STRATEGIC INFORMATION AND NON-COOPERATIVE GAMES

BY MINORU SAKAGUCHI

The concept of information is one of the chief essentials in the game theory and it is quite distinct from that of Wiener and Shannon. If one speaks of information in Wiener-Shannon's sense as 'selective', then von Neuman's sense could be called 'strategic'.

The theory of games may be viewed as a formal model embodying three principal elements: (1) the preferences of the players of a game; (2) the choices or decisions open to them at each move; (3) their information regarding the choices made by the opponent player at previous moves.

Strategic information in a game-situation is represented by partitions in a finite set of possibilities, or 'plays'. It is the means of expressing a player's state of knowledge, at any move of a game, regarding the choices which have been made at earlier moves. The problem of rational choice of a plan of action and the existence of equilibrium situations are both closely related to the nature of the information pattern of the game.

If we set in a game-situation both the choices and preferences for the players symmetrically, then if, moreover, the information pattern is fair for all players in the game, that is, if each player, for example, is completely ignorant of the choices of his opponents, the value of the game is zero and the optimal strategy, when it existed, is common to all players. If we set the choices and preferences of the players symmetrically in a game-situation, and if we let the information pattern be unfair, then symmetry of the game disappears. Consider, for example, the case where the player I must take the first move in the game and his choice is told to the player II who can use this information and act optimally at the second move. It is, as our common sense tells us, clear that the player II stands in favor.

We shall, in this paper, show somewhat numerically this type of information-unbalance by examples of continuous poker models. Our method of the analysis owes to Karlin and Restrepo [1].

EXAMPLE 1. *La relance* (two-person stud poker with a single bet).

In our model the unit interval is taken as the representation of all possible hand that can be dealt to a player. Each hand is considered equally likely and therefore the operation of dealing a hand to a player may be considered as equivalent to selecting a random number from the unit interval according to the uniform distribution. The game proceeds as follows: An ante of 1 unit is required by each of the two players I and II. At the beginning of a play

Received January 29, 1960

they receive fixed hands, x and y, chosen at random from the unit intervals $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$. Then I either bets an amount a or drops out, losing his ante. If I bets, then II can either see the bet or fold, losing his ante. If II sees a bet the hands are compared with the higher card winning the total wager $1 + a$.

The above procedure is summarized in the following diagram:

Player	Hand	1st Move	2nd Move	Payoff to I
I	x	drops out	....	-1
		bets a →		
II	y		folds	1
			sees	$(1+a)\,\mathrm{sgn}(x-y)$

A mixed strategy for I can be described as a function $\phi(x)$ which represents the probability with which I will bet the amount a when his hand is x. A strategy for II can be represented by a function $\psi(y)$ which expresses the probability with which the player II will see a bet when he folds the hand y.

THEOREM 1. *Let $b = a/(2+a)$. The optimal strategies in this model are as follows:*

$$\phi^*(x) = \begin{cases} \text{arbitrary, but subject to the constraint that} & \\ \int_0^b \phi^*(x)\,dx = b(1-b), & \text{if } 0 \leqq x < b, \\ 1, & \text{if } b \leqq x \leqq 1, \end{cases}$$

$$\psi^*(y) = \begin{cases} 0, & \text{if } 0 \leqq y < b, \\ 1, & \text{if } b \leqq y \leqq 1. \end{cases}$$

The value of the game is $-(a/(2+a))^2$.

Proof. By enumerating all the possibilities we find that the expected payoff to the player I is

$$M(\phi, \psi) = -\int (1-\phi(x))\,dx + \int\int \phi(x)(1-\psi(y))\,dxdy + (1+a)\int\int \phi(x)\psi(y)\,\mathrm{sgn}(x-y)\,dxdy. \tag{1}$$

The ranges of integrations are always from 0 to 1. Hence they are omitted here and hereafter. Common sense tells us to guess that the optimal $\psi^*(y)$ is of the form

$$\psi^*(y) = \begin{cases} 0, & \text{if } y < b, \\ 1, & \text{if } y \geqq b \end{cases} \qquad \text{for some } b. \tag{2}$$

