.*

In fact, let z_0 be an arbitrary point in R and ε be an arbitrary positive number. We can find a neighborhood U of z_0 with compact closure in R and a positive constant M such that $g(z_0, w)\leqq M$ if $w\notin U$. We can also find a neighborhood V of z_0 such that $\overline{V}\subset U$ and $c^{-1}g(z_0, w)\leqq g(z, w)\leqq cg(z_0, w)$ for any z in V and w in $R-U$, where $c=1+\varepsilon/M$. Hence $|g(z, w)-g(z_0, w)|\leqq\varepsilon$ for any z in V and w in $R-U$. Letting $w\to p$, we get

$$|g(z, p)-g(z_0, p)|\leqq\varepsilon$$

for any z in V. This shows that $g(z, p)$ is continuous on R with respect to z.

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3) Functions or continuous functions considered in this paper are all assumed to be $[-\infty, \infty]$-valued Bounded functions are functions whose ranges are compact in $(-\infty, \infty)$.

4) See Čech [1].

5) See p. 163 in [5].

6) See pp. 154–157 in [4].

Next take a countable dense subset $(z_m)_{m=1}^{\infty}$ of R. By induction, we can find sequences $(U_{m,n})_{n=1}^{\infty}$ $(m=1, 2, \cdots)$ of neighborhoods of p such that

$$U_{m,n} \supset U_{m,n+1},\ U_{m+1,n}, \qquad \bigcap_{n=1}^{\infty} (R \cap U_{1,n}) = \phi$$

and

$$\lim_n \sup_{w \in U_{m,n}} |g(z_m, w) - g(z_m, p)| = 0.$$

This is possible, since $g(z_m, w) \to g(z_m, p)$ as $w \to p$ for each $m=1, 2, \cdots$. Set $V_n = U_{n,n} \cap R$ and fix a point w_n in V_n. Then

$$\lim_n g(z, w_n) = g(z, p)$$

for $z = z_m$ $(m=1, 2, \cdots)$. On the other hand, by Harnack type inequality, $(g(z, w_n))_{n=1}^{\infty}$ is a bounded sequence of solutions on each compact subdomain except a finite number of terms. Hence by choosing a suitable subsequence, we may assume $(g(z, w_n))_{n=1}^{\infty}$ converges to a solution $u(z)$ of (E) on R. Thus $u(z) = q(z, p)$ on the dense subset $(z_n)_{n=1}^{\infty}$ of R. Since $g(z, p)$ is continuous on R, we conclude that $g(z, p) \equiv u(z)$ on R, which shows that $g(z, p)$ is a solution of (E) on R. Q. E. D.

DEFINITION. The *Green kernel* $G(p, q)$ on R^* is defined by

$$G(p, q) = \lim_{R \ni z \to p} \left(\lim_{R \ni w \to q} g(z, w) \right) \qquad (p, q \in R^*).$$

PROPOSITION 1. *The Green kernel $G(p, q)$ on $R^* \times R^*$ possesses the following properties:*

(G. 1) *$G(z, w) = g(z, w)$ for z and w in R;*
(G. 2) *$G(z, p) = G(p, z)$ if z is in R;*
(G. 3) *$G(z, p)$ is a solution of (E) on R except p;*
(G. 4) *$G(p, q)$ is continuous in $p \in R^*$ for fixed q in R^*.*

This is a simple consequence of hitherto considerations. Notice that we do not claim the symmetricity $G(p, q) = G(q, p)$ for p and q in Γ and the continuity of $G(p, q)$ with respect to q at Γ for fixed p in Γ.

From Proposition 1 and Harnack type inequality, it is easily seen that $G(z, p)$ is finitely continuous on $R \times \Gamma$ and hence continuous on $R \times R^*$.

§ 2. Quantities $D(K)$ and $E(K)$.

