DEPENDENCE PROPERTIES OF SOLUTIONS ON THE RETARDATION AND INITIAL VALUES IN THE THEORY OF DIFFERENCE-DIFFERENTIAL EQUATIONS

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Introduction.

In [1], Bellman and Cooke have discussed the behavior of solutions of a particular type of the equation

$$x'(t)=f(t, x(t), x(t-h)) \tag{0.1}$$

as the retardation h tends to zero, and stated that the same method they used can be applied to demonstrate the corresponding result for more general differential-difference equations. The author [6] has discussed the same problems as above for general equations (0.1), in which $f(t, x, y)$ is a continuous function defined in a bounded and closed domain and satisfies Lipschitz condition, and he obtained some results as direct consequences of the dependence properties of solutions on the retardation h, as well as the behavior of solutions as h tends to zero.

The purpose of this paper is to discuss the problems of dependence properties of solutions of (0.1) on retarded arguments and initial values for the case where t varies in the infinite interval.

§ 1. Existence of solutions.

In order to consider the problems stated above, it is useful to establish the existence ana uniqueness of solutions of (0.1), which are defined for $-\infty<t<\infty$. Hence, we first prove an existence theorem. The uniqueness, however, will be proved in § 2 by means of a result concerning the continuity property with respect to retarded arguments and initial values.

On the other hand, Doss and Nasr [3] have discussed the problem similar to the above one for the equation (0.1), in which $h<0$ and $f(t, x, y)$ is continuous in the region $t_0\leqq t<\infty$, $|x|<\infty$, $|y|<\infty$.

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Now, we shall prove the following

THEOREM 1. *In the equation*

$$x'(t)=f(t, x(t), x(t-h)), \tag{1.1}$$

it is supposed that the following conditions are satisfied:

(i) *$f(t, x, y)$ is continuous in the region $I\times D\times D$, where D is a domain in R^n and $I=(-\infty, \infty)$;*

(ii) *$f(t, x, y)$ satisfies Lipschitz condition such that*

$$|f(t, x_1, y_1)-f(t, x_2, y_2)|\leqq k(t)(|x_1-x_2|+|y_1-y_2|), \tag{1.2}$$

where $k(t)$ is continuous in I and

$$\int_{-\infty}^{\infty} k(t)dt=A<\frac{1}{2}; \tag{1.3}$$

(iii) *for a given x_0 in D,*

$$\int_{-\infty}^{\infty} |f(t, x_0, x_0)|dt=B<\infty. \tag{1.4}$$

Then, we obtain the following results:

(a) *the existence and uniqueness of solutions under the initial condition $x(0)=x_0$ are guranteed in the interval I, provided that every point such that $|x-x_0|\leqq 2AB/(1-2A)$ lies in D;*

(b) *the solution is a continuous function of the initial value;*

(c) *the limits* $\lim_{t\to\pm\infty} x(t)$ *exist, and they are one-to-one corresponding to the solution;*

(d) *if the integral in* (ii) *is not convergent, the uniqueness in* (a) *is no longer true.*

In this theorem, it is not necessary for h to be positive. For the sake of simplicity, however, the proof will be proceeded for the case $h>0$. The slight modification can be applied to prove the corresponding results for the case $h<0$.

Proof. (a) We define a sequence $\{x_n(t)\}_{n=0}^{\infty}$ in the interval I as follows:

$$\begin{aligned} &x_0(t)=x_0, \\ &x_{n+1}(t)=x_0+\int_0^t f(s, x_n(s), x_n(s-h))ds \qquad (n=0, 1, 2, \cdots). \end{aligned} \tag{1.5}$$

Now, we consider two cases:

I. The case $0 \leqq t < \infty$. It follows from (1.5) that

$$
\begin{aligned}
|x_{n+1}(t)-x_n(t)| &\leqq \int_0^t k(s)(|x_n(s)-x_{n-1}(s)|+|x_n(s-h)-x_{n-1}(s-h)|)ds \\
&\leqq \int_{-h}^t \lambda(s)|x_n(s)-x_{n-1}(s)|ds,
\end{aligned} \tag{1.6}
$$

where $\lambda(s)=k(s)+k(s+h)$.

