ON A UNIQUENESS CONDITION FOR SOLUTIONS OF THE DIRICHLET PROBLEM CONCERNING A QUASI-LINEAR EQUATION OF ELLIPTIC TYPE

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§1. Introduction.

In the present paper, we are concerned with a uniqueness condition for solutions of the Dirichlet problem concerning a quasi-linear elliptic equation of the second order

$$\sum_{i,j=1}^{m} a_{ij}(x, \nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \nabla u).^{1)} \tag{1.1}$$

Recently, Kusano [1][2] has established the maximum principle for quasi-linear elliptic equations of the general form, and as its application, he has given a uniqueness condition for solutions of the equation (1.1). We will here show that the uniqueness of solutions may be established under a weaker condition, by the method adopted in author's previous note [2].

In this paper, x denotes a point $(x_1, x_2, \cdots, x_m)$ in the m-dimensional Euclidean space, and we use the notations $\partial_i u$ for $\partial u/\partial x_i$, and $\partial_i \partial_j u$ for $\partial^2 u/\partial x_i \partial x_j$. Furthermore we introduce a differential operator $L[v; u]$ of elliptic type by the expression

$$L[v; u] = \sum_{i,j=1}^{m} a_{ij}(x, \nabla v)\partial_i \partial_j u,$$

and then the equation (1.1) can be written as follows:

$$L[u; u] = f(x, u, \nabla u). \tag{1.2}$$

By a solution of the equation (1.2) in a domain D, we mean a function belonging to $C^2[D]$ and satisfying the equation (1.2) in D.

§2. Hypotheses on the functions $a_{ij}(x, p)$ and $f(x, u, p)$.

Let D be a bounded domain in the m-dimensional Euclidean space and let $\dot{D}$ be the boundary of D.

We define a domain $\mathfrak{D}_0$ in the $2m$-dimensional Euclidean space as follows:

$$\mathfrak{D}_0 = \{(x, p);\ x \in D,\ |p| < +\infty\},^{3)}$$

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1) ∇u denotes the vector $(\partial u/\partial x_1, \partial u/\partial x_2, \cdots, \partial u/\partial x_m)$.

2) The numbers in brackets refer to the list of references at the end of this paper.

3) The notation $|p|$ means $\{\sum_{i=1}^{m} |p_i|^2\}^{1/2}$, where $p = (p_1, \cdots, p_m)$.

and regarding the coefficients $a_{ij}(x, p)$, we assume the following hypotheses.

HYPOTHESIS 1: The functions $a_{ij}(x, p)$ are continuous in the domain $\mathfrak{D}_0$ and the quadratic form $\sum_{i,j=1}^{m} a_{ij}(x, p)\xi_i\xi_j$ is positive definite for any $(x, p) \in \mathfrak{D}_0$.

HYPOTHESIS 2: The functions $a_{ij}(x, p)$ satisfy a Lipschitz condition

$$\left\{\sum_{i,j=1}^{m}[a_{ij}(x, p) - a_{ij}(x, q)]^2\right\}^{1/2} \leqq H(x, p, q)\,|\,p - q\,|$$

for any (x, p), $(x, q) \in \mathfrak{D}_0$, and $H(x, p, q)$ is a positive function of (x, p, q), bounded in any domain

$$\{(x, p, q);\ x \in S,\ |\,p\,|,\ |\,q\,| \leqq M\},$$

where S is any compact subdomain of D, and M is any positive quantity.

REMARK. In the subsequent section, as a matter of fact, the following hypothesis will suffice for our discussion.

HYPOTHESIS 2′: For at least an index k, the functions $a_{ij}(x, p)$ satisfy a Lipschitz condition

$$\left\{\sum_{i,j=1}^{m}[a_{ij}(x, p) - a_{ij}(x, \bar{p})]^2\right\}^{1/2} \leqq H(x, p, \bar{p})\,|\,p_k - \bar{p}_k\,|,$$

where $p = (p_1, \cdots, p_k, \cdots, p_m)$, $\bar{p} = (p_1, \cdots, \bar{p}_k, \cdots, p_m)$.

We now define a domain $\mathfrak{D}_1$ in the $(2m+1)$-dimensional Euclidean space as follows:

$$\mathfrak{D}_1 = \{(x, u, p);\ x \in D,\ |\,u\,| < +\infty,\ |\,p\,| < +\infty\},$$

and regarding the function $f(x, u, p)$, we assume the following hypotheses.

HYPOTHESIS 3: The function $f(x, u, p)$ is defined in the domain $\mathfrak{D}_1$ and is non-decreasing with respect to u.