Under this assumption the part of M which involves ϕ can be reduced to

$$\int \phi(x) L(x)\,dx, \tag{3}$$

where

$$L(x) \equiv \begin{cases} -a + b(2+a), & \text{if } 0 \leqq x < b, \\ 2(a+1)x - a(b+1), & \text{if } b \leqq x \leqq 1. \end{cases}$$

If we set $b = a/(2+a)$, we have

$$L(x) = \begin{cases} 0, & 0 \leqq x < b, \\ 2(1+a)(x-b), & b \leqq x \leqq 1. \end{cases}$$

Thus it is clear that if I wants to maximize (3) he must take

$$\phi^*(x) = \begin{cases} \text{arbitrary}, & 0 \leqq x < b, \\ 1, & b \leqq x \leqq 1. \end{cases}$$

The part of M which involves ψ can be reduced to

$$\int \psi(y) K(y)\, dy, \tag{4}$$

where

$$\begin{aligned} K(y) &\equiv -\left(\int_0^y + \int_y^1\right)\phi^*(x)\,dx + (1+a)\left\{\int_y^1 \phi^*(x)\,dx - \int_0^y \phi^*(x)\,dx\right\} \\ &= -(2+a)\int_0^y \phi^*(x)\,dx + a\int_y^1 \phi^*(x)\,dx. \end{aligned}$$

It is clear that since $K(y)$ is monotonously decreasing and the function ψ which minimizes (4) is of the form $\psi = 1$ if $K(y) \leqq 0$, and 0 if $K(y) > 0$, we must have the expression

$$-(2+a)\int_0^b \phi^*(x)\,dx + a\int_b^1 \phi^*(x)\,dx = 0,$$

in order to have the minimum of (4) with $\psi = \psi^*$ given by (2). We easily have from (1) and (3)

$$M(\phi^*, \psi^*) = -\left(\frac{a}{2+a}\right)^2.$$

Thus we have shown that $\max_\phi M(\phi, \psi^*) = M(\phi^*, \psi^*) = \min_\psi M(\phi^*, \psi)$, completing the proof of the theorem.

EXAMPLE 2. *Le her* (two-person draw poker).

This game proceeds as follows: Before the play the two players I and II receive fixed hands, x and y, each being randomly (and independently) chosen from the unit interval.

Now if I is content with his hand he may keep it. But if I is not content with his hand he is allowed to change it for another taken out of the unit interval at random. The rule of the play is the same for player II, and I has to take the first move. The main object is for each to obtain a higher card than his opponent.

This procedure is summarized in the following diagram:

Player	Hand	1st Move	2nd Move	Payoff to I
I	x	keeps x		
		changes to u		
II	y		keeps y	$\operatorname{sgn}(x-y)$
			changes to v	$\operatorname{sgn}(x-v)$
			keeps y	$\operatorname{sgn}(u-y)$
			changes to w	$\operatorname{sgn}(u-w)$

Let $\alpha(x)$ be the probability that if I receives x he keeps it. Let $\beta(y)$ be the probability that if I keeps his hand and II receives y II keeps it. Let $\gamma(y)$ be the probability that II keeps his hand if he receives y and I has changed his hand. Clearly a mixed strategy for I can be represented by $\alpha(x)$ and that for II by $\beta(y)$ and $\gamma(y)$.

THEOREM 2. *The optimal strategies in this model are as follows*:

$$\alpha^*(x)=\begin{cases}0, & x<x_0,\\ 1, & x\geqq x_0,\end{cases}$$

$$\beta^*(y)=\begin{cases}0, & y<b=(1+x_0^2)/2\fallingdotseq 0.65,\\ 1, & y\geqq b,\end{cases}$$

$$\gamma^*(y)=\begin{cases}0, & y<1/2,\\ 1, & y\geqq 1/2,\end{cases}$$

where $x_0\fallingdotseq 0.56$ *is the unique root lying in the unit interval of the equation* $4x^3+4x-3=0$.

