A NOTE ON THE EXISTENCE OF SOLUTIONS OF DIFFERENCE-DIFFERENTIAL EQUATIONS

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Introduction. In [2] and [3], the author has discussed the existence of bounded and periodic solutions of a difference-differential equation such that

$$\frac{dx(t+1)}{dt} = ax(t+1) + bx(t) + f(x(t+1),\ x(t),\ t), \tag{0.1}$$

where a and b are constant, corresponding respectively to the cases where $f(0, 0, t)$ is bounded and periodic in t. His discussion proceeded there was essentially based on the assumption that every root of the characteristic equation $e^s(s-a)-b=0$ lies to the left of the straight line $\Re s = -\delta$, where δ is a positive constant.

The purpose of this note is to discuss the existence of solutions, not necessary to be bounded, of (0.1) under the condition for the roots of the characteristic equation weaker than that stated above, that is, the condition that some roots of $e^s(s-a)-b=0$ lie to the right of the imaginary axis. However, the assumptions upon $f(x, y, t)$ may be made strong.

1. Kernel functions. In (0.1), we suppose that every real part of all the roots of the characteristic equation $e^s(s-a)-b=0$ is less than δ (>0). Applying for (0.1) a transformation $e^{2\delta t}y(t+1)=x(t+1)$, (0.1) is transformed into an equation

$$\frac{dy(t+1)}{dt} = (a-2\delta)y(t+1) + be^{-2\delta}y(t) + e^{-2\delta t}f(e^{2\delta t}y(t+1),\ e^{2\delta(t-1)}y(t),\ t). \tag{1.1}$$

Then, we find that every real part of all the roots of the characteristic equation

$$e^s(s-a+2\delta) - be^{-2\delta} = 0 \tag{1.2}$$

corresponding to the linear equation

$$\frac{dy(t+1)}{dt} = (a-2\delta)y(t+1) + be^{-2\delta}y(t) \tag{1.3}$$

is less than $-\delta$.

Now, we shall define a kernel function for (1.3). Let $K_y(t)$ be a solution of (1.3) for $0 \leqq t < \infty$ under the initial conditions $K_y(t)=0$ $(-1 \leqq t < 0)$ and $K_y(0)=1$. Then, we call $K_y(t)$ the *kernel function* of (1.3) and it is useful to summarize the results concerning $K_y(t)$ which will be used later:

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(i) *$K_y(t)$ is a continuous function of t for $0 \leqq t < \infty$ and is determined uniquely*;

(ii) *$K_y(t)$ is differentiable for $0 < t < 1$ and $1 < t < \infty$*;

(iii) *$|K_y(t)| \leqq ce^{-\delta t}$ for $0 \leqq t < \infty$, where c is a constant*;

(iv) *$K_y'(t+1) = (a-2\delta)K_y(t+1) + be^{-2\delta}K_y(t)$ for $0 < t < \infty$*;

(v) *$K_y'(t) = (a-2\delta)K_y(t)$ for $0 < t < 1$.*

If we put $K_x(t) = e^{2\delta(t-1)}K_y(t)$, $e^{2\delta}K_x(t)$ is the unique solution of the linear equation

$$\frac{dx(t+1)}{dt} = ax(t+1) + bx(t) \tag{1.4}$$

under the initial conditions $e^{2\delta}K_x(t)=0$ $(-1 \leqq t < 0)$ and $e^{2\delta}K_x(0)=1$. Then, we call $e^{2\delta}K_x(t)$ the kernel function of (1.4).

