ON THE EXISTENCE OF AN ESSENTIAL PICARD'S PERFECT SET

By Mitsuru Ozawa

1. Introduction.

Let E be a perfect set and D a complementary domain of E. If any meromorphic function in D with its singularities at each point of E admits at most n Picard's exceptional values at any neighborhood of each point of E, then E is said to be an n-Picard's perfect set. A 2-Picard's perfect set is simply said to be a Picard's perfect set.

Recently Matsumoto [5] proved the existence of n ($\geqq 3$)-Picard's perfect set E. Further he constructed a 3-Picard's perfect set E in any neighborhood of any point of which there is a meromorphic function with just 3 Picard's exceptional values. In his construction E is of zero capacity. At the same time Carleson [2] proved independently the existence of 3-Picard's perfect set E in a class $N_{\mathfrak{B}}$ but cap $E > 0$.

In the present paper we shall extend the notion of Picard's perfect set and prove the existence of a Picard's perfect set in a new sense. We shall make use of the standard notions of the Nevanlinna theory [6].

Hayman [3] developed the Nevanlinna theory in a great extent in a case of the unit disc. Our main idea is due to the nice theorems I and II in [3].

2. Definition of an essential Picard's perfect set.

Let $\mathfrak{L}(X)$ be a class of meromorphic functions which are Lindelöfian in a domain X in Heins' sense [4]. This is the same as a class of meromorphic functions of bounded type in X. Let E be a perfect set lying on a simple closed curve γ and D a complementary domain of E. Let D_1 and D_2 be two domains bounded by γ. Let $N(p)$ be a generic neighborhood of any generic point p of E. Let $\mathfrak{M}$ be a class of meromorphic functions in D with essential singularities on E.

If any element of f in $\mathfrak{M} - \mathfrak{L}(N(p) \frown D_1) - \mathfrak{L}(N(p) \frown D_2)$ has n-Picard's exceptional values at most in any $N(p)$ of each point p of E, then E is said to be an essential n-Picard's perfect set. If $n = 2$, then E is simply said to be an essential Picard's perfect set.

This modification of the definition of n-Picard's perfect set E brings us an advantage. In fact, if $E \notin N_{\mathfrak{B}}$, then there exists a bounded analytic function

Received June 14, 1962.

in D and hence any such E is not n-Picard's perfect set in the former definition. However such E may be an essential n-Picard's perfect set.

3. Hayman's theorems.

We shall say that a domain B *properly cantains* a set of arcs $z=e^{i\theta}$, $\alpha<\theta<\beta$, if these arcs lie in B and we have uniformly on all the arcs

$$(1)\qquad d(\theta)\geqq C_1[(\beta-\theta)(\theta-\alpha)]^{C_2},$$

where C_1, C_2 are positive constants and $d(\theta)$ is the smallest distance from $e^{i\theta}$ to the boundary of B.

THEOREM 1. (*A modification of Hayman's theorem* II) *Suppose that B is a bounded domain containing $|z|<1$ and properly containing a set of arcs $z=e^{i\theta}$, $\alpha_\nu<\theta<\beta_\nu$, where*

$$(2)\qquad \sum(\beta_\nu-\alpha_\nu)=2\pi,$$

$$(3)\qquad \sum(\beta_\nu-\alpha_\nu)\log\frac{1}{\beta_\nu-\alpha_\nu}<\infty.$$

Suppose that $f(z)$ is regular in B and $f(z)\neq 0$, 1 in B. Then $f(z)$ is of bounded type in $|z|<1$.

LEMMA 1. (*Hayman's theorem* I) *Suppose that $f(z)$ is meromorphic in a bounded domain B containing $|z|\leqq R$. Let $d_R(\theta)$ denote the distance of $z=Re^{i\theta}$ from the boundary of B, and $n_R(\theta)$ the total number of roots of the equations $f(z)=0,1$, distant at least $d_R(\theta)/2$ from the boundary of B. Then we have*

$$m\left(R,\frac{f^{(p)}}{f}\right)\leqq A(p)\left[\log^+ m(R,f)+\log^+ m\left(R,\frac{1}{f}\right)+I+\log^+\frac{1}{R}+1\right]$$

where

$$I=I(R)\equiv\frac{1}{2\pi}\int_0^{2\pi}\left[\log^+ n_R(\theta)+\log^+\frac{1}{d_R(\theta)}\right]d\theta.$$

Here $m(R,F)$ is the so-called "Schmiegungsfunktion" of F and $A(p)$ depends only on p.

The following two lemmas are also due to Hayman.