The value of the game is

$$-\frac{1}{4}x_0^4-\frac{1}{2}x_0^2+\frac{3}{4}x_0-\frac{1}{4}\fallingdotseq -0.015.$$

Proof. By enumerating all the possibilities we find the expected payoff to player I is

$$\begin{aligned} M(\alpha;\beta,\gamma)=&\iint\alpha(x)\beta(y)\operatorname{sgn}(x-y)\,dxdy+\iint\alpha(x)(1-\beta(y))\,dxdy\int\operatorname{sgn}(x-v)\,dv\\ &+\iint(1-\alpha(x))\gamma(y)\,dxdy\int\operatorname{sgn}(u-y)\,du\\ &+\iint(1-\alpha(x))(1-\beta(y))\,dxdy\iint\operatorname{sgn}(u-w)\,dudw. \end{aligned}\tag{5}$$

It is natural to guess that the optimal β^* and γ^* are of the forms

$$\beta^*(y)=\begin{cases}0, & y<b,\\ 1, & y\geqq b,\end{cases}\qquad \gamma^*(y)=\begin{cases}0, & y<1/2,\\ 1, & y\geqq 1/2,\end{cases}\tag{6}$$

for some b, since player II has no opportunity to bluff.

After some calculation the part of $M(\alpha;\beta^*,\gamma^*)$ which involves α becomes expressible as follows:

$$\int \alpha(x) L(x)\, dx, \tag{7}$$

where

$$L(x) \equiv \begin{cases} 2bx - 3/4 & x \leqq b, \\ 2(b+1)x - 2b - 3/4 & x \geqq b. \end{cases}$$

Thus it is clear that if I wants to maximize (7) he must take

$$\alpha^*(x) = \begin{cases} 0, & \text{if } x < x_0, \\ 1, & \text{if } x \geqq x_0, \end{cases} \qquad \text{for some } 0 < x_0 < 1.$$

Now let us look at the part of $M(\alpha^*; \beta^*, \gamma^*)$ which involves β^*. This is found to be

$$\int \beta^*(y) K(y)\, dy, \tag{8}$$

where

$$K(y) \equiv \begin{cases} x_0^2 - 2x_0 + 1, & y \leqq x_0, \\ x_0^2 + 1 - 2y, & y \geqq x_0. \end{cases}$$

It is easily seen that we must have $b > x_0$, since II wants to minimize by the optimal choice of β^*. Hence x_0 and b must satisfy the equations $2bx_0 - 3/4 = 0$ and $x_0^2 + 1 - 2b = 0$ respectively.

From the derivations of α^* and β^* we know that

$$M(\alpha^*; \beta^*, \gamma^*) = \max_{\alpha} M(\alpha; \beta^*, \gamma^*) = \min_{\beta} M(\alpha^*; \beta, \gamma^*),$$

but we must also check that

$$M(\alpha^*; \beta^*, \gamma^*) = \min_{\beta, \gamma} M(\alpha^*; \beta, \gamma).$$

This is found out from (5) by reducing the part of $M(\alpha^*; \beta, \gamma)$ involving γ to

$$\int (1 - \alpha^*(x))\, dx \int (1 - 2y)\gamma(y)\, dy.$$

This completes the proof of the theorem.

EXAMPLE 3. *La relance with three players.*

The procedure of this game is described in the following diagram:

Player	Hand	1st Move	2nd Move	3rd Move	Payoffs
I	x	folds			$(-1, \frac{1}{2}, \frac{1}{2})$
		bets a →			
II	y		folds		$(\frac{1}{2}, -1, \frac{1}{2})$
			bets a →		
III	z			folds	$(\frac{1}{2}, \frac{1}{2}, -1)$
				sees	$(1+a)(U(x; y, z),$ $U(y; z, x),\ U(z; x, y))$

Here the function U is defined by

$$U(x;\, y,\, z) \equiv \begin{cases} 2, & \text{if } \; x > \max(y,\, z), \\ -1, & \text{if } \; x \leqq \max(y,\, z). \end{cases}$$

Let $\alpha(x)$ be the probability that if I receives x, he bets.

Let $\beta(y)$ be the probability with which II will bet when his hand is y and I has bet at the first move. Let $\gamma(z)$ be the probability with which III will see when his hand is z and I and II have bet at the previous two moves. Mixed strategies for I, II and III can be represented by the functions $\alpha(x)$, $\beta(y)$ and $\gamma(z)$ respectively.

We shall consider the Nash equilibrium strategy-points [2] of this three-person non-cooperative game.