HYPOTHESIS 4: The function $f(x, u, p)$ satisfies one of the following conditions:

$$\text{(I)}\quad -G(x, u, p, \bar{p}_k)(\bar{p}_k - p_k) \leqq f(x, u, \bar{p}) - f(x, u, p),$$

$$\text{(II)}\quad f(x, u, \bar{p}) - f(x, u, p) \leqq G(x, u, p, \bar{p}_k)(\bar{p}_k - p_k),$$

where $p = (p_1, \cdots, p_k, \cdots, p_m)$, $\bar{p} = (p_1, \cdots, \bar{p}_k, \cdots, p_m)$, $p_k < \bar{p}_k$, and k is a fixed index.

Furthermore $G(x, u, p, \bar{p}_k)$ is a function of $(x, u, p, \bar{p}_k)$, bounded in any domain

$$\{(x, u, p, \bar{p}_k),\ x \in S,\ |\,u\,| \leqq M,\ |\,p\,| \leqq M,\ |\,\bar{p}_k\,| \leqq M\},$$

where S and M have the same meaning as in Hypothesis 2.

§3. Main theorem.

LEMMA. *If* $\varphi(x)$, $\psi(x) \in C^2$, *then we have*

$$L[v; \varphi\psi] = \varphi L[v; \psi] + \psi L[v; \varphi] + 2 \sum_{i,j=1}^{m} a_{ij}(x, \nabla v)\partial_i \varphi \partial_j \psi.$$

The proof is omitted.

The principal part of this paper is to prove the following

THEOREM 1. *Let Hypotheses* 1–4 *be fulfilled, and let* $u_1(x)$ *and* $u_2(x)$ *be solutions of the equation*

$$L[u; u] = f(x, u, \nabla u). \tag{3.1}$$

If the inequality

$$\overline{\lim_{x \to \dot{x}}} |u_1(x) - u_2(x)| \leqq \varepsilon \tag{3.2}$$

holds for any boundary point $\dot{x} \in \dot{D}$ *and for a non-negative real number* ε, *then we have*

$$|u_1(x) - u_2(x)| \leqq \varepsilon \quad \text{in } D. \tag{3.3}$$

Proof. We will first prove the inequality

$$u_2(x) - u_1(u) \leqq \varepsilon \quad \text{in } D, \tag{3.3'}$$

by assuming Hypothesis (4, I), and to this end, we show that, a contradiction arises, if the inequality (3.3′) does not hold.

Suppose that the inequality (3.3′) is not true. Then, since there exists a point $\bar{x} \in D$, such that $\varepsilon < u_2(\bar{x}) - u_1(\bar{x})$, we see

$$\operatorname*{Inf}_{D} \{\varepsilon - (u_2(x) - u_1(x))\} < 0,$$

and the inequality (3.2) implies that there exists a point $x^{(0)} \in D$, such that

$$\begin{aligned} \operatorname*{Inf}_{D} \{\varepsilon - (u_2(x) - u_1(x))\} &= \{\varepsilon - (u_2(x^{(0)}) - u_1(x^{(0)}))\} \\ &\equiv -\delta. \end{aligned}$$

The function $\{\varepsilon - (u_2(x) - u_1(x))\}$ assumes therefore the negative minimum $-\delta$ in D, which is attained at the point $x^{(0)} \in D$.

Put

$$G(x) \equiv G(x, u_2(x), \nabla u_1(x), \partial_k u_2(x)),$$
$$H(x) \equiv H(x, \nabla u_1(x), \nabla u_2(x)),$$

and let $\{D_n\}$ be a sequence of domains, such that $\bar{D}_n \subset D_{n+1}$ and $\cup_{n=1}^{\infty} D_n = D$, then we can choose four sequences $\{G_n\}$, $\{H_n\}$, $\{U_n\}$ and $\{\alpha_n\}$ of positive numbers, such that

$$\operatorname*{Sup}_{\bar{D}_n} G(x) < G_n, \qquad \operatorname*{Sup}_{\bar{D}_n} H(x) < H_n,$$

$$\operatorname*{Sup}_{\bar{D}_n} \left\{ \sum_{i,j=1}^{m} |\partial_i \partial_j u_1(x)|^2 \right\}^{1/2} < U_n, \qquad \operatorname*{Inf}_{\bar{D}_n} a_{kk}(x, \nabla u_2(x)) > \alpha_n,$$

and the sequence $\{(G_n+H_nU_n)/\alpha_n\}$ tends monotonely to infinity for $n\to\infty$.

The existence of such sequences $\{G_n\}$, $\{H_n\}$, $\{U_n\}$, $\{\alpha_n\}$ can be verified by virtue of the hypotheses of the theorem.

Define functions $\varphi_n(x)$ and $v_n(x)$ as follows:

$$\varphi_n(x)=P-\exp\left[-\frac{1}{\alpha_n}(G_n+H_nU_n)(x_k-\beta)\right],$$
$$v_n(x)=\frac{1}{\varphi_n(x)}\{\varepsilon-(u_2(x)-u_1(x))\},$$

where x_k is the k-component of $x\in\bar{D}$, and β is a real number such that $x_k-\beta$ is positive for any $x\in\bar{D}$, and P is a real number greater than unity.