For each subset K of R^*, we set

$$\binom{n}{2} D_n(K) = \inf_{p_1, \cdots, p_n \in K} \sum_{i<j}^{1, \cdots, n} G(p_i, p_j).$$

It is easy to see that $(D_n(K))_{n=1}^{\infty}$ is non-decreasing and so we can define

$$D(K)=\lim_{n\to\infty} D_n(K).$$

Similarly, we set

$$nE_n(K)=\sup_{p_1,\cdots,p_n\in K}\ \inf_{p\in K}\sum_{i=1}^{n} G(p, p_i).$$

Since the sequence $(E_n(K))_{n=1}^{\infty}$ satisfies

$$(n+m)E_{n+m}(K)\geqq nE_n(K)+mE_m(K) \qquad (n, m=1, 2, \cdots),$$

we can define

$$E(K)=\lim_{n\to\infty} E_n(K).$$

PROPOSITION 2. $$E(K)\geqq D(K).$$

Proof. Let n be an arbitrary positive integer. We set $r=1/(n-1)$ and choose n points $p_n, p_{n-1},\cdots, p_2, p_1$ in K satisfying

$$\sum_{j=n-i+1}^{n} G(p_{n-i}, p_j)\leqq\inf_{p\in K}\sum_{j=n-i+1}^{n} G(p, p_j)+r \qquad (j=1, 2,\cdots, n-1). \tag{2.1}$$

We choose these n points inductively. Let p_n be an arbitrary point in K. Assume that $p_n, p_{n-1},\cdots, p_{n-i+1}$ $(i\leqq n-1)$ have been already chosen. Consider

$$h(p)=\sum_{j=n-i+1}^{n} G(p, p_j).$$

Since $\inf_{p\in K} h(p)\geqq 0$, we can find a point p_{n-i} in K such that $h(p_{n-i})\leqq h(p)+r$ on K. This is nothing but (2.1).[7]

By the definition of $E_i(K)$, we can easily see that

$$\inf_{p\in K}\sum_{j=n-i+1}^{n} G(p, p_j)\leqq iE_i(K).$$

Hence by (2.1), we get

$$\sum_{j=n-i+1}^{n} G(p_{n-i}, p_j)\leqq iE_i(K)+r \qquad (i=1, 2,\cdots, n-1).$$

Summing up these $n-1$ inequalities, we get by the definition of $D_n(K)$

$$\binom{n}{2}D_n(K)\leqq\sum_{i=1}^{n-1} iE_i(K)+(n-1)r$$

or

$$D_n(K)\leqq\left(\sum_{i=1}^{n-1} iE_i(K)\right)\Big/\binom{n}{2}+1\Big/\binom{n}{2} \tag{2.2}$$

7) Since $h(p)$ is continuous on R^*, we can choose a point p_{n-i} in K satisfying $h(p_{n-i})\leqq h(p)$ on K. Hence r in (2.1) is superfluous in this case. We want to emphasize here that the above proof to show the relation $E(K)\geqq D(K)$ is valid for any kernel bounded from below.

Since $\lim_n E_n(K)=E(K)$, it is easy to see that

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n-1} iE_i(K)\right)\Big/\binom{n}{2}=E(K).$$

Hence by making $n\nearrow\infty$ in (2.2), we get $D(K)\leqq E(K)$.

§ 3. Evaluation of $D(\Gamma)$.

In this section, we assume the condition (Ω):

$$\sigma=\inf_{R-R_0\ni z}\Omega_0(z)>0.$$

Under this assumption, we shall show that $D(\Gamma)=\infty$, where $\Gamma=R^*-R$.

Let $(R_n)_{n=0}^{\infty}$ be a normal exhaustion of R. Set $\Gamma_n=\bar{R}_n-R_n$. We denote by $M(\Gamma_n)$ the totality of unit positive Borel measures on Γ_n. For each measure μ in $M(\Gamma_n)$, we put

$$I(\mu)=\int G(z, w)d\mu(z)d\mu(w).$$

Set

$$W_n=\inf_{\mu\in M(\Gamma_n)} I(\mu).$$

Then we have[8]

LEMMA 3.1. *There exists a unique measure μ_n in $M(\Gamma_n)$ such that*

$$I(\mu_n)=W_n$$

and the function $U_n(z)$ on R defined by

$$U_n(z)=\int G(z, w)d\mu_n(w)$$

is the solution of (E) on $R-\Gamma_n$ and $U_n(z)\leqq W_n$ on R and $U_n(z)=W_n$ on Γ_n.