LEMMA 2. *Under the hypotheses of Theorem 1, let $0\leqq r<1$ and let $d_r(\theta)$ denote the distance of $z=re^{i\theta}$ from the nearest frontier point of B. Then we have uniformly in ν and r*

$$\int_{\alpha_\nu}^{\beta_\nu}\log^+\frac{1}{d_r(\theta)}d\theta=O(\beta_\nu-\alpha_\nu)\left(\log\frac{1}{\beta_\nu-\alpha_\nu}+1\right).$$

LEMMA 3. *Suppose that B satisfies the conditions of theorem 1, then $I(r)$ is uniformly bounded for $0\leqq r<1$.*

We should here remark that, under the hypotheses of Theorem 1, $I(r)$ reduces to the following form

$$\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{d_r(\theta)}\,d\theta,$$

since $n_r(\theta)=0$ for any $0\leqq r<1$. By Lemma 2 and the relations (2) and (3) we have immediately Lemma 3. Thus we do not use the finiteness of the order of $f(z)$ in B in Hayman's sense, although Hayman's original one is based on that property.

Proof of Theorem 1. We make use of the first and second fundamental theorems of Nevanlinna. Let $F(z)$ be a regular function in B defined by the relation

$$F(z)=\frac{f(z)}{f(z)-1}.$$

Then $F(z)\neq 0,1,\infty$ in B. By Lemmas 1, 2 and 3 we have

$$\begin{aligned} m\left(R,\frac{f'}{f}-\frac{f'}{f-1}\right)&=m\left(R,\frac{F'}{F}\right)\\ &\leqq C_1\left(\log^+ m(R,F)+\log^+ m\left(R,\frac{1}{F}\right)\right)+O(1)\\ &\leqq C_2\log^+ T\left(R,\frac{f}{f-1}\right)+O(1)=C_2\log^+ T(R,f)+O(1). \end{aligned}$$

Similarly we have

$$m\left(R,\frac{f'}{f}\right)\leqq C_3\log^+ T(R,f)+O(1).$$

Since $\log^+|b|\leqq\log^+|a-b|+\log^+|a|+\log 2$, we can say that

$$m\left(R,\frac{f'}{f-1}\right)\leqq m\left(R,\frac{f'}{f}-\frac{f'}{f-1}\right)+m\left(R,\frac{f'}{f}\right)+\log 2.$$

Thus we have

$$\begin{aligned} m\left(R,\frac{f'}{f-1}+\frac{f'}{f}\right)&\leqq m\left(R,\frac{f'}{f}\right)+m\left(R,\frac{f'}{f-1}\right)+O(1)\\ &\leqq C_4\log^+ T(R,f)+O(1). \end{aligned}$$

By the second fundamental theorem of Nevanlinna we have

$$m\left(R,\frac{1}{f}\right)+m\left(R,\frac{1}{f-1}\right)+m(R,f)<2T(R,f)-N_1(R)+S(R,f),$$

$$S(R,f)<m\left(R,\frac{f'}{f}\right)+m\left(R,\frac{f'}{f}+\frac{f'}{f-1}\right)+O(1).$$

For the last term $S(R,f)$ we have the following estimation

$$S(R,f)<C_5\log^+ T(R,f)+O(1).$$

If $T(R,f)$ is unbounded as $R\to 1$, then we have

$$\varliminf_{R\to 1}\frac{S(R,f)}{T(R,f)}=0.$$

This implies the famous defect relation

$$\Theta(0)+\Theta(1)+\Theta(\infty)\leqq 2,$$

where

$$\Theta(a)=1-\varlimsup_{R\to 1}\frac{\bar{N}(R,1/(f-a))}{T(R,f)}.$$

By our assumption $f(z)\neq 0,1,\infty$ the left hand side is equal to three, which is a contradiction.

4. Proof of the existence of an essential Picard's perfect set E.

Let E_z be a Cantor set on $|z|=1$ satisfying two conditions (2) and (3) in Theorem 1. This is easy to construct as Hayman said. By (2) the one-dimensional measure of E_z is equal to zero and hence E_z belongs to the class $N_{\mathfrak{B}}$; see Ahlfors-Beurling [1].

Let $U(p)$ be a symmetric disc neighborhood of a point $p\in E_z$ with respect to $|z|=1$. We may assume that any two intersection points M, N of the circumference of $U(p)$ and $|z|=1$ do not lie on E_z. The perfectness of E_z implies that $d(\mathrm{M},E_z)$ and $d(\mathrm{N},E_z)$ is bounded away from zero, where $d(A,B)$ indicates the Euclidean distance of two sets A and B. We shall map $U(p)\frown\{|z|<1\}$ conformally onto $|w|<1$ in such a manner that M, N correspond to two points i, $-i$, respectively. Then the remaining part of $U(p)$ is conformally mapped onto $|w|>1$ by reflection of $|w|<1$ through the semi-circle $\{|w|=1\}\frown\{-\pi/2<\arg w<\pi/2\}$. The image of E_z is denoted by E_w. Then the derivative of the Riemann mapping function $w(z)$ has its maximum and minimum moduli in a fixed arc γ_ε which is defined by $\{|z|=1\}\frown U(p)-(U^\varepsilon_{\mathrm{M}}+U^\varepsilon_{\mathrm{N}})$, where $U^\varepsilon_{\mathrm{M}}$ and $U^\varepsilon_{\mathrm{N}}$ are two ε-neighborhoods of M and N and satisfy $d(U^\varepsilon_{\mathrm{M}},E_z)>\delta>0$, $d(U^\varepsilon_{\mathrm{N}},E_z)>\delta>0$. We denote these maximum and minimum of $|w'(z)|$ on γ_ε by Ω and ω, respectively. Further any arc $z=e^{i\theta}$, $\alpha_\nu<\theta<\beta_\nu$ lying on γ_ε is distorted into a comparable arc, that is,