Then we have

$$P>\varphi_n(x)>0 \qquad \text{in} \quad D$$

and

$$\varliminf_{x\to\dot{x}} v_n(x)\geqq 0$$

for any boundary point $\dot{x}\in\dot{D}$. Therefore each of functions $v_n(x)$ may assume negative values smaller than $-\delta/P$ in D.

On the other hand, the inequality (3.2) implies that, for any boundary point $\dot{x}\in\dot{D}$, there exists an open neighborhood $V(\dot{x})$ of the point $\dot{x}$, such that

$$\varepsilon-(u_2(x)-u_1(x))\geqq-\frac{\delta}{2} \qquad \text{in} \quad V(\dot{x})\frown D,$$

and then we obtain

$$\varepsilon-(u_2(x)-u_1(x))\geqq-\frac{\delta}{2} \qquad \text{in} \quad V\frown D,$$

where

$$V=\bigcup_{\dot{x}\in\dot{D}}V(\dot{x}).$$

Now, it follows from the definitions of the functions $\varphi_n(x)$ and the sequence $\{D_n\}$, that there exists a natural number n_0, such that

$$V\supset D-D_{n_0} \quad \text{and} \quad \varphi_{n_0}(x)>\frac{P}{2},$$

hence we see

$$v_{n_0}(x)\geqq-\frac{\delta}{2}\frac{2}{P}=-\frac{\delta}{P} \qquad \text{in} \quad D-D_{n_0}.$$

The function $v_{n_0}(x)$ therefore assumes the negative minimum in D which is attained at a point $x^{(n_0)}$ belonging to the domain D_{n_0}.

After all, renewing the notations, we arrive at the following conclusion.

If we choose adequately a bounded domain D_0 ($\bar{D}_0\subset D$) and four positive numbers G, H, U and α, such that

$$\sup_{\bar{D}_0}G(x)<G, \qquad \sup_{\bar{D}_0}H(x)<H,$$

$$\sup_{\bar{D}_0}\left\{\sum_{i,j=1}^{m}|\partial_i\partial_j u|^2\right\}^{1/2}<U, \qquad \inf_{\bar{D}_0} a_{kk}(x, \nabla u_2(x))>\alpha,$$

and if we put

$$\varphi(x)=P-\exp\left[-\frac{1}{\alpha}(G+HU)(x_k-\beta)\right],$$

then the function

$$v(x)\equiv\frac{1}{\varphi(x)}\{\varepsilon-(u_2(x)-u_1(x))\}$$

assumes the negative minimnm in D, which is attained at a point ξ belonging to D_0.

Thus we have

$$v(\xi)<0, \qquad \nabla v(\xi)=0, \tag{3.4}$$

$$L[u_2(\xi);\, v(\xi)]\geqq 0. \tag{3.5}$$

On the other hand, since

$$v(x)\varphi(x)=\varepsilon-(u_2(x)-u_1(x)),$$

by virtue of Lemma, we obtain

$$L[u_2(x);\, v(x)]=\frac{1}{\varphi(x)}\Big\{-L[u_2(x);\, u_2(x)]+L[u_2(x);\, u_1(x)]$$
$$-2\sum_{i,j=1}^{m}a_{ij}(x, \nabla u_2(x))\partial_i\varphi(x)\partial_j v(x)-v(x)L[u_2(x);\, \varphi(x)]\Big\},$$

and hence

$$L[u_2(\xi);\, v(\xi)]=\frac{1}{\varphi(\xi)}\Big\{-f(\xi, u_2(\xi), \nabla u_2(\xi))+f(\xi, u_1(\xi), \nabla u_1(\xi))$$
$$+\sum_{i,j=1}^{m}\{a_{ij}(\xi, \nabla u_2(\xi))-a_{ij}(\xi, \nabla u_1(\xi))\}\partial_i\partial_j u_1(\xi)$$
$$-v(\xi)L[u_2(\xi);\, \varphi(\xi)]\Big\}.$$