Corresponding to (i), (ii), (iii), (iv) and (v), it is observed that the following results are obtained respectively:

(i)′ *$K_x(t)$ is a continuous function of t for $0 \leqq t < \infty$ and is uuiquely determined*;

(ii)′ *$K_x(t)$ is differentiable for $0 < t < 1$ and $1 < t < \infty$*;

(iii)′ *$|K_x(t)| \leqq ce^{\delta(t-2)}$ for $0 \leqq t < \infty$*;

(iv)′ *$K_x'(t+1) = aK_x(t+1) + bK_x(t)$ for $0 < t < \infty$*;

(v)′ *$K_x'(t) = aK_x(t)$ for $0 < t < 1$.*

2. **Existence of solutions.** We shall consider the existence of solutions of the equation

$$\frac{dx(t+1)}{dt} = ax(t+1) + bx(t) + f(x(t+1), x(t), t) \tag{2.1}$$

for $|t| < \infty$ under the following conditions:

(i) *$f(x, y, t)$ is continuous in x, y, t for $|x| < \infty$, $|y| < \infty$, $|t| < \infty$*;

(ii) *$|f(0, 0, t)| \leqq Me^{2\delta t}$ for $|t| < \infty$, where M is a constant*;

(iii) *$f(x, y, t)$ satisfies Lipschitz condition, that is, there exists a constant k such that*

$$|f(x_1, y_1, t) - f(x_2, y_2, t)| \leqq k(|x_1 - x_2| + |y_1 - y_2|)$$

for $|x_i| < \infty$, $|y_i| < \infty$ $(i = 1, 2)$ and $|t| < \infty$;

(iv) *every real part of all the roots of the characteristic equation*

$$e^s(s-a) - b = 0$$

is less than δ, where δ is a positive constant.

In order to establish the existence of a solution of the equation (2.1), it is sufficient to prove that a solution of an integral equation

$$y(t+1) = \int_{-\infty}^{t} e^{-2\delta s} f(e^{2\delta s}y(s+1), e^{2\delta(s-1)}y(s), s) K_y(t-s)\, ds \tag{2.2}$$

is also that of the difference-differential equation

$$\text{(2.3)} \quad \frac{dy(t+1)}{dt} = (a-2\delta)y(t+1) + be^{-2\delta}y(t) + e^{-2\delta t}f(e^{2\delta t}y(t+1),\ e^{2\delta(t-1)}y(t),\ t).$$

Then, it is observed that $x(t+1) = e^{2\delta t}y(t+1)$ is to be a solution of (2.1). To this end, it is useful to apply for (2.2) the successive approximation method, so that we define a sequence $\{y_n(t+1)\}_{n=0}^{\infty}$ for $|t| < \infty$ as follows:

$$y_0(t+1) = 0,$$

$$\text{(2.4)} \quad y_{n+1}(t+1) = \int_{-\infty}^{t} e^{-2\delta s} f(e^{2\delta s}y_n(s+1),\ e^{2\delta(s-1)}y_n(s),\ s)K_y(t-s)\,ds \qquad (n = 0, 1, 2, \cdots).$$

Then, it follows from (2.4), (iii), and (iii)′ in the preceding section that

$$\text{(2.5)} \quad \begin{aligned} &|y_{n+1}(t+1) - y_n(t+1)| \\ &\leqq \int_{-\infty}^{t} e^{-2\delta s} |f(e^{2\delta s}y_n(s+1),\ e^{2\delta(s-1)}y_n(s),\ s) \\ &\qquad - f(e^{2\delta s}y_{n-1}(s+1),\ e^{2\delta(s-1)}y_{n-1}(s),\ s)|\,|K_y(t-s)|\,ds \\ &\leqq ck\int_{-\infty}^{t} (|y_n(s+1) - y_{n-1}(s+1)| + |y_n(s) - y_{n-1}(s)|)\, e^{-\delta(t-s)}\,ds. \end{aligned}$$

In particular, for $n = 0$, we obtain that

$$|y_1(t+1) - y_0(t+1)| \leqq \int_{-\infty}^{t} e^{-2\delta s} |f(0,\ 0,\ s)|\,|K_y(t-s)|\,ds.$$