THEOREM 3. *The non-cooperative solution of this three-person model is as follows*:

(i) *If* $0 < a \leqq 3\sqrt{2}/4$, *there is the unique Nash equilibrium strategy-point*:

$$\alpha^*(x) \equiv 1, \qquad 0 \leqq x \leqq 1,$$
$$\beta^*(y) \equiv 1, \qquad 0 \leqq y \leqq 1,$$
$$\gamma^*(z) \equiv \begin{cases} 0, & 0 \leqq z \leqq c^*, \\ 1, & c^* < z \leqq 1, \end{cases}$$

where $c^* = \sqrt{a/(3(1+a))}$. *Payoffs to three players are*

$$M_1(\alpha^*,\, \beta^*,\, \gamma^*) = M_2(\alpha^*,\, \beta^*,\, \gamma^*) = -ac^*/3, \qquad M_3(\alpha^*,\, \beta^*,\, \gamma^*) = 2ac^*/3.$$

(ii) *If* $3\sqrt{2}/4 \leqq a < \infty$, *there are two Nash equilibrium strategy-points. These are* $(\alpha^\circ,\, \beta^\circ,\, \gamma^\circ)$ *and* $(\alpha^+,\, \beta^+,\, \gamma^+)$ *with*

$$\alpha^\circ(x) = \begin{cases} \text{arbitrary, but subject to the constraint that} & \\ \quad \displaystyle\int^c \alpha^\circ(x)\,dx = \frac{1-c}{1+c}, & 0 \leqq x \leqq c, \\ 1, & c < x \leqq 1, \end{cases}$$
$$\beta^\circ(y) \equiv 1, \qquad 0 \leqq y \leqq 1,$$
$$\gamma^\circ(z) = \begin{cases} 0, & 0 \leqq z \leqq c, \\ 1, & c < z \leqq 1, \end{cases}$$

where $c = 2a/(2a+3)$, and

$$\alpha^+(x) \equiv 1, \qquad 0 \leqq x \leqq 1,$$
$$\beta^+(y) = \begin{cases} \text{arbitrary, but subject to the constraint that} & \\ \quad \displaystyle\int_0^c \beta^+(y)\,dy = \frac{1-c}{1+c}, & 0 \leqq y \leqq c, \\ 1, & c < y \leqq 1 \end{cases}$$
$$\gamma^+(z) = \begin{cases} 0, & 0 \leqq z \leqq c, \\ 1, & c < z \leqq 1 \end{cases}$$

where again $c = 2a/(2a+3)$.

Payoffs to three players corresponding to the two strategy-points are

$$M_1(\alpha^\circ, \beta^\circ, \gamma^\circ) = -\frac{3}{4}c^2 - \frac{1}{4}c^3,$$

$$M_2(\alpha^\circ, \beta^\circ, \gamma^\circ) = -\frac{3}{4}c + \frac{3}{4}c^2 + \frac{1}{2}c^3,$$

$$M_3(\alpha^\circ, \beta^\circ, \gamma^\circ) = \frac{3}{4}c - \frac{1}{4}c^3,$$

and

$$M_1(\alpha^+, \beta^+, \gamma^+) = -\frac{3}{4}c + \frac{3}{4}c^2 + \frac{1}{2}c^3,$$

$$M_2(\alpha^+, \beta^+, \gamma^+) = -\frac{3}{4}c^2 - \frac{1}{4}c^3,$$

$$M_3(\alpha^+, \beta^+, \gamma^+) = \frac{3}{4}c - \frac{1}{4}c^3.$$

We have

$$\begin{aligned} M_1(\alpha^\circ, \beta^\circ, \gamma^\circ) &= M_2(\alpha^+, \beta^+, \gamma^+) \leqq M_2(\alpha^\circ, \beta^\circ, \gamma^\circ) \\ &= M_1(\alpha^+, \beta^+, \gamma^+) < 0 < M_3(\alpha^\circ, \beta^\circ, \gamma^\circ) \\ &= M_3(\alpha^+, \beta^+, \gamma^+), \end{aligned}$$

the left inequality becoming the equality when $a = 3\sqrt{2}/4$. *We of course have*

$$\sum_1^3 M_i(\alpha^\circ, \beta^\circ, \gamma^\circ) = \sum_1^3 M_i(\alpha^+, \beta^+, \gamma^+) = 0.$$