$$\frac{2}{\pi}\omega(\beta_\nu-\alpha_\nu)\leqq B_\nu-A_\nu\leqq\frac{\pi}{2}\Omega(\beta_\nu-A_\nu),\quad A_\nu=\arg w(e^{i\alpha_\nu}),\quad B_\nu=\arg w(e^{i\beta_\nu}).$$

Therefore we can say that

$$\sum(B_\nu-A_\nu)\log\frac{1}{B_\nu-A_\nu}<\infty.$$

Further we must add two points i, $-i$ to the set E_w. The series is then still convergent. We can easily construct a domain G_w which contains each arc $|w|=1$, $A_\nu<\Theta<B_\nu$ and three arcs related to two points i, $-i$ properly and contains $|w|<1$.

Let $f(z)$ be a meromorphic function in D_z which is the complement of E_z and has its essential singularities at each point of E_z. We assume that $f(z)$ excludes three distinct values in $U(p)$. We may assume that $f(z)$ excludes three values 0, 1 and ∞. Then $f\circ z(w)$ also excludes 0, 1 and ∞ in G_w. Therefore we can apply Theorem 1 and conclude that $f\circ z(w)$ is of bounded type in $|w|<1$. Similarly we can say that $f\circ z(w)$ is of bounded type in $|w|>1$. Therefore we have the desired fact.

THEOREM 2. *There is an essential Picard's perfect set.*

REMARK. Matsumoto's original problem is still open. Further the following problem is also still open.

Does there exist an essential n-Picard's perfect set E for any n which does not belong to the class $N_{\mathfrak{B}}$?

We can further impose a condition due to Matsumoto to our Cantor set. This condition guarantees the existence of at most 3 Picard's exceptional values for any meromorphic functions. Thus we can say that there exists an essential Picard's perfect set which belongs to a class of 3-Picard's perfect set.

5. The linear meaure $m(E)$ and its effect to the value distribution.

We shall discuss the effect of the linear measure $m(E)$ of E to the value distribution of some meromorphic functions.

THEOREM 3. *Suppose that B is a bounded domain containing $|z|<1$ and properly containing a set of arcs $z=e^{i\theta}$, $\alpha_\nu<\theta<\beta_\nu$, where*

$$\sum(\beta_\nu-\alpha_\nu)\leqq 2\pi,$$

$$\sum(\beta_\nu-\alpha_\nu)\log\frac{1}{\beta_\nu-\alpha_\nu}<\infty.$$

Suppose that $f(z)$ is regular and $f(z)\neq 0$, 1 in B and $f(z)$ is not of bounded type in $|z|<1$. Then

$$\lambda\leqq\frac{3m(E)}{2\pi},\quad \lambda\equiv\varlimsup_{r\to1}\frac{T(r,f)}{\log\dfrac{1}{1-r}}.$$

LEMMA 4. *Under the assumptions of Theorem 3, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{d_r(\theta)}d\theta\leqq\frac{1}{2\pi}m(E)\log^+\frac{1}{1-r}+O(1).$$

Proof of Lemma 4. By Lemma 2, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{d_r(\theta)}d\theta\leqq\frac{1}{2\pi}\int_E\log^+\frac{1}{d_r(\theta)}d\theta+\sum_\nu\frac{1}{2\pi}\int_{\alpha_\nu}^{\beta_\nu}\log^+\frac{1}{d_r(\theta)}d\theta$$

$$=\frac{1}{2\pi}m(E)\log^+\frac{1}{1-r}+O(1).$$

Proof of Theorem 3. By a quite similar method as in Theorem 1 and by Lemma 4 instead of Lemma 3, we have

$$S(r,f) < m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f} + \frac{f'}{f-1}\right) + O(1)$$

$$\leqq \frac{3m(E)}{2\pi} \log \frac{1}{1-r} + O(\log T(r,f)).$$

Then we have

$$\Theta(0) + \Theta(1) + \Theta(\infty) \leqq 2 + \varliminf \frac{S(r,f)}{T(r,f)}$$

$$\leqq 2 + \frac{3m(E)}{2\pi\lambda}.$$

By the assumption the left hand side sum is equal to 3. Thus we have the desired result.

COROLLARY. Under the assumptions of Theorem 3, if $\lambda > 3m(E)/2\pi$, then $f(z)$ has the Picard property.

REMARK. It is very plausible to explain a conjecture that the best possible numerical factor is 1 instead of 3 in the above theorem and its corollary. Further we can obtain a formal extension of Hayman's original theorem in our case. From this extension we can also say that there occurs an effect of $m(E)$ to the value distribution of a sort of meromorphic functions.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.