Especially, for $n=0$, we obtain from (1.3) and (1.5) that

$$
|x_1(t)-x_0| \leqq \int_0^t |f(s, x_0, x_0)|ds \leqq \int_{-\infty}^{\infty} |f(s, x_0, x_0)|ds = B. \tag{1.7}
$$

Then, from (1.6) together with (1.3) and (1.7), it follows that

$$
|x_{n+1}(t)-x_n(t)| \leqq B(2A)^n \qquad (n=0, 1, 2, \cdots), \tag{1.8}
$$

which implies the uniform convergence of the sequence $\{x_n(t)\}_{n=0}^{\infty}$ in the interval $0 \leqq t < \infty$.

It is noted that the upper bound $2AB/(1-2A)$ of the sequence (1.5) is not dependent on the retarded argument h.

II. The case $-\infty < t \leqq 0$. It follows from (1.5) that

$$
\begin{aligned}
|x_{n+1}(t)-x_n(t)| &\leqq \int_t^0 k(s)(|x_n(s)-x_{n-1}(s)|+|x_n(s-h)-x_{n-1}(s-h)|)ds \\
&\leqq \int_{t-h}^0 \lambda(s)|x_n(s)-x_{n-1}(s)|ds.
\end{aligned}
$$

On account of the same reason as in the case I, we obtain just the same estimation as (1.8), which leads us to the uniform convergence of $\{x_n(t)\}_{n=0}^{\infty}$ in the interval $-\infty < t \leqq 0$.

Putting $x(t)=\lim_{n\to\infty} x_n(t)$ $(-\infty < t < \infty)$, it follows by the uniform convergence that $x(t)$ is a continuous solution of an integral equation

$$
x(t)=x_0+\int_0^t f(s, x(s), x(s-h))ds, \tag{1.9}
$$

provided that the condition in (a) is fulfilled. It is apparent from (1.9) that $x(t)$ satisfies the equation (1.1) in the interval I and $x(0)=x_0$.

The proof of the uniqueness of solutions and continuity dependence on initial conditions will be found in § 2.

(c) In order to prove (c), it is sufficient to establish the absolute integrability of $f(t, x(t), x(t-h))$ over the interval I. It follows from (ii) and (iii) that

$$\int_{-\infty}^{\infty} |f(s, x(s), x(s-h))| ds$$

$$\leqq \int_{-\infty}^{\infty} |f(s, x(s), x(s-h)) - f(s, x_0, x_0)| ds + \int_{-\infty}^{\infty} |f(s, x_0, x_0)| ds$$

$$\leqq \int_{-\infty}^{\infty} k(s)(|x(s)-x_0| + |x(s-h)-x_0|) ds + \int_{-\infty}^{\infty} |f(s, x_0, x_0)| ds$$

$$\leqq \frac{4A^2B}{1-2A} + B.$$

Hence, the limits $\lim_{t\to\pm\infty} x(t)$ exist and the uniqueness of solutions implies the one-to-one correspondence of the limits to the solution.

(d) To prove (d), it is sufficient to consider an equation

$$x'(t) = \frac{e^t}{\exp((1-e^{-1})e^t)-1}(x(t-1)-x(t)) \tag{1.10}$$

under the condition $x(0)=e^{-1}$.

It is easily observed that $x=\exp(-e^t)$ is a bounded solution of (1.10) with $x(0)=e^{-1}$ in the interval I. It is evident that for any constants α and β, $\alpha \exp(-e^t)+\beta$ satisfies the equation (1.10). From the equation $\alpha e^{-1}+\beta=e^{-1}$, however, we can find an infinite number of solutions with the same initial value e^{-1} at $t=0$.

Furthermore, by means of a simple calculation, we have

$$\int_{-\infty}^{\infty} \frac{e^t}{\exp((1-e^{-1})e^t)-1} dt = +\infty.$$

§ 2. Dependence properties of solutions on initial values and retarded arguments.