Proof. The expected payoffs to players I, II and III are

$$\begin{aligned} M_1(\alpha, \beta, \gamma) = &-\int(1-\alpha(x))\,dx + \frac{1}{2}\iint\alpha(x)(1-\beta(y))\,dxdy \\ &+ \frac{1}{2}\iiint\alpha(x)\beta(y)(1-\gamma(z))\,dxdydz \\ &+ (1+a)\iiint\alpha(x)\beta(y)\gamma(z)U(x;\, y,\, z)\,dxdydz, \end{aligned} \tag{9}$$

$$\begin{aligned} M_2(\alpha, \beta, \gamma) = &\frac{1}{2}\int(1-\alpha(x))\,dx - \iint\alpha(x)(1-\beta(y))\,dxdy \\ &+ \frac{1}{2}\iiint\alpha(x)\beta(y)(1-\gamma(z))\,dxdydz \\ &+ (1+a)\iiint\alpha(x)\beta(y)\gamma(z)U(y;\, z,\, x)\,dxdydz \end{aligned} \tag{10}$$

and

$$\begin{aligned} M_3(\alpha, \beta, \gamma) = &\frac{1}{2}\int(1-\alpha(x))\,dx + \frac{1}{2}\iint\alpha(x)(1-\beta(y))\,dxdy \\ &- \iiint\alpha(x)\beta(y)(1-\gamma(z))\,dxdydz \\ &+ (1+a)\iiint\alpha(x)\beta(y)\gamma(z)U(z;\, x,\, y)\,dxdydz, \end{aligned} \tag{11}$$

respectively.

The analysis proceeds as in the previous examples and we may guess that the form of the optimal strategy for III is

$$\gamma^*(z) = \begin{cases} 0, & z < c, \\ 1, & z \geqq c, \end{cases} \qquad \text{for some } c.$$

Under this assumption we have

$$(9)' \quad \begin{aligned} M_1(\alpha, \beta, \gamma^*) = &-\int(1-\alpha(x))\,dx + \frac{1}{2}\iint\alpha(x)(1-\beta(y))\,dxdy \\ &+ \frac{c}{2}\iint\alpha(x)\beta(y)\,dxdy + (1+a)\iint\alpha(x)\beta(y)\,V_1(x, y)\,dxdy, \end{aligned}$$

$$(10)' \quad \begin{aligned} M_2(\alpha, \beta, \gamma^*) = &\frac{1}{2}\int(1-\alpha(x))\,dx - \iint\alpha(x)(1-\beta(y))\,dxdy \\ &+ \frac{c}{2}\iint\alpha(x)\beta(y)\,dxdy + (1+a)\iint\alpha(x)\beta(y)\,V_2(x, y)\,dxdy \end{aligned}$$

and

$$(11)' \quad \begin{aligned} M_3(\alpha, \beta, \gamma^*) = &\frac{1}{2}\int(1-\alpha(x))\,dx + \frac{1}{2}\iint\alpha(x)(1-\beta(y))\,dxdy \\ &- c\iint\alpha(x)\beta(y)\,dxdy + (1+a)\iint\alpha(x)\beta(y)\,V_3(x, y)\,dxdy, \end{aligned}$$

where

$$V_1(x, y) \equiv \int_c^1 U(x;\, y, z)\,dz = \begin{cases} 3x-2c-1, & \text{if } \max(c, y) < x, \\ -(1-c), & \text{otherwise}, \end{cases}$$

$$V_2(x, y) \equiv \int_c^1 U(y;\, z, x)\,dz = \begin{cases} 3y-2c-1, & \text{if } \max(c, x) < y, \\ -(1-c), & \text{otherwise}, \end{cases}$$

$$V_3(x, y) \equiv \int_c^1 U(z;\, x, y)\,dz = \begin{cases} 2+c-3\max(x, y), & \text{if } c < \max(x, y), \\ 2(1-c), & \text{if } c \geqq \max(x, y). \end{cases}$$