Furthermore, since

$$u_1(\xi)=u_2(\xi)+v(\xi)\varphi(\xi)-\varepsilon<u_2(\xi), \qquad \nabla v(\xi)=0,$$

by virtue of the fact that the function $f(x, u, p)$ is non-decreasing with respect to u, we have

$$L[u_2(\xi);\, v(\xi)]$$
$$\leqq\frac{1}{\varphi(\xi)}\Big\{-f(\xi, u_2(\xi), \nabla u_2(\xi))+f(\xi, u_2(\xi), \nabla u_2(\xi)+v(\xi)\nabla\varphi(\xi))$$
$$+\left[\sum_{i,j=1}^{m}|a_{ij}(\xi, \nabla u_2(\xi))-a_{ij}(\xi, \nabla u_1(\xi))|^2\right]^{1/2}\left[\sum_{i,j=1}^{m}|\partial_i\partial_j u_1(\xi)|^2\right]^{1/2}$$
$$-v(\xi)L[u_2(\xi);\, \varphi(\xi)]\Big\},$$

and therefore it follows from Hypothesis (4, I) and the relation

$$\begin{aligned}\nabla u_1(\xi) &= \nabla u_2(\xi) + v(\xi)\nabla\varphi(\xi)\\ &= (\partial_1 u_2(\xi), \partial_2 u_2(\xi), \cdots, \partial_m u_2(\xi))\\ &\quad + (0, \cdots, 0, v(\xi)\partial_k\varphi(\xi), 0, \cdots, 0),\end{aligned}$$

that

$$\begin{aligned}&L[u_2(\xi); v(\xi)]\\ &\leqq \frac{1}{\varphi(\xi)}\Big\{G(\xi, u_2(\xi), \nabla u_1(\xi), \partial_k u_2(\xi))\partial_k\varphi(\xi)\\ &\qquad + H(\xi, \nabla u_1(\xi), \nabla u_2(\xi))\Big[\sum_{i,j=1}^{m} |\partial_i\partial_j u_1(\xi)|^2\Big]^{1/2}\partial_k\varphi(\xi)\\ &\qquad + L[u_2(\xi); \varphi(\xi)]\Big\}(-v(\xi)),\\ &\leqq \frac{1}{\varphi(\xi)}\Big\{\frac{1}{\alpha}(G+HU)^2 - a_{kk}(\xi, \nabla u_2(\xi))\Big[\frac{1}{\alpha}(G+HU)\Big]^2\Big\}\\ &\qquad \cdot(-v(\xi))\exp\Big[-\frac{1}{\alpha}(G+HU)(x_k-\beta)\Big] < 0.\end{aligned}$$

Thus we obtain

$$L[u_2(\xi); v(\xi)] < 0,$$

which contradicts the inequality (3.5).

Hence we have proved the inequality (3.3′), and similarly we can verify the validity of the inequality

$$-\varepsilon \leqq u_2(x) - u_1(x) \qquad \text{in} \quad D.$$

The theorem may be proved under Hypothesis (4, II), by letting β be a real number such that $x_k - \beta$ is negative for any $x = (x_1, \cdots, x_k, \cdots, x_m) \in \bar{D}$, and by using the functions

$$\varphi_n(x) = P - \exp\Big[\frac{1}{\alpha_n}(G_n + H_n U_n)(x_k - \beta)\Big]$$

and

$$\varphi(x) = P - \exp\Big[\frac{1}{\alpha}(G + HU)(x_k - \beta)\Big]$$

in stead of the functions adopted in the above process of the proof.

Remark. In the case of $\varepsilon = 0$, Theorem 1 gives a theorem of uniqueness for solutions of the Dirichlet problem concerning the equation (3.1).

§4. Harnack's first theorem.

As a corollary of Theorem 1, we have Harnack's first theorem for solutions of the equation (3.1).

Theorem 2. *Let Hypotheses* 1–4 *be fulfilled, and let* $\{u_n(x)\}$ *be a sequence of solutions of the equation* (3.1) *which are all continuous in* $\bar{D}$.

If the sequence $\{u_n(x)\}$ *converges uniformly on the boundary* $\dot{D}$ *of* D, *then* *this sequence converges to a continuous function* $u(x)$ *uniformly in* $\bar{D}$.

Proof. For any positive number ε, there exists a natural number N such that

$$(4.1) \qquad |u_n(x)-u_{n'}(x)| \leqq \varepsilon$$

for any n, $n'>N$, and $x \in \dot{D}$. This fact derives from the uniform convergence of the sequence $\{u_n(x)\}$ on $\dot{D}$.

It follows therefore from Theorem 1, that the inequality (4.1) holds for any n, $n'>N$ and any $x \in \bar{D}$, which implies the fact that the sequence $\{u_n(x)\}$ converges to a continuous function $u(x)$ uniformly in $\bar{D}$, q.e.d.

ADDENDUM. In the discussion of the present paper, it is obvious that Hypothesis 1 may be replaced by the following hypothesis which is assumed in Kusano's paper:

HYPOTHESIS 1′: There exists a positive lower semi-continuous function $h(x, p)$ such that

$$\sum_{i,j=1}^{m} a_{ij}(x, p)\xi_i\xi_j \geqq h(x, p)|\xi|^2$$

for any $(x, p) \in \mathfrak{D}_0$ and any real vector ξ.

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