From (ii), it follows that

$$\text{(2.6)} \quad |y_1(t+1) - y_0(t+1)| \leqq \frac{Mc}{\delta}.$$

Successively applying (2.5) and (2.6), we inductively obtain the inequalities

$$\text{(2.7)} \quad |y_{n+1}(t+1) - y_n(t+1)| \leqq \frac{M}{2k}\left(\frac{2ck}{\delta}\right)^{n+1} \qquad (n=0, 1, 2, \cdots)$$

for $|t| < \infty$. Then, the inequality (2.7) shows us that the sequence $\{y_n(t+1)\}_{n=0}^{\infty}$ uniformly converges to a function $y(t+1)$ for $|t| < \infty$, provided that $2ck/\delta$ is less than one. The uniform convergence yields that $y(t+1)$ is a solution of (2.2) for $|t| < \infty$. Furthermore, we obtain an upper bound of $|y(t+1)|$ for $|t| < \infty$, that is, it follows from (2.7) that

$$\begin{aligned} |y(t+1)| = \lim_{n\to\infty} |y_{n+1}(t+1)| &\leqq \sum_{n=0}^{\infty} |y_{n+1}(t+1) - y_n(t+1)| \\ &\leqq \sum_{n=0}^{\infty} \frac{M}{2k}\left(\frac{2ck}{\delta}\right)^{n+1} = \frac{Mc}{\delta - 2ck}, \end{aligned}$$

which implies the boundedness of $y(t+1)$ for $|t| < \infty$.

Next, it is proved that the solutions of (2.2) are uniquely determined. In fact, if there exist two solutions $y(t+1)$ and $z(t+1)$ of (2.2), we have from (2.2) and (iii) that

$$|y(t+1)-z(t+1)|$$
$$\leqq \int_{-\infty}^{t} e^{-2\delta s}|f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s)-f(e^{2\delta s}z(s+1),\ e^{2\delta(s-1)}z(s),\ s)||K_y(t-s)|\,ds$$
$$\leqq ck\int_{-\infty}^{t}(|y(s+1)-z(s+1)|+|y(s)-z(s)|)e^{-\delta(t-s)}ds.$$

Let $A(t)$ be the supremum of $|y(s+1)-z(s+1)|$ over $-\infty<s\leqq t$. Then, it follows that

$$A(t)\leqq 2ckA(t)\int_{-\infty}^{t}e^{-\delta(t-s)}ds=\frac{2ck}{\delta}A(t),$$

which is a contradiction, unless $A(t)$ vanishes identically, since $2ck/\delta$ is less than one. This proves the uniqness of solutions of (2.2).

Finally, we shall prove that the solution of (2.2) is also that of (2.3) for $|t|<\infty$. To this end, differentiating both sides of (2.2), we obtain that

$$\frac{dy(t+1)}{dt}=e^{-2\delta t}f(e^{2\delta t}y(t+1),\ e^{2\delta(t-1)}y(t),\ t)$$
$$+\left(\int_{-\infty}^{t-1}+\int_{t-1}^{t}\right)e^{-2\delta s}f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s)K_y'(t-s)\,ds,$$

since $K_y(0)=1$. It follows, from (iv) and (v) in the preceding section, that

$$\begin{aligned}\frac{dy(t+1)}{dt}&=e^{-2\delta t}f(^{2\delta t}y(t+1),\ e^{2\delta(t-1)}y(t),\ t)\\&\quad+\int_{-\infty}^{t-1}e^{-2\delta s}f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s)((a-2\delta)K_y(t-s)\\&\qquad\qquad+be^{-2\delta}K_y(t-1-s))\,ds\\&\quad+\int_{t-1}^{t}e^{-2\delta s}f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s)(a-2\delta)K_y(t-s)\,ds\\&=e^{-2\delta t}f(e^{2\delta t}y(t+1),\ e^{2\delta(t-1)}y(t),\ t)\\&\quad+(a-2\delta)\int_{-\infty}^{t}e^{-2\delta s}f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s)K_y(t-s)\,ds\\&\quad+be^{-2\delta}\int_{-\infty}^{t-1}e^{-2\delta s}f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s)K_y(t-1-s)\,ds\\&=(a-2\delta)y(t+1)+be^{-2\delta}y(t)+e^{-2\delta t}f(e^{2\delta t}y(t+1),\ ^{2\delta(t-1)}y(t),\ t),\end{aligned}$$

which is the desired result.