Let $w_n(z)$ be the continuous function on R such that w_n is a solution of (E) in R_n and $w_n=1$ on $R-R_n$. We set

$$A_n=\int_R w_n(z)P(z)dxdy.$$

LEMMA 3.2. $$\lim_{n\to\infty} A_n=0.$$

Proof. The condition $\sigma>0$ implies that R is of null boundary.[9] Hence (E) does not possess bounded solution except the constant zero.[10] By the maximum

8) See pp. 157-165 in [4].

9) See Theorem 1 in Ozawa [8].

10) See Ozawa [6] or Theorem 1 in Royden [9].

principle, (w_n) is a monotone decreasing sequence converging to a bounded solution of (E) on R which must reduce to the constant zero. Hence $P(z) \geqq w_n(z)P(z) \searrow 0$. On the other hand, $\sigma > 0$ implies

$$\int_R P(z)dxdy < \infty.$$ [11]

Hence by Lebesgue's convergence theorem

$$\lim_{n\to\infty} A_n = \int_R \lim_{n\to\infty} w_n(z)P(z)dxdy = 0.$$

LEMMA 3.3. $\lim_{n\to\infty} W_n = \infty.$

Proof. Let $G_n(z, w)$ be the Green function on R_n with respect to (E). Then $G_n(z, w) \nearrow G(z, w)$ on $R \times R$. By Green's formula

$$2\pi w_n(w) = \int_{\Gamma_n} {}^*d_z G_n(z, w) = -\int_{R_n} d_z({}^*d_z G_n(z, w)) + 2\pi.$$

Since $d_z({}^*d_z G_n(z, w)) = \Delta_z G_n(z, w) = P(z)G_n(z, w)$, we get

$$\int_{R_n} G_n(z, w)P(z)dxdy = 2\pi(1 - w_n(z)).$$

Hence by making $n \nearrow \infty$, we get

$$\int_R G(z, w)P(z)dxdy = 2\pi.$$ [12]

From this, by Fubini's theorem

$$\int_R U_n(z)P(z)dxdy = \int_{\Gamma_n}\left(\int_R G(z, w)P(z)dxdy\right)d\mu_n(w) = 2\pi.$$

Thus

(3.1) $$\int_R U_n(z)P(z)dxdy = 2\pi.$$

By the maximum principle, $U_n(z) \leqq W_n w_n(z)$ on R. From this

$$\int_R U_n(z)P(z)dxdy \leqq W_n \int_R w_n(z)P(z)dxdy.$$

Hence by (3.1), we get $2\pi \leqq W_n A_n$ and so

$$\liminf_{n\to\infty} W_n \geqq \lim_{n\to\infty} 2\pi / A_n = \infty.$$

LEMMA 3.4. $D(\Gamma) \geqq \sigma^2 W_m \qquad (m = 1, 2, \cdots).$

11) See Corollary 1 in Ozawa [8].

12) This relation is due to L. Myrberg [3] and Ozawa [7].

Proof. Let $\Omega_{m,k}(z)$ $(k>m)$ be a continuous function on $\bar{R}_k-R_m$ such that $\Omega_{m,k}$ is the solution of (E) in $R_k-\bar{R}_m$ with $\Omega_{m,k}=1$ on Γ_m and $\Omega_{m,k}=0$ on Γ_k. Clearly $(\Omega_{m,k})_{k=m+1}^{\infty}$ is an increasing sequence and so there exists a continuous function Ω_m on $R-R_m$ which is a solution of (E) on $R-\bar{R}_m$ with $\Omega_m=1$ on Γ_m and

$$\lim_{k\to\infty}\Omega_{m,k}(z)=\Omega_m(z)$$

on $R-R_m$. Since $\Omega_{m,k}\geqq\Omega_{0,k}$ on R_k-R_m by the maximum principle, we have