After some tedious calculations we obtain: The part of $M_1(\alpha, \beta, \gamma^*)$ which involves α is $\int\alpha(x)A(x)\,dx$, where

$$(12) \quad A(x) \equiv \begin{cases} \dfrac{3}{2} - (1-c)\left(\dfrac{3}{2}+a\right)\displaystyle\int\beta(y)\,dy, & x \leqq c, \\ \dfrac{1}{2} - \dfrac{1-c}{2}\displaystyle\int\beta(y)\,dy & \\ \qquad + (1+a)\left\{(3x-2c-1)\displaystyle\int_0^x\beta(y)\,dy - (1-c)\displaystyle\int_x^1\beta(y)\,dy\right\}, & x \geqq c, \end{cases}$$

the part of $M_2(\alpha, \beta, \gamma^*)$ which involves β is $\int\beta(y)B(y)\,dy$, where

$$(13) \quad B(y) \equiv \begin{cases} \left(\dfrac{3c}{2} - a(1-c)\right)\displaystyle\int\alpha(x)\,dx, & y \leqq c, \\ \dfrac{2+c}{2}\displaystyle\int\alpha(x)\,dx & \\ \qquad + (1+a)\left\{(3y-2c-1)\displaystyle\int_0^y\alpha(x)\,dx - (1-c)\displaystyle\int_y^1\alpha(x)\,dx\right\}, & y \geqq c, \end{cases}$$

and the part of $M_3(\alpha, \beta, \gamma^*)$ involving γ^* is

$$C(c) \equiv -c\left(\int \alpha(x)\,dx\right)\left(\int \beta(y)\,dy\right) + (1+a)\cdot$$

$$\cdot\left[\left(\int_0^c \beta(y)\,dy\right)\left\{2(1-c)\int_0^c \alpha(x)\,dx + \int_c^1 (2+c-3x)\alpha(x)\,dx\right\}\right. \tag{14}$$

$$\left. + \int_c^1 \beta(y)\left\{(2+c-3y)\int_0^y \alpha(x)\,dx + \int_y^1 (2+c-3x)\,\alpha(x)\,dx\right\} dy\right].$$

A strategy-point $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is a Nash equilibrium point [2] if and only if

$$\begin{aligned} M_1(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) &= \max_{\alpha} M_1(\alpha, \bar{\beta}, \bar{\gamma}), \\ M_2(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) &= \max_{\beta} M_2(\bar{\alpha}, \beta, \bar{\gamma}), \\ M_3(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) &= \max_{\gamma} M_3(\bar{\alpha}, \bar{\beta}, \gamma). \end{aligned} \tag{15}$$

By differentiating $C(c)$ partially with respect to c and putting the derivative equal to zero, we first obtain the relation

$$\left(\int_0^c \alpha(x)\,dx\right)\left(\int_0^c \beta(y)\,dy\right) = \frac{a}{3(1+a)}\left(\int \alpha(x)\,dx\right)\left(\int \beta(y)\,dy\right). \tag{16}$$

By (12) $A(x)$ is constant for $x \leqq c$, non-decreasing for $x \geqq c$ and $A(0) \gtreqless 0$ and $A(1) > 0$. By (13) $B(y)$ has the similar property as the function of $0 \leqq y \leqq 1$. By examining carefully all the possibilities for every pair of signs of $A(0)$ and $B(0)$, we can find that there do not exist equilibrium points satisfying (15) except for the cases

(i) $A(0),\ B(0) > 0$; (ii) $A(0) = B(0) = 0$, (iii) $A(0) > 0 = B(0)$.

Let us consider these three cases.

(i) If $A(0),\ B(0) > 0$, then maximizing strategies for I and II must be

$$\alpha^*(x) \equiv \beta^*(y) \equiv 1, \qquad 0 \leqq x,\ y \leqq 1,$$

since $A(x)$ and $B(x)$ are constant for $x \leqq c$ and non-decreasing for $x \geqq c$. Take $c = c^* = \sqrt{a/(3(1-a))}$ which satisfies (16) with $\alpha = \alpha^*$ and $\beta = \beta^*$. With this choice of c^* we must have $a < 3\sqrt{2}/4$ in order to have $B(0) > 0$. It is easily seen from (9)–(11) and (9)′–(11)′ that $\alpha^*(x)$, $\beta^*(y)$ and $\gamma^*(z)$ with $c = c^*$ satisfy the conditions (15).

(ii) Consider the case $A(0) = B(0) = 0$. Since $B(0) = 0$ and $A(1) > 0$, we have $c = 2a/(2a+3)$. It follows that $\int \beta^\circ(y)\,dy = 1$ for the equilibrium strategy $\beta^\circ(y)$ of player II since $A(0) = 0$.