Returning to the original equation (2.1) by the transformation $x(t+1)=e^{2\delta t}y(t+1)$ for $|t|<\infty$, we obtain the following

THEOREM 1. *Under the conditions* (i), (ii), (iii) *and* (iv), *there exists a solution of* (2.1) *for* $|t|<\infty$, *and the inequality*

$$|x(t+1)|\leqq\frac{Mc}{\delta-2ck}e^{2\delta t}$$

remains valid for $|t|<\infty$, *provided that* $2ck/\delta$ *is less than one.*

REMARK. If we are simply concerned with the existence of solutions of an integral equation

$$x(t+1) = e^{2\delta}\int_{-\infty}^{t} f(x(s+1),\ x(s),\ s)K_x(t-s)\,ds, \tag{2.8}$$

which is equivalent to (2.2), we can proceed the same discussions as above. It seems, however, to be difficult to prove the uniqueness of solutions of (2.8). Hence, although the descriptions were not simple, we used the equation (2.2) instead of (2.8).

If the uniqueness of solutions of (2.1) is guaranteed under certain initial conditions (cf. [1]), Theorem 1 asserts that the equation (2.1) is equivalent to (2.8).

3. Equations with a parameter. As to an equation whose perturbed term has a parameter μ such that

$$\frac{dx(t+1)}{dt} = ax(t+1) + bx(t) + f(x(t+1),\ x(t),\ t,\ \mu), \tag{3.1}$$

we can apply for (3.1) the same methods as used in the preceding section under the following conditions:

(i) *$f(x, y, t, \mu)$ is a continuous function of x, y, t, μ for $|x|<\infty$, $|y|<\infty$, $|t|<\infty$, and small $|\mu|$;*

(ii) *$|f(0, 0, t, \mu)| \leqq Me^{2\delta t}$ for $|t|<\infty$, where M is a constant independent of μ;*

(iii) *$f(x, y, t, \mu)$ satisfies Lipschitz condition such that*

$$|f(x_1, y_1, t, \mu_1) - f(x_2, y_2, t, \mu_2)| \leqq k(|x_1 - x_2| + |y_1 - y_2| + |\mu_1 - \mu_2|)$$

for $|x_i|<\infty$, $|y_i|<\infty$, $|t|<\infty$, and small $|\mu_i|$ $(i=1, 2)$, where k is a constant independent of μ;

(iv) *the condition* (iv) *in the preceding section is still retained.*

Then, by just the same reason as before, we can establish, under the conditions (i), (ii), (iii) and (iv), the existence of the unique solution of the integral equation

$$y(t+1) = \int_{-\infty}^{t} e^{-2\delta s} f(e^{2\delta s}y(s+1),\ e^{2\delta(s-1)}y(s),\ s,\ \mu)K_y(t-s)\,ds \tag{3.2}$$

for $|t|<\infty$, provided that $2ck/\delta$ is less than one. Furthermore, it is also observed that the solution of (3.2) is bounded and is also that of the equation

$$\frac{dy(t+1)}{dt} = (a-2\delta)\,y(t+1) + be^{-2\delta}y(t) + e^{-2\delta t}f(e^{2\delta t}y(t+1),\ e^{2\delta(t-1)}y(t),\ t,\ \mu) \tag{3.3}$$

for $|t|<\infty$.

Since the equation (3.2) has a parameter μ, the solution may depend on μ. Hence, we denote it by $y(t, \mu)$. The solution corresponding to $\mu=0$ is simply denoted by $y(t)$.