$$\Omega_m(z)\geqq\Omega_0(z)$$

on $R-R_m$. Hence in particular

$$\inf_{z\in R-R_m}\Omega_m(z)\geqq\sigma.$$

Let n be an arbitrary positive integer larger than 4 and $p_1, p_2,\cdots, p_n$ be in Γ. We choose n points $q_1, q_2,\cdots, q_n$ in Γ_m inductively as follows. Let

$$h_1(z)=\sum_{i=2}^{n}G(z, p_i)$$

and q_1 be in Γ_m such that

$$h_1(q_1)=\min_{z\in\Gamma_m}h_1(z).$$

Since $h_1(z)\geqq 0$ on R_k-R_m, we have by the maximum principle, $h_1(z)\geqq h_1(q_1)\Omega_{m,k}(z)$ for z in R_k-R_m. Hence on R^*-R_m

$$h_1(p)\geqq h_1(q_1)\Omega_m(p).$$

Hence in particular $h_1(p_1)\geqq h_1(q_1)\Omega_m(p_1)\geqq\sigma h_1(q_1)$ and so

$$\sigma\sum_{i=2}^{n}G(q_1, p_i)+\sum_{i<j}^{2,\cdots,n}G(p_i, p_j)\leqq\sum_{i<j}^{1,\cdots,n}G(p_i, p_j)\equiv a. \tag{3.2}$$

Next we choose $q_2, q_3,\cdots, q_{n-2}$ in Γ_m satisfying

$$\sigma^2\sum_{i<j}^{1,\cdots,k}G(q_i, q_j)+\sigma\sum_{i=1}^{k}\sum_{j=k+1}^{n}G(q_i, p_j)+\sum_{i<j}^{k+1,\cdots,n}G(p_i, p_j)\leqq a\quad(k=2, 3,\cdots, n-2). \tag{3.3}$$

First let

$$h_2(z)=\sum_{j=3}^{n}G(z, p_j)+\sigma G(q_1, z)$$

and q_2 be in Γ_m such that

$$h_2(q_2)=\min_{z\in\Gamma_m}h_2(z).$$

Similarly as above, we have $h_2(p)\geqq h_2(q_2)\Omega_m(p)$ on R^*-R_m and so

$$\sum_{j=3}^{n}G(p_2, p_j)+\sigma G(q_1, p_2)\geqq\sigma\sum_{j=3}^{n}G(q_2, p_j)+\sigma^2G(q_1, q_2).$$

From this with (3.2), we get

$$\sigma^2 G(q_1, q_2)+\sigma \sum_{i=1}^{2} \sum_{j=3}^{n} G(q_i, p_j)+\sum_{i<j}^{3,\cdots,n} G(p_i, p_j)\leqq a.$$

This is nothing but (3.3) for $k=2$. Next assume that $q_2,\cdots, q_k$ $(k\leqq n-3)$ have been chosen in Γ_m satisfying (3.3). Let

$$h_{k+1}(z)=\sum_{j=k+2}^{n} G(z, p_j)+\sigma \sum_{i=1}^{k} G(q_i, z)$$

and q_{k+1} be in Γ_m such that

$$h_{k+1}(q_{k+1})=\min_{z\in\Gamma_m} h_{k+1}(z).$$

Similarly as before, we have $h_{k+1}(p)\geqq h_{k+1}(q_{k+1})\Omega_m(p)$ on R^*-R_m and so

$$\sum_{j=k+2}^{n} G(p_{k+1}, p_j)+\sigma \sum_{i=1}^{k} G(q_i, p_{k+1})$$

$$\geqq \sigma \sum_{j=k+2}^{n} G(q_{k+1}, p_j)+\sigma^2 \sum_{i=1}^{k} G(q_i, q_{k+1}).$$