In order to study how the solution of (1.1) depends upon the retarded arguments h under the analogous assumptions to those in Theorem 1, we suppose that h varies on the interval $[a, b]$. For the sake of simplicity, it is supposed that a is non-negative. Instead of the assumptions in Theorem 1, we suppose that the following conditions are satisfied:

(i) *$f(t, x, y)$ is a continuous function defined in $I \times D \times D$;*

(ii) *$f(t, x, y)$ satisfies Lipschitz condition such that*

$$|f(t, x_1, y_1)-f(t, x_2, y_2)| \leqq k(t)(|x_1-x_2|+|y_1-y_2|),$$

where $k(t)$ is continuous in I, $(t, x, y) \in I \times D \times D$, and

$$\int_{-\infty}^{\infty} k(t)dt = A < \frac{1}{2};$$

(iii) *for any constant α in D_0,*

$$\int_{-\infty}^{\infty} |f(t, \alpha, \alpha)|dt < B < \infty,$$

where B is an absolute constant and D_0 is a subdomain in D which is chosen so as to satisfy that every x satisfying the inequality $|x-\alpha| \leqq 2AB/(1-2A)$ is contained in D;

(iv) *for sufficiently large T such that $b<T$, there exists a constant K such that $|f(t, x, y)| \leqq K$ in $I_0 \times D \times D$, where I_0 represents the interval $[-T, T]$.*

Then, as proved in § 1, the existence of solutions of

$$x'(t)=f(t, x(t), x(t-h_i)), \qquad a \leqq h_i \leqq b \tag{2.1}$$

under the condition $x(0)=x_i$, where $x_i \in D_0$, is guaranteed in the interval I, and (2.1) is equivalent to an integral equation

$$x(t)=x_i+\int_0^t f(s, x(s), x(s-h_i))ds \tag{2.2}$$

in the interval I.

Denoting by $x(t, x_i, h_i)$ the solution of (2.2), we obtain from (2.2) that

$$\begin{aligned}
&|x(t, x_1, h_1)-x(t, x_2, h_2)| \\
&\leqq |x_1-x_2| + \left| \int_0^t |f(s, x(s, x_1, h_1), x(s-h_1, x_1, h_1)) \right. \\
&\qquad\qquad \left. -f(s, x(s, x_2, h_2), x(s-h_1, x_2, h_2))|ds \right| \\
&\qquad + \left| \int_0^t |f(s, x(s, x_2, h_2), x(s-h_1, x_2, h_2)) \right. \\
&\qquad\qquad \left. -f(s, x(s, x_2, h_2), x(s-h_2, x_2, h_2))|ds \right| \\
&\leqq |x_1-x_2| + \left| \int_0^t k(s)(|x(s, x_1, h_1)-x(s, x_2, h_2)| \right. \\
&\qquad\qquad \left. +|x(s-h_1, x_1, h_1)-x(s-h_1, x_2, h_2)|)ds \right|
\end{aligned} \tag{2.3}$$

$$+\left|\int_0^t k(s)|x(s-h_1, x_2, h_2)-x(s-h_2, x_2, h_2)|ds\right|.$$

On account of the assumptions mentioned before, it follows from (2.2) that

$$\begin{aligned}&|x(s-h_1, x_2, h_2)-x(s-h_2, x_2, h_2)|\\ &\leqq\int_{s-h_2}^{s-h_1}|f(u, x(u, x_2, h_2), x(u-h_2, x_2, h_2))|du\\ &\leqq K(h_2-h_1),\end{aligned}$$

provided that $-T\leqq s-h_2\leqq u\leqq s-h_1\leqq T$. Then, if we consider the case $0\leqq t<\infty$, we obtain from (2.3) that

$$\begin{aligned}&|x(t, x_1, h_1)-x(t, x_2, h_2)|\\ &\leqq|x_1-x_2|+KA(h_2-h_1)+\int_{-h_1}^t \lambda(s)|x(s, x_1, h_1)-x(s, x_2, h_2)|ds,\end{aligned}$$

where $\lambda(s)=k(s)+k(s+h_1)$, which yields an estimation

$$|x(t, x_1, h_1)-x(t, x_2, h_2)|\leqq\frac{1}{1-2A}(|x_1-x_2|+KA(h_2-h_1)). \tag{2.4}$$

For the case $-\infty<t\leqq 0$, we obtain just the same inequality as (2.4) by means of a slight modification of the above method.