We get then $\beta^\circ(y) \equiv 1$, $(0 \leqq y \leqq 1)$, and with this choice of $\beta(y)$ we obtain $A'(c) > 0$ and hence

$$\alpha^\circ(x) = \begin{cases} \text{arbitrary}, & 0 \leqq x \leqq c, \\ 1, & c < x \leqq 1, \end{cases}$$

for the equilibrium strategy of player I. The condition (16) for the optimal choice of c for III requires the constraint

$$\int_0^c \alpha^\circ(x)\,dx = \frac{1-c}{1+c}.$$

Moreover the obvious condition $0 \leqq \alpha^\circ(x) \leqq 1$ requires that $(1-c)/(1+c) \leqq c$ from which we get $a \geqq 3\sqrt{2}/4$. It is easily seen from (9)–(11) and (9)′–(11)′ that $\alpha^\circ(x)$, $\beta^\circ(y)$ and $\gamma^\circ(z)$ with $c = 2a/(2a+3)$ satisfy the equilibrium conditions (15).

(iii) We shall at last consider the case $A(0) > 0 = B(0)$. Since $B(0) = 0$ and $A(1) > 0$ we have $c = 2a/(2a+3)$. By $A(0) > 0$ we must have $\alpha^+(x) \equiv 1$ for the maximizing strategy of I. With this choice of $\alpha(x)$ we obtain $B'(c) > 0$ and hence

$$\beta^+(y) = \begin{cases} \text{arbitrary}, & 0 \leqq y \leqq c, \\ 1, & c < y \leqq 1, \end{cases}$$

for the equilibrium strategy of II. From the relation (16) we must have the constraint

$$\int_0^c \beta^+(y)\,dy = \frac{1-c}{1+c},$$

for c which corresponds to the optimal choice for player III. The obvious condition $0 \leqq \beta^+(y) \leqq 1$ requires that $(1-c)/(1+c) \leqq c$, i.e., $a \geqq 3\sqrt{2}/4$. It is easily seen from (9)–(11) and (9)′–(11)′ that $\alpha^+(x)$, $\beta^+(y)$ and $\gamma^+(z)$ with $c = 2a/(2a+3)$ satisfy the equilibrium conditions (15).

It should be added that the bordering case between (i) and (iii) corresponds to the case where $a = 3\sqrt{2}/4$. In this case the two strategy-points $(\alpha^*, \beta^*, \gamma^*)$ and $(\alpha^+, \beta^+, \gamma^+)$ coincide and we have

$$\begin{aligned} \alpha^*(x) &\equiv \alpha^+(x) \equiv 1, && 0 \leqq x \leqq 1, \\ \beta^*(y) &\equiv \beta^+(y) \equiv 1, && 0 \leqq y \leqq 1, \\ \gamma^*(z) &\equiv \gamma^+(z) = \begin{cases} 0, & 0 \leqq z \leqq \sqrt{2}-1, \\ 1, & \sqrt{2}-1 < z \leqq 1. \end{cases} \end{aligned}$$

The proof of the remaining parts of the theorem will be omitted. As to payoffs to three players corresponding to the three strategy-points $(\alpha^*, \beta^*, \gamma^*)$, $(\alpha^\circ, \beta^\circ, \gamma^\circ)$ and $(\alpha^+, \beta^+, \gamma^+)$, straight-forward calculations will yield the result. Thus we have completed the proof.

Theorems 1 and 2 show that player I stands unfavorable who must move first and inevitably gives information about his true hand to his opponent. Our Theorem 3 shows an interesting point. In this three-person game model player I must move first, player II moves next, and player III moves at last. Of course, the choices and preferences of the players are quite symmetrically defined in the game, and so, if they must move simultaneously values of the game to the players are (0, 0, 0). Theorem 3 shows that in this non-cooperative game model, players I and II symmetrically have an unfavorability against player III.

References

[1] KARLIN, S., AND R. RESTREPO, Multi-stage poker model. Contribution to the Theory of Games III (1957), 337–364.

[2] NASH, J., Non-cooperative games. Ann. of Math. 54 (1951), 286–295.

UNIVERSITY OF ELECTRO-COMMUNICATIONS,
CHŌFU, TOKYO.