ON A CORRESPONDENCE BETWEEN CLASSES OF ANALYTIC FUNCTIONS WITH POSITIVE REAL PART IN ANNULI

Dedicated to Professor Marston Morse

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1. Introduction.

The class of analytic functions regular in a circle and with the positive real part has been extensively investigated from various points of view. It has been shown to possess several interesting properties which are now mostly classical. A natural analogue of this class in the doubly-connected domain is the class consisting of analytic functions regular, single-valued and with the positive real part in an annulus. Many propositions on the former class have been transferred to the latter and the results thus obtained may be regarded as extensions of original ones.

Let $\mathfrak{R}=\{\Phi(z)\}$ denote the class consisting of analytic functions, regular in the unit circle $|z|<1$, satisfying

$$\Re\Phi(z)>0 \qquad \text{for} \quad |z|<1$$

and normalized by

$$\Phi(0)=1.$$

A straightforward extension of $\mathfrak{R}$ is the class $\mathfrak{R}_q=\{\Phi(z; q)\}$, depending on a parameter q with $0\leqq q<1$, which consists of analytic functions regular, single-valued in the annulus $q<|z|<1$, satisfying

$$\Re\Phi(z; q)>0 \qquad \text{for} \quad q<|z|<1$$

and normalized by

$$\Re\Phi(z; q)=1 \quad \text{along} \quad |z|=q \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(qe^{i\theta}; q)d\theta=1.$$

According to the above-mentioned normalization, any $\Phi(z; q)$ may be supposed to be continuous on the semi-closed annulus $q\leqq|z|<1$. Consequently, based on this boundary behavior, $\Phi(z; q)$ is moreover analytically prolongable beyond the circumference $|z|=q$ by means of the functional equation

$$\Phi(z; q)+\overline{\Phi\left(\frac{q^2}{\bar{z}}; q\right)}=2$$

so that it becomes a function regular throughout a wider annulus $q^2<|z|<1$. The class $\mathfrak{R}_0$ coincides with $\mathfrak{R}$. In fact, the origin is only a removable sin-

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gularity for every function $\Phi(z; 0) \in \mathfrak{R}_0$.

We could observe here more generally a wider class $\hat{\mathfrak{R}}_q = \{\hat{\Phi}(z; q)\}$ which is defined as an extension of $\mathfrak{R}_q$ by releasing its normalization on $|z| = q$ in such a way that the constant term in the Laurent expansion is equal to unity, i. e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\Phi}(re^{i\theta}; q)\, d\theta = 1 \qquad (q < r < 1).$$

However, the extended class $\hat{\mathfrak{R}}_q$ is related to the restricted class in a very simple way and can be really constructed in terms of the latter. In fact, as was shown in a previous paper [7], any $\hat{\Phi}(z; q) \in \hat{\mathfrak{R}}_q$ is expressible in the form

$$\hat{\Phi}(z; q) = \Phi(z; q) + \Psi\left(\frac{q}{z}; q\right) - 1; \qquad \Phi(z; q) \in \mathfrak{R}_q,\ \Psi(z; q) \in \mathfrak{R}_q$$

and moreover this decomposition is unique.

In the present paper, we shall establish a one-to-one correspondence between the classes $\mathfrak{R}_q$ and $\mathfrak{R}$ which will be found to be of remarkable nature. Functional equations determining this correspondence will be obtained in explicit forms. When a correspondence is established between the classes $\mathfrak{R}_q$ and $\mathfrak{R}$, it will be readily transferred to that between two classes with any values of the parameter q. In view of the decomposition theorem referred to above, it can be further transferred to those between a class $\hat{\mathfrak{R}}_q$ and a cross-class $(\mathfrak{R}_{q'}, \mathfrak{R}_{q''})$ as well as those between two classes $\hat{\mathfrak{R}}_{q'}$ and $\hat{\mathfrak{R}}_{q''}$ with any values of the associated parameter.

2. Correspondence in terms of series expansions.

To establish the desired correspondence between $\mathfrak{R}_q$ and $\mathfrak{R}$, we may choose, as the starting point, any definition of the correspondence among several possible ones which are equivalent each other. As we shall actually do so in the following lines, it seems rather intuitive to take as the starting definition, the definition which is formulated in terms of the power series expansions of functions of the respective classes. Thus, we begin with the following theorem.