If $h_1=h_2$ in the inequality (2.4), it implies the equicontinuity of solutions with respect to initial values. Furthermore, if $x_1=x_2$ and $h_1=h_2$, we can establish the uniqueness of solutions of (2.1) from (2.4), which implies the proof of the uniqueness in (a) of Theorem 1.

On the other hand, if $x_1=x_2=x_0$, any solution of (2.1) is an equicontinuous function of the retarded argument h. Furthermore, since any solution is bounded, it follows by a well known theorem that any sequence of solutions $\{x(t, x_0, h_n)\}_{n=0}^{\infty}$ such that $h_n\to 0$ $(n\to\infty)$ contains a subsequence which is uniformly convergent as $n\to\infty$. The limiting function $x(t, x_0)$ will be expected to be a solution of the differential equation

$$x'(t)=f(t, x(t), x(t)), \qquad x(0)=x_0. \tag{2.5}$$

To this end, consider the inequality (2.4) which corresponds to the case where $x_1=x_2=x_0$, and $h_2=h$, $h_1=0$. Then, by virtue of the same reason as before, we obtain an inequality

$$|x(t, x_0, h)-x(t, x_0)|\leqq\frac{Kh}{1-2A},$$

which implies the uniform convergence of $x(t, x_0, h)$ to $x(t, x_0)$ as $h\to 0$. Here, it is noted that the interval of the uniform convergence is $-T\leqq t\leqq T$ for any large T. Thus, we obtain the following

THEOREM 2. *Under the same assumptions* (i), (ii), (iii), (iv), *we obtain the following results*:

(i) *for a fixed retardation, any solution of* (2.1) *is an equicontinuous function of the initial value uniformly in t*;

(ii) *for a fixed initial value, any solution of* (2.1) *is an equicontinuous function of h in the interval $[a, b]$ uniformly in t, if t belongs to an interval I_0*;

(iii) *as the retardation approaches zero, the solution of* (2.1) *with a fixed initial value tends to the solution of* (2.5) *uniformly in $t\in I_0$.*

§ 3. ε-approximate solutions.

Let $u_i(t)$ $(i=1, 2)$ be functions which are continuous in $0\leqq t<\infty$, differentiable in $0<t<\infty$ and satisfy the inequalities

$$|u_i'(t)-f(t, u_i(t), u_i(t-h_i))|\leqq\varepsilon_i \qquad (i=1, 2) \tag{3.1}$$

for given constants ε_i $(i=1, 2)$, where it is supposed that for a given function $\varphi_i(t)$ the initial conditions

$$u_i(t-h_i)=\varphi_i(t) \quad (0\leqq t<h_i,\ i=1, 2) \tag{3.2}$$

are fulfilled. Then, we call $u_i(t)$ $(i=1, 2)$ the *ε_i-approximate solutions* with respect to the difference-differential equation

$$x'(t)=f(t, x(t), x(t-h_i)) \qquad (i=1, 2), \tag{3.3}$$

respectively, where $x(t-h_i)=\varphi_i(t)$ $(0\leqq t<h_i)$.