Now, we shall prove that $y(t, \mu)$ uniformly converges to $y(t)$ for $|t|<\infty$ as

μ tends to zero. For two solutions $y(t, \mu)$ and $y(t)$ of (3.2), it follows from (3.2) that

$$\begin{aligned} &|y(t+1, \mu)-y(t+1)| \\ &\leqq \int_{-\infty}^{t} e^{-2\delta s}|f(e^{2\delta s}y(s+1, \mu), e^{2\delta(s-1)}y(s, \mu), s, \mu) \\ &\qquad\qquad -f(e^{2\delta s}y(s+1), e^{2\delta(s-1)}y(s), s, 0)||K_y(t-s)|ds \\ &\leqq ck\int_{-\infty}^{t}(|y(s+1, \mu)-y(s+1)|+|y(s, \mu)-y(s)|+|\mu|)e^{-\delta(t-s)}ds. \end{aligned}$$

Let $B(t)$ be the supremum of $|y(s, \mu)-y(s)|$ over $-\infty < s \leqq +1$. Then, it follows that

$$B(t) \leqq ck(2B(t)+|\mu|)\int_{-\infty}^{t} e^{-\delta(t-s)}ds = \frac{ck}{\delta}(2B(t)+|\mu|).$$

Hence, we obtain

$$B(t)\left(1-\frac{2ck}{\delta}\right) \leqq \frac{ck}{\delta}|\mu|.$$

Since $2ck/\delta$ is less than one, the above inequality shows us that $y(t, \mu)$ uniformly converges to $y(t)$ for $|t|<\infty$, as μ tends to zero. This is the desired result.

Thus, returning to the equations

$$x(t+1) = e^{2\delta}\int_{-\infty}^{t} f(x(s+1), x(s), s, \mu)K_x(t-s)ds \tag{3.4}$$

and

$$\frac{dx(t+1)}{dt} = ax(t+1)+bx(t)+f(x(t+1), x(t), t, \mu) \tag{3.5}$$

by means of the transformations $x(t+1)=e^{2\delta t}y(t+1)$ and $K_x(t+1)=e^{2\delta t}K_y(t+1)$, we can assert that there exists a solution of (3.5) (or (3.4)) for $|t|<\infty$ converging uniformly to that of (3.5) (or (3.4)) corresponding to $\mu=0$ as μ tends to zero.

For different values μ_1 and μ_2, it follows from (3.2) that

$$\begin{aligned} &|y(t+1, \mu_1)-y(t+1, \mu_2)| \\ &\leqq ck\int_{-\infty}^{t}(|y(s+1, \mu_1)-y(s+1, \mu_2)|+|y(s, \mu_1)-y(s, \mu_2)|+|\mu_1-\mu_2|)e^{-\delta(t-s)}ds. \end{aligned}$$

Let $M(t)$ be the supremum of $|y(s, \mu_1)-y(s, \mu_2)|$ over $-\infty < s \leqq t+1$. Then, it follows that

$$M(t) \leqq \frac{ck}{\delta}(2M(t)+|\mu_1-\mu_2|).$$

Hence, we have

$$|y(t+1, \mu_1)-y(t+1, \mu_2)| \leqq \frac{ck}{\delta-2ck}|\mu_1-\mu_2|,$$

which implies that $y(t+1, \mu)$ is a equi-continuous function of μ. Furthermore, it was proved that $y(t+1, \mu)$ is bounded for small $|\mu|$ and $|t|<\infty$. Hence, by means of a well-known theorem in the theory of normal families, any family of solutions $\{y(t+1, \mu_n)\}_{n=0}^{\infty}$ such that $\mu_n \to 0$ contains a subsequence converging uniformly to a function $y_0(t+1)$ for $|t|<\infty$ as $n\to\infty$. It is expected that $y_0(t+1)$ will be a solution of (3.2). This fact was already proved before. Thus, we obtain the following

THEOREM 2. *Under the conditions* (i), (ii), (iii) *and* (iv), *there exists a solution of* (3.5) *for* $|t|<\infty$. *Furthermore, the solution is bounded for* $|t|<\infty$, *an equi-continuous function of* μ *for small* $|\mu|$, *and it converges uniformly to that of the equation corresponding to* $\mu=0$ *as* μ *tends to zero.*