THEOREM 1. *Let*

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$$

be any function belonging to the classs $\mathfrak{R}$. Then, the function defined by

$$\Phi(z; q) = 1 + \sum_{n=1}^{\infty} \frac{1}{1-q^{2n}}(C_n z^n - q^{2n}\bar{C}_n z^{-n})$$

belongs to the class $\mathfrak{R}_q$. Conversely, let

$$\Phi(z; q) = 1 + \sum_{n=-\infty}^{\infty}{}' C_n(q) z^n$$

be any function belonging to the class $\mathfrak{R}_q$ *where the prime attached to the summation symbol means the exception of the summand with* $n=0$. *Then, its Laurent coefficients satisfy the quasi-symmetry relation*

$$C_{-n}(q) = -q^{2n}\overline{C_n(q)} \qquad (n=1, 2, \cdots)$$

and the function defined by

$$\Phi(z) = 1 + \sum_{n=1}^{\infty}(1-q^{2n})C_n(q)z^n$$

belongs to the class $\mathfrak{R}$.

Proof. In order to justify the first part of the theorem, we note that the series defining $\Phi(z; q)$ converges in $q<|z|<1$ (and really in $q^2<|z|<1$) and hence it represents a function regular and single-valued in this annulus. We observe the difference

$$\Phi(z; q) - \Phi(z) = \sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}(C_n z^n - \bar{C}_n z^{-n}).$$

The last expression converges in the annulus $q^2<|z|<q^{-2}$ including the circumference $|z|=1$ in its interior and the value

$$\lim_{z\to e^{i\theta}}(\Phi(z; q) - \Phi(z)) = 2i\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\mathfrak{J}(C_n e^{in\theta}) \qquad (-\pi \leqq \theta < \pi)$$

is purely imaginary. On the other hand, the boundary value of $\Phi(z; q)$ along $|z|=q$ is given by

$$\Phi(qe^{i\theta}; q) = 1 + 2i\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\mathfrak{J}(C_n e^{in\theta}) \qquad (-\pi \leqq \theta < \pi)$$

so that $\mathfrak{R}\Phi(z; q)$ is equal to unity everywhere along $|z|=q$. Consequently, in view of the maximum principle, the real part of $\Phi(z; q)$ remains positive throughout $q<|z|<1$. Finally, it is evident that the mean value of $\Phi(z; q)$ along $|z|=q$ is equal to unity since this quantity is nothing but the constant term of its Laurent expansion.

We next consider the second part. The series defining $\Phi(z)$ converges evidently for $|z|<1$ and hence it represents a function regular there. Remembering the quasi-symmery property of the Laurent coefficients of $\Phi(z; q)$, we get the relation

$$\Phi(z) - \Phi(z; q) = -\sum_{n=1}^{\infty}q^{2n}(C_n(q)z^n - \overline{C_n(q)}z^{-n}).$$

The last expression converges in $q^2<|z|<q^{-2}$ and has the real part vanishing everywhere along $|z|=1$. Further we have $\Phi(0)=1$. Consequently, $\Phi(z)$ belongs to $\mathfrak{R}$, as claimed.

3. Correspondence in terms of integrals of Herglotz type.

The correspondence between the classes $\mathfrak{R}_q$ and $\mathfrak{R}$ formulated in Theorem 1 will become more clear if it is transformed into the form expressed in connection with integral representations of Herglotz type of the respective classes. These representations are classical, but for the sake of completeness we re-formulate them here as lemmas; cf., for instance, [4, 5, 6].

LEMMA 1. *It is necessary and sufficient for $\Phi(z) \in \mathfrak{R}$ that $\Phi(z)$ is representable by means of the Herglotz integral*

$$\Phi(z) = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\rho(\varphi)$$

where $\rho(\varphi)$ is a real-valued function satisfying

$$d\rho(\varphi) \geqq 0 \ (-\pi \leqq \varphi \leqq \pi) \quad \textit{and} \quad \int_{-\pi}^{\pi} d\rho(\varphi) = 1.$$

LEMMA 2. *It is necessary and sufficient for $\Phi(z; q) \in \mathfrak{R}_q$ that $\Phi(z; q)$ is representable by means of the integral of Villat-Stieltjes type*

$$\Phi(z; q) = \int_{-\pi}^{\pi} \frac{2}{i}\left(\zeta(i \lg z + \varphi) - \frac{\eta_1}{\pi}(i \lg z + \varphi)\right) d\rho(\varphi)$$

where $\rho(\varphi)$ is a real-valued function satisfying

$$d\rho(\varphi) \geqq 0 \ (-\pi \leqq \varphi \leqq \pi) \quad \textit{and} \quad \int_{-\pi}^{\pi} d\rho(\varphi) = 1$$

and the elliptic zeta-function depends on the Weierstrassian theory with the primitive periods $2\omega_1 = 2\pi$ and $2\omega_3 = -2i \lg q$.