On the function $f(t, x, y)$ and others, we impose the following conditions which are more general than those in the preceding sections:

(i) *$f(t, x, y)$ is continuous in the region $I^+\times D\times D$, where I^+ represents the interval $0\leqq t<\infty$, and $\sup|f|\leqq K$ in $I_0^+\times D\times D$, where I_0^+ the interval $0\leqq t\leqq T$ for any large T*;

(ii) *$f(t, x, y)$ satisfies the condition such that*

$$|f(t, x_1, y_1)-f(t, x_2, y_2)|\leqq k(t)(M(|x_1-x_2|)+M(|y_1-y_2|)),$$

where $k(t)$ is continuous on I^+ and $M(r)$ is piecewise continuous, non-negative, non-decreasing for $r\geqq 0$, and $M(0)=0$ if and only if $r=0$;

(iii) $\qquad u_i(t-h_i)=\varphi_i(t)\ (0\leqq t<h_i,\ i=1, 2),$

where $\varphi_i(t)$ $(i=1, 2)$ are given continuous functions and $\lim_{t\to h_i-0}\varphi_i(t)$ $(i=1, 2)$ exist;

(iv) $$0<h_1\leqq h_2.$$

Now, we consider three cases.

I. The case $0\leqq t\leqq h_1$. Then, we obtain from (3.1) that

$$\begin{aligned}
&|u_1(t)-u_2(t)|\\
(3.4)\quad &\leqq|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)t+\int_0^t|f(s, u_1(s), u_1(s-h))-f(s, u_2(s), u_2(s-h))|ds\\
&\leqq|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)h_1+\int_0^{h_1}k(s)M(|\varphi_1(s)-\varphi_2(s)|)ds+\int_0^t k(s)M(|u_1(s)-u_2(s)|)ds.
\end{aligned}$$

Then, by means of Bihari-Langenhop's inequality (cf. [2], [4]), it follows from (3.4) that

$$\begin{aligned}
&|u_1(t)-u_2(t)|\\
(3.5)\quad &\leqq G^{-1}\left(G\left(|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)h_1+\int_0^{h_1}k(s)M(|\varphi_1(s)-\varphi_2(s)|)ds\right)+\int_0^t k(s)ds\right),
\end{aligned}$$

where $G^{-1}(r)$ is defined as the inverse function of

$$(3.6)\qquad G(r)=\int_{r_0}^r\frac{d\rho}{M(\rho)}\qquad(0<r_0\leqq r),$$

and the constant in the bracket of G in (3.5) is supposed to be positive.

II. The case $h_1\leqq t\leqq h_2$. Then, it follows that

$$\begin{aligned}
&|u_1(t)-u_2(t)|\\
&\leqq|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)h_2+2K(h_2-h_1)\\
&\quad+\int_0^{h_1}k(s)M(|\varphi_1(s)-\varphi_2(s)|)ds+\int_0^t\lambda(s)M(|u_1(s)-u_2(s)|)ds,
\end{aligned}$$

where $\lambda(s)=k(s)+k(s+h_1)$. Then, by means of the same inequality as above, we obtain

$$\begin{aligned}
&|u_1(t)-u_2(t)|\leqq G^{-1}\Big(G\Big(|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)h_2+2K(h_2-h_1)\\
(3.7)\quad &\qquad+\int_0^{h_1}k(s)M(|\varphi_1(s)-\varphi_2(s)|)ds\Big)+\int_0^t\lambda(s)ds\Big).
\end{aligned}$$

III. The case $h_2\leqq t\leqq T$. Then, it follows that

$$\begin{aligned}
&|u_1(t)-u_2(t)|\\
&\leqq|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)T+2K(h_2-h_1)+AM((2K+\varepsilon_2)(h_2-h_1))\\
&\quad+\int_0^{h_1}k(s)M(|\varphi_1(s)-\varphi_2(s)|)ds+\int_0^t\lambda(s)M(|u_1(s)-u_2(s)|)ds,
\end{aligned}$$

which implies the inequality

$$|u_1(t)-u_2(t)|$$

$$\leqq G^{-1}\Big(G\Big(|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)T+2K(h_2-h_1)+AM((2K+\varepsilon_2)(h_2-h_1)) \tag{3.8}$$

$$+\int_0^{h_1} k(s)M(|\varphi_1(s)-\varphi_2(s)|)ds\Big)+\int_0^t \lambda(s)ds\Big).$$

Thus, we obtain the following

THEOREM 3. *Under the conditions* (i), (ii), (iii), (iv), *we obtain the estimations* (3.5), (3.7), (3.8) *for* $|u_1(t)-u_2(t)|$, *where* $u_i(t)$ $(i=1,2)$ *are* ε_i*-approximate solutions defined by* (3.1), *and* $G^{-1}(r)$ *is defined as the inverse function of* (3.6).