From this with (3.3), we have

$$\sigma^2 \sum_{i<j}^{1,\cdots,k+1} G(q_i, q_j)+\sigma \sum_{i=1}^{k+1} \sum_{j=k+2}^{n} G(q_i, p_j)+\sum_{i<j}^{k+2,\cdots,n} G(p_i, p_j)\leqq a.$$

This is (3.3) for $k+1$. Thus we have constructed the system $q_2,\cdots, q_{n-2}$. Next let

$$h_{n-1}(z)=G(z, p_n)+\sigma \sum_{i=1}^{n-2} G(q_i, z)$$

and q_{n-1} be in Γ_m such that

$$h_{n-1}(q_{n-1})=\min_{z\in\Gamma_m} h_{n-1}(z).$$

Similarly as before, we have $h_{n-1}(p)\geqq h_{n-1}(q_{n-1})\Omega_m(p)$ on R^*-R_m and so

$$G(p_{n-1}, p_n)+\sigma \sum_{i=1}^{n-2} G(q_i, p_{n-1})\geqq \sigma G(q_{n-1}, p_n)+\sigma^2 \sum_{i=1}^{n-2} G(q_i, q_{n-1}).$$

From this with (3.3) for $k=n-2$, we have

$$\sigma^2 \sum_{i<j}^{1,\cdots,n-1} G(q_i, q_j)+\sigma \sum_{i=1}^{n-1} G(q_i, p_n)\leqq a. \tag{3.4}$$

Finally let

$$h_n(z)=\sigma \sum_{i=1}^{n-1} G(q_i, z)$$

and q_n be in Γ_m such that

$$h_n(q_n)=\min_{z\in\Gamma_m} h_n(z).$$

Similarly as before, we have $h_n(p)\geqq h_n(q_n)\Omega_m(p)$ on R^*-R_m and so

$$\sigma\sum_{i=1}^{n-1} G(q_i, p_n)\geqq\sigma^2\sum_{i=1}^{n-1} G(q_i, q_n).$$

From this with (3.4), we get

$$\sigma^2\sum_{i<j}^{1,\cdots,n} G(q_i, q_j)\leqq a=\sum_{i<j}^{1,\cdots,n} G(p_i, p_j).$$

From this inequality, we get by the definition of $D_n(\Gamma_m)$

$$\sigma^2\binom{n}{2}D_n(\Gamma_m)\leqq\sum_{i<j}^{1,\cdots,n} G(p_i, p_j).$$

Since $p_1,\cdots, p_n$ are arbitrary in Γ, we get

$$\sigma^2\binom{n}{2}D_n(\Gamma_m)\leqq\binom{n}{2}D_n(\Gamma)\quad\text{or}\quad D_n(\Gamma)\geqq\sigma^2 D_n(\Gamma_m).$$

Hence by making $n\nearrow\infty$, we get

$$D(\Gamma)\geqq\sigma^2 D(\Gamma_m). \tag{3.5}$$

Now let $q_1^{(n)},\cdots, q_n^{(n)}$ be in Γ_m with

$$\binom{n}{2}D_n(\Gamma_m)+1\geqq\sum_{i<j}^{1,\cdots,n} G(q_i^{(n)}, q_j^{(n)}) \tag{3.6}$$

and let μ_n be in $M(\Gamma_m)$ with $\mu_n(q_i^{(n)})=1/n$ $(i=1,\cdots, n)$. Since $M(\Gamma_m)$ is vaguely compact,[13] there exists a subsequence $(\mu_{n'})$ of (μ_n) and a measure μ in $M(\Gamma_m)$ with $\mu=\lim_{n'\to\infty}\mu_{n'}$ (vaguely). Let c be an arbitrary positive number. From (3.6)

$$\begin{aligned}\binom{n'}{2}D_{n'}(\Gamma_m)+1&\geqq\sum_{i<j}^{1,\cdots,n'}\min(c, G(q_i^{(n')}, q_j^{(n')}))\\&=\frac{n'^2}{2}\int\min(c, G(z, w))d\mu_{n'}(z)d\mu_{n'}(w)-\frac{n'c}{2}.\end{aligned}$$