COROLLARY. *Let* $f(x, y, t, \mu)$ *be of the form* $\mu f(x, y, t)$, *where* $f(x, y, t)$ *satisfies the following conditions*:

(i) $f(x, y, t)$ *is a continuous function of* x, y, t *for* $|x|<\infty$, $|y|<\infty$, $|t|<\infty$;

(ii) $|f(0, 0, t)|\leqq Me^{2\delta t}$ *for* $|t|<\infty$, *where* M *is a constant*;

(iii) $f(x, y, t)$ *satisfies Lipschitz condition, that is, there exists a constant* k *such that*

$$|f(x_1, y_1, t)-f(x_2, y_2, t)|\leqq k(|x_1-x_2|+|y_1-y_2|)$$

for $|x_i|<\infty$, $|y_i|<\infty$ $(i=1, 2)$ *and* $|t|<\infty$;

Then, there exists a solution of (3.1) *for* $|t|<\infty$, *provided that* $|\mu|$ *is less than* $\delta/2ck$.

4. Equations with forcing functions.

Now, we consider an equation

$$\frac{dx(t+1)}{dt}=ax(t+1)+bx(t)+f(x(t+1), x(t), u(t), t) \tag{4.1}$$

for $|t|<\infty$, where $u(t)$ is a given function of t, which is called a forcing function.

In [2], the author discussed the problems of the boundedness of solutions and existence of periodic solutions of (4.1) under the condition that every real part of all the roots of the characteristic equation is negative. On the contrary, in this section, there will appear some roots having positive real part. That is, we shall consider (4.1) under the following conditions:

(i) $f(x, y, z, t)$ *is a continuous function of* x, y, z, t *for* $|x|<\infty$, $|y|<\infty$, $|z|<\infty$, $|t|<\infty$;

(ii) $|f(0, 0, u(t), t)|\leqq Me^{2\delta t}$ *for* $|t|<\infty$, *where* M *is a constant*;

(iii) $f(x, y, z, t)$ *satisfies Lipschitz condition such that*

$$|f(x_1, y_1, z_1, t)-f(x_2, y_2, z_2, t)|\leqq k(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|)$$

for $|x_i|<\infty$, $|y_i|<\infty$, $|z_i|<\infty$ $(i=1, 2)$ *and* $|t|<\infty$;

(iv) *every real part of all the roots of the characteristic equation* $e^s(s-a)-b=0$ *is less than* δ, *where* δ *is a positve constant*;

(v) $u(s)$ *is a continuous function of* s *for* $|s|<\infty$ *and is integrable over* $-\infty<s\leqq t$ *for any* t.

Then, by means of the same reason as in the preceding sections, it is observed that there exists a solution of (4.1) for $|t|<\infty$, provided that $2ck/\delta$ is less than one.

Now, let $y_i(t)$ $(i=1, 2)$ be solutions of

$$y(t+1)=\int_{-\infty}^{t} e^{-2\delta s} f(e^{2\delta s} y(s+1),\ e^{2\delta(s-1)} y(s),\ u_i(s),\ s) K_y(t-s)\, ds \tag{4.2}$$

for $|t|<\infty$, where $u_i(t)$ $(i=1, 2)$ are forcing functions satisfying the conditions stated above. Then, (4.2) yields that

$$\begin{aligned} &|y_1(t+1)-y_2(t+1)| \\ &\leqq ck\int_{-\infty}^{t}(|y_1(s+1)-y_2(s+1)|+|y_1(s)-y_2(s)|+|u_1(s)-u_2(s)|)e^{-\delta(t-s)}ds. \end{aligned}$$

Let $N(t)$ be the supremum of $|y_1(s)-y_2(s)|$ over $-\infty<s\leqq t+1$. Then, it follows that

$$N(t)\leqq \frac{2ck}{\delta}N(t)+ck\int_{-\infty}^{t}|u_1(s)-u_2(s)|e^{-\delta(t-s)}ds.$$