If we consider the case $M(r)=r$, that is, if $f(t,x,y)$ satisfies Lipschitz condition, it follows from (3.5) that

$$G(r)=\log\frac{r}{r_0} \quad \text{and} \quad G^{-1}(r)=r_0\exp r.$$

Hence, we obtain the following

COROLLARY. *Under the conditions* (i), (iii), (iv) *in Theorem* 3, *and the condition*

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leqq k(t)(|x_1-x_2|+|y_1-y_2|), \tag{ii$'$}$$

where $k(t)$ *is a continuous function.*

Then, we have the following estimations:

I. *The case* $0\leqq t\leqq h_1$.

$$|u_1(t)-u_2(t)|$$

$$\leqq\exp\Big(\int_0^t k(s)ds\Big)\Big(|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)h_1+\int_0^{h_1} k(s)|\varphi_1(s)-\varphi_2(s)|ds\Big).$$

II. *The case* $h_1\leqq t\leqq h_2$.

$$|u_1(t)-u_2(t)|$$

$$\leqq\exp\Big(\int_0^t \lambda(s)ds\Big)\Big(|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)h_2+2K(h_2-h_1)$$

$$+\int_0^{h_1} k(s)|\varphi_1(s)-\varphi_2(s)|ds\Big),$$

where $\lambda(s)=k(s)+k(s+h_1)$.

III. *The case* $h_2 \leqq t \leqq T$.

$$|u_1(t)-u_2(t)|$$

$$\leqq \exp\left(\int_0^t \lambda(s)ds\right)\Big(|u_1(0)-u_2(0)|+(\varepsilon_1+\varepsilon_2)T+2K(A+1)+\varepsilon_2(h_2-h_1)$$

$$+\int_0^{h_1} k(s)|\varphi_1(s)-\varphi_2(s)|ds\Big).$$

§ 4. Osgood conditions.

The author [5] has obtained the general theorems concerning the uniqueness problems of difference-differential equations. In this section, as an application of Theorem 3, we shall establish a uniqueness theorem very similar to Osgood conditions in the theory of differential equations. Furthermore, as an application of a fixed point theorem, a result which asserts the existence of solutions will be proved.

We first prove the following

THEOREM 4.[1] *In the equation*

$$x'(t)=f(t, x(t), x(t-h)), \quad h>0, \tag{4.1}$$

we suppose that the following conditions are satisfied:

(i) $f(t, x, y)$ *is continuous in* $0 \leqq t<\infty$, $|x|<\infty$, $|y|<\infty$;

(ii) $f(t, x, y)$ *satisfies the condition such that*

$$|f(t, x, y)| \leqq k(t)(M(|x|)+M(|y|)), \tag{4.2}$$

where $k(t)$ *is continuous in* $0 \leqq t<\infty$, $M(r)$ *is defined as in Theorem* 3.

Then, there exists a solution of (4.1) *under the initial conditions* $x(t)=0$ $(-h \leqq t<0)$ *and* $x(0)=x_0$ *in the interval* $0 \leqq t<\infty$, *provided that* $|x_0| \leqq r_0$, *where* r_0 *is a given constant.*

Proof. For a given $r_0>0$, we consider an equation

$$\int_{r_0}^{r} \frac{d\rho}{M(\rho)}=\int_0^t \lambda(s)ds \qquad (0<r_0 \leqq r,\ 0 \leqq t<\infty), \tag{4.3}$$

where $\lambda(t)=k(t)+k(t+h)$. Since both of the right and left hand sides of (4.3) are monotone increasing, r is uniquely determined as a function of t in the interval $0 \leqq t<\infty$, and $r(0)=r_0$. Furthermore, (4.3) is equivalent to a differential equation

$$r'=\lambda(t)M(r)$$

under the initial condition $r(0)=r_0$.