Hence

$$D(\Gamma_m)+1\Big/\binom{n'}{2}\geqq\int\min(c, G(z, w))d\mu_{n'}(z)d\mu_{n'}(w)-\frac{c}{n'}.$$

As $\mu_{n'}\times\mu_{n'}$ converges to $\mu\times\mu$ vaguely and $\min(c, G(z, w))$ is continuous on $\Gamma_m\times\Gamma_m$, so by making $n'\nearrow\infty$,

$$D(\Gamma_m)\geqq\int\min(c, G(z, w))d\mu(z)d\mu(w).$$

13) See Selection theorem, p. 162 in [4].

Making $c\nearrow\infty$, we have $D(\Gamma_m)\geqq I(\mu)\geqq\inf_{\nu\in M(\Gamma_m)}I(\nu)$, i. e.

$$D(\Gamma_m)\geqq W_m.$$

From this with (3.5), we finally get

$$D(\Gamma)\geqq\sigma^2 W_m.$$

PROPOSITION 3. $\qquad D(\Gamma)=\infty.$

This follows from Lemmas 3.3 and 3.4.

§ 4. Proof of Theorem 1.

Assume that the condition (Ω) is satisfied:

$$\inf_{z\in R-R_0}\Omega_0(z)>0.$$

We have to prove the existence of an Evans' solution of (E) on R.[14]

By Propositions 2 and 3, we have $E(\Gamma)=\infty$. Since $E(\Gamma)=\lim_n E_n(\Gamma)$, we can find an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that

$$E_{n_k}(\Gamma)>2^k \qquad (k=1,2,\cdots).$$

By the definition of $E_{n_k}(\Gamma)$, we can find n_k points $p_{k,i}$ $(i=1,2,\cdots,n_k)$ in Γ such that

$$\inf_{p\in\Gamma}\sum_{i=1}^{n_k}G(p,p_{k,i})>2^k n_k.$$

Then the function

$$e_k(p)=2^{-k-1}n_k^{-1}\sum_{i=1}^{n_k}G(p,p_{k,i})$$

is continuous on R^* and a solution of (E) on R and $e_k(p)>1/2$ on Γ. Thus we can find a compact set K_k in R such that

$$e_k(p)>\frac{1}{2}\quad\text{on } R^*-K_k \text{ and a fortiori on } R-K_k.$$

Let z_0 be a point in R and V_0 be a neighborhood of z_0 with $\bar{V}_0$ compact in R. Then there exists a constant c_0 such that $G(z_0,w)\leqq c_0$ $(w\in R-V_0)$. Hence $G(z_0,p)\leqq c_0$ for any p in Γ. From this, $e_k(z_0)\leqq c_0/2^{k+1}$. Thus the sequence $(\sum_{k=1}^{n}e_k(z))_{n=1}^{\infty}$ is a monotone increasing sequence of solutions of (E) on R such that $\sum_{k=1}^{n}e_k(z_0)\leqq c_0$. Hence

$$e(z)=\sum_{k=1}^{\infty}e_k(z)$$

14) The following method of construction is the standard one originally due to Evans. The following proof is contained only for the sake of completeness.

is a solution of (E) on R and $e(z)\geqq n/2$ on $R-\cup_{k=1}^{n}K_k$. Since $\cup_{k=1}^{n}K_k$ is compact in R, the above inequality shows that $\lim_{R\ni z\to A_\infty}e(z)=\infty$. Thus the function $e(z)$ is a required Evans' solution of (E) on R.

REMARK. The Evans' solution $e(z)$ of (E) on R constructed above satisfies the following condition:

$$(*)\qquad \int_R e(z)P(z)dxdy<\infty.$$

In fact, by the footnote[12]

$$\int_R G(z, w)P(z)dxdy=2\pi$$

for any w in R. Hence by Fatou's lemma

$$\int_R G(z, p)P(z)dxdy\leqq\liminf_{R\ni w\to p}\int_R G(z, w)P(z)dxdy=2\pi.$$

Thus

$$\int_R e_k(z)P(z)dxdy=2^{-k-1}n_k^{-1}\sum_{i=1}^{n_k}\int_R G(z, p_{k,i})P(z)dxdy\leqq 2^{-k-1}2\pi$$

and so

$$\int_R e(z)P(z)dxdy=\sum_{k=1}^{\infty}\int_R e_k(z)P(z)dxdy\leqq 2\pi.$$

It is the writer's conjecture that the condition (Ω) is equivalent to the existence of Evans' solution satisfying the condition (*).