Hence, we have an estimation

$$|y_1(t+1)-y_2(t+1)|\leqq \frac{\delta ck}{\delta-2ck}\int_{-\infty}^{t}|u_1(s)-u_2(s)|e^{-\delta(t-s)}ds \tag{4.3}$$

for $|t|<\infty$. Thus, we obtain the following

THEOREM 3. *Under the conditions* (i), (ii), (iii), (iv) *and* (v), *there exists a solution of* (4.1) *for* $|t|<\infty$. *Furthermore, the estimation* (4.3) *for the difference of two solutions corresponding to two forcing functions holds good.*

COROLLARY. *Let* $f(x, y, u(t), t)$ *be of the form* $f(x, y, t)+u(t)$, *where* $f(x, y, t)$ *and* $u(t)$ *satisfy the following conditions*:

(i) $f(x, y, t)$ *is a continuous function of* x, y, t *for* $|x|<\infty$, *and* $f(0, 0, t)=0$ *for* $|t|<\infty$;

(ii) $f(x, y, t)$ *satisfies Lipschitz condition such that*

$$|f(x_1, y_1, t)-f(x_2, y_2, t)|\leqq k(|x_1-x_2|+|y_1-y_2|)$$

for $|x_i|<\infty$, $|y_i|<\infty$ $(i=1, 2)$, $|t|<\infty$;

(iii) $u(s)$ *is a continuous function of* s *for* $|s|<\infty$, $|u(s)|$ *is integrable over* $-\infty<s\leqq t$ *for any* t, *and there exists a constant* M *such that* $|u(t)|\leqq Me^{2\delta t}$ *for* $|t|<\infty$.

Then, there exists a solution of (4.1) *for* $|t|<\infty$, *if* $2ck<\delta$, *and the inequality* (4.3) *remains valid for two forcing functions.*

REMARK. Let us suppose that all the roots of the characteristic equation lie to the left of the straight line $\Re s=-\delta$, where δ is a positive constant. Then, in Theorem 1, the assumptions on $f(x, y, t)$ are replaced by the following ones:

(i) *$f(x, y, t)$ is a continuous function of x, y, t for* $|x|<R$, $|y|<R$, $|t|<\infty$;

(ii) $|f(0, 0, t)|\leqq M$ *for* $|t|<\infty$, *where M is a constant;*[1]

(iii) *$f(x, y, t)$ satisfies Lipschitz condition such that*

$$|f(x_1, y_1, t)-f(x_2, y_2, t)|\leqq k(|x_1-x_2|+|y_1-y_2|)$$

for $|x_i|<R$, $|y_i|<R$ $(i=1, 2)$, $|t|<\infty$.

Then, the uniform convergence is guaranteed, if the inequality $2ck/\delta<1$ is fulfilled. Furthermore, if the inequality $Mc/(\delta-2ck)<R$ holds good, the limiting function will be a solution of the equation (2.1). If $R=+\infty$, we need only the first inequality, which has been used before.[2]

The similar remarks to the above ones may be applied for Theorem 3, if the forced term is of the form $f(x, y, t)+u(t)$ for $|t|<\infty$.

REFERENCES

[1] SUGIYAMA, S., On the existence and uniqueness theorems of difference-differential equations. Kōdai Math. Sem. Rep. 12 (1960), 179–190.

[2] SUGIYAMA, S., A note on difference-differential equations with a parameter. Mem. School Sci. Eng., Waseda Univ. 25 (1961). (To appear shortly)

[3] SUGIYAMA, S., On the existence of periodic solutions of difference-differential equations. Proc. Japan Acad. 37 (1961), 432–436.

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1) If $f(0, 0, t)$ is a periodic function of t for $|t|<\infty$, the boundedness condition (ii) remains valid for $|t|<\infty$. The periodicity was assumed in [2] and [3].

2) Cf. [3].