1) The result was simply stated in [7].

Let A be a family of functions $x(t)$ which are continuous and $|x(t)|\leqq r(t)$ in $0\leqq t<\infty$, and $x(t)=0$ $(-h\leqq t<0)$. Then, we define a transformation T such that

$$Tx(t)=x_0+\int_0^t f(s, x(s), x(s-h))ds \tag{4.4}$$

for any $x(t)$ in A. Then, it follows from the hypotheses that

$$\begin{aligned}|Tx(t)| &\leqq |x_0|+\int_0^t k(s)(M(|x(s)|)+M(|x(s-h)|))ds \\ &\leqq |x_0|+\int_0^t \lambda(s)M(|x(s)|)ds \\ &\leqq r_0+\int_0^t \lambda(s)M(r(s))ds=r(t)\end{aligned}$$

for any $x(t)$ in A, which implies $TA\subset A$. Thus, by using Tychonov's fixed point theorem in the topology suitably chosen, it follows that there exists a fixed point in A, which corresponds to a solution of (3.1) under the conditions $x(0)=x_0$ and $x(t)=0$ $(-h\leqq t<0)$, if $|x_0|\leqq r_0$.

This completes the proof of Theorem 4.

COROLLARY. *Under the same assumptions as in Theorem* 4, *if*

$$\int_0^\infty k(t)dt$$

is convergent, but

$$\int_{r_0}^r \frac{d\rho}{M(\rho)}$$

diverges as $r\to+\infty$, *then there exists a bounded solution of* (4.1) *under the conditions* $x(t)=0$ $(-h\leqq t<0)$ *and* $x(0)=x_0$, *provided that* $|x_0|\leqq r_0$.

Proof. From the equation (4.3) and the above assumptions, r is determined as a bounded function of t in the interval $0\leqq t<\infty$. Then, proceeding the same method as in the proof of Theorem 4, it is observed that A defined before is a family of bounded functions in $0\leqq t<\infty$. Hence, a fixed point in A, that is, a solution of (4.1) is also bounded in $0\leqq t<\infty$.

In order to apply Theorem 3 to the uniqueness problems, we use the notation $G(r, r_0)$ instead of $G(r)$ and impose on it the assumption $\lim_{r_0\to+0} G(r, r_0)=+\infty$ for fixed r. Then, we obtain $\lim_{r_0\to+0} G^{-1}(r, r_0)=0$. In fact, if the result does not hold, there exists a positive constant ε such that $G^{-1}(r, r_0)\geqq\varepsilon$ for $0<r_0<\eta$, where η is any sufficiently small number. By the monotonicity of the function $G(r, r_0)$ with respect to r, it follows that

$$r_1 = G(G^{-1}(r_1, r_0), r_0) \geqq G(\varepsilon, r_0) \tag{4.5}$$

for any fixed $r_1(>r_0)$. On the other hand, it follows from the hypothesis that $\lim_{r_0\to+0} G(\varepsilon, r_0)=+\infty$, which contradicts the inequality (4.5), since r_1 is fixed. It is noted that such a condition as above corresponds to Osgood condition concerning the uniqueness of solutions in the theory of differential equations. Thus, we obtain the following

THEOREM 5. *In Theorem* 3, *we suppose an additional condition such that* $\lim_{r_0\to+0} G(r)=+\infty$. *Then, we have the following results*:

(a) $u_1(0)=u_2(0)$, $\varphi_1(t)\equiv\varphi_2(t)$, *and* $h_1=h_2$. *Then, the uniqueness of solutions is obtained*;

(b) $\varphi_1(t)\equiv\varphi_2(t)$, $h_1=h_2$. *Then, the equicontinuity of solutions with respect to initial values is obtained*;

(c) $\varphi_1(t)\equiv\varphi_2(t)$, $u_1(0)=u_2(0)$. *Then, the equicontinuity of solutions with respect to retarded arguments is obtained.*

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