§ 5. Proof of Theorem 2.

To see the equivalence of (a) and (b), we have only to show that (a) is equivalent to

$$\inf_{z\in R} G(z, w)>0$$

for a fixed w in R_0. Let $c>1$ and satisfy $c>G(z, w)>c^{-1}$ for any z in ∂R_0 and $G_n(z, w)$ be Green's function of (E) on R_n. Then for sufficiently large n, $c>G_n(z, w)>c^{-1}$ for z in ∂R_0. Hence by the maximum principle

$$c\Omega_{0,n}(z)>G_n(z, w)>c^{-1}\Omega_{0,n}(z)$$

on $R_n-\bar{R}_0$. From this we get

$$c\Omega_0(z)>G(z, w)>c^{-1}\Omega_0(z).$$

This shows that the equivalence of (a) and (b).

The implication (a)→(c) is nothing but Proposition 3

Finally we show that the implication (c)→(b). Contrary to the assertion, assume that

$$\inf_{z\in R} G(z, w)=0$$

for a point w in R. Then there exists a point p in Γ such that $G(w, p)=G(p, w)=0$ for a point w in R and so for every point w in R. Thus for any q in Γ, $G(q, p)=0$. Hence by putting $q_1=q_2=\cdots=q_n=p$, we get

$$0\leqq \inf_{p_1,\cdots,p_n\in\Gamma} \sum_{i<j}^{1,\cdots,n} G(p_i, p_j)\leqq \sum_{i<j}^{1,\cdots,n} G(q_i, q_j)=0$$

or

$$D_n(\Gamma)=0.$$

Thus $D(\Gamma)=\lim_n D_n(\Gamma)=0$, which is a contradiction. Q. E. D.

From the above proof, we also get

THEOREM 2′. *The following conditions are mutually equivalent*:

(a) $$\inf_{z\in R-R_0} \Omega_0(z)=0;$$

(b) $$\inf_{z\in R} G(z, w)=0 \quad \textit{for every } w \textit{ in } R;$$

(c) $$D(\Gamma)=0.$$

REMARK. The condition (Ω) is a sufficient but not necessary condition for the existence of Evans' solution of (E) on R. As an example,[15] let

$$R=(z;\ |z|<1)$$

and

$$P(z)=\frac{4(1+|z|^2)}{(1-|z|^2)^2}.$$

Then (E) possesses an Evans' solution

$$e(z)=\frac{1}{1-|z|^2}$$

of (E) on R. To show that the condition (Ω) is not satisfied, let $(R_n)_{n=0}^\infty$ be the exhaustion of R such that $R_n=(z;\ |z|<1-1/(n+2))$ and $w_n(z)$ be the harmonic function on $R_n-\bar{R}_0$ $(n\geqq 1)$ with boundary value $w_n=1$ on ∂R_0 and $w_n=0$ on ∂R_n. Then by the maximum principle

$$\Omega_{0,n}(z)\leqq w_n(z).$$

Clearly w_n converges to the harmonic function w on $R-\bar{R}_0$ with boundary value $w=1$ on ∂R_0 and $w=0$ on $\partial R=(z;\ |z|=1)$. Hence

15) This example is due to Royden [9], p. 10.

$$0<\Omega_0(z)\leqq w(z)$$

and so

$$0\leqq \inf_{R-R_0\ni z} \Omega_0(z)\leqq \inf_{R-R_0\ni z} w(z)=0.$$

It is the Prof. Ozawa's conjecture that the non-existence of non-zero bounded solution of (E) on R is equivalent to the existence of Evans' solution of (E) on R.[16]

§ 6. Proof of Theorem 3.

The "only if" part of Theorem 3 is easily seen and well known. So we have only to show the "if" part. Take a density $P(z)$ with $P(z)dxdy\not\equiv 0$ on R and $P(z)dxdy\equiv 0$ outside a fixed compact set in R. Let $(R_n)_{n=0}^{\infty}$ be a normal exhaustion of R. We assume that $P(z)dxdy\equiv 0$ on $R-R_0$. Then the function $\Omega_{0,n}(z)$ $(n\geqq 1)$ is harmonic in $R_n-\bar{R}_0$ with boundary value $\Omega_{0,n}=1$ on ∂R_0 and $\Omega_{0,n}=0$ on ∂R_n. Hence Ω_0 is the positive harmonic function on $R-\bar{R}_0$ with boundary value $\Omega_0=1$ on ∂R_0. Since R is of null boundary, by well known Mori's theorem,

$$\sigma=\inf_{z\in R-R_0} \Omega_0(z)>0,$$

i. e. the condition (Ω) is satisfied. Hence by Theorem 1, there exists an Evans' solution $e(z)$ of (E) on R. As

$$\Delta e(z)=P(z)e(z)=0$$

on $R-R_1$, so $e(z)$ is harmonic on $R-R_1$. Let $a>\sup_{z\in R_1}e(z)$ and $U=(z\in R; e(z)>a)$. We can find a positive number b such that

$$b\int_{\partial U} {}^*de(z)=2\pi.$$

Let the singularity function $s(z)$ on $R-\partial U$ be defined as follows:

$$s(z)=\begin{cases} be(z)-ab & \text{on } U, \\ -k(z,w) & \text{on } R-\overline{U}, \end{cases}$$

where $k(z,w)$ is the harmonic Green's function on $R-\overline{U}$ with its pole w in $R-\overline{U}$. Let L be a normal operator of Sario.[17] Consider the equation

$$L(h-s)=h-s.$$

By the definition of $s(z)$,

$$\int_{\partial U+\partial(R-U)} {}^*ds(z)=0$$

and so the above equation has a solution $h(z)$ harmonic on R which is a desired Evans-Selberg's potential on R with one negative logarithmic pole at w.[17]

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16) Compare this with the conjecture in the remark of § 4.

17) See Sario [10].

REFERENCES

[1] ČECH, E., On bicompact spaces. Ann. of Math. **38** (1937), 823–844.

[2] KURAMOCHI, Z., Mass distributions on the ideal boundaries of abstract Riemann surfaces, I. Osaka Math. J. 8 (1956), 119–137.

[3] MYRBERG, L., Über die Existenz der Greenschen Funktion der Gleichung $\Delta u=c(P)u$ auf Riemannschen Flächen. Ann. Acad. Sci. Fenn. AI **170** (1954), 1–8.

[4] NAKAI, M., The space of non-negative solutions of the equation $\Delta u=pu$ on a Riemann surface. Kōdai Math. Sem. Rep. **12** (1960), 151–178.

[5] ———, Algebraic criterion on quasiconformal equivalence of Riemann surfaces. Nagoya Math. J. **16** (1960), 157–184.

[6] OZAWA, M., Classification of Riemann surfaces. Kōdai Math. Sem. Rep. **4** (1952), 63–76.

[7] ———, Some classes of positive solutions of $\Delta u=Pu$ on Riemann surfaces, I. Kōdai Math. Sem. Rep. **6** (1954), 121–126.

[8] ———, A set of capacity zero and the equation $\Delta u=Pu$. Kōdai Math. Sem. Rep. **12** (1960), 76–81.

[9] ROYDEN, H. L., The equation $\Delta u=Pu$, and the classification of open Riemann surfaces. Ann. Acad. Sci. Fenn. AI **271** (1959), 1–26.

[10] SARIO, L., A Linear operator method on arbitrary Riemann surfaces. Trans. Amer. Math. Soc. **72** (1952), 281–295.

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