Now, the Lemmas 1 and 2 are supplemented by the uniqueness assertion of the increasing function $\rho(\varphi)$ associated with a given analytic function of every class.

LEMMA 3. *The increasing function $\rho(\varphi)$ associated with a given function $\Phi(z; q) \in \mathfrak{R}_q$ as in Lemma 2 is substantially unique, i. e., for any given $\Phi(z; q) \in \mathfrak{R}_q$, $\rho(\varphi)$ is uniquely determined for $-\pi \leqq \varphi \leqq \pi$ under the normalization*

$$\rho(-\pi) = 0 \quad \textit{and} \quad \rho(\varphi) = \frac{1}{2}(\rho(\varphi - 0) + \rho(\varphi + 0)) \quad \textit{for} \quad -\pi < \varphi < \pi.$$

Moreover, it is expressed by the limit equation

$$\rho(\varphi) = \lim_{r \to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\varphi} \Re\Phi(re^{i\theta}; q)\, d\theta \qquad (-\pi \leqq \varphi \leqq \pi).$$

Proof. The proposition for the class $\mathfrak{R} = \mathfrak{R}_0$ has been fully proved in a previous paper [8]. Though the case of $\mathfrak{R}_q$ with any q can be dealt with quite similarly as in this particular case, it is more convenient to reduce the proof to this case. In fact, the integral representation stated in Lemma 2 implies

$$\Re\Phi(re^{i\theta}; q) = \int_{-\pi}^{\pi} \left(\frac{1 - r^2}{1 - 2r\cos(\psi - \theta) + r^2} + 2\sum_{\nu=1}^{\infty} \frac{q^{\nu}(r^{\nu} - r^{-\nu})}{1 - q^{2\nu}} \cos\nu(\psi - \theta)\right) d\rho(\psi);$$

cf. [6]. The infinite series involved in the integrand represents a function which is regular harmonic in $re^{i\theta}$ throughout the annulus $q < r < q^{-1}$ and which vanishes everywhere along $r = 1$, $-\pi \leqq \theta \leqq \pi$. Hence, we get

$$\lim_{r\to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\varphi} \Re\Phi(re^{i\theta}; q)\, d\theta = \lim_{r\to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\varphi} d\theta \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\psi - \theta) + r^2}\, d\rho(\psi),$$

provided that the limit exists, which is really the case. Namely, the right hand member of the last equation represents nothing but the corresponding expression for the particular case of the class $\Re$ and has been shown to be equal to $\rho(\varphi)$.

It should be noted that, for the particular class $\Re$, the uniqueness assertion formulated in Lemma 3 has been implicitly described also in [3] from the standpoint of the trigonometric moment problem.

Now, by means of the lemmas enumerated above, it follows from Theorem 1 the following proposition formulated in terms of the associated functions in the integral representations of the respective classes.

THEOREM 2. *The one-to-one correspondence between $\Re$ and $\Re_q$ introduced in Theorem* 1 *is of the nature such that the increasing functions $\rho(\varphi)$ associated with $\Phi(z) \in \Re$ and with $\Phi(z; q) \in \Re_q$ in Lemmas* 1 *and* 2 *respectively are the same.*

Proof. Let the Taylor expansion of $\Phi(z)$ and the Laurent expansion of $\Phi(z; q)$ defined with the same function $\rho(\varphi)$ in Lemma 1 and Lemma 2 respectively, be expressed by

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n,$$

$$\Phi(z; q) = 1 + \sum_{n=-\infty}^{\infty}{}' C_n(q) z^n.$$

Expanding the kernels of the respective integral representations, we get

$$C_n = 2\int_{-\pi}^{\pi} e^{-in\varphi} d\rho(\varphi) \qquad (n = 1, 2, \cdots),$$

$$C_n(q) = \frac{2}{1 - q^{2n}} \int_{-\pi}^{\pi} e^{-in\varphi} d\rho(\varphi) \qquad (n = \pm 1, \pm 2, \cdots);$$

cf. [6, 9]. Hence, it follows readily the relations

$$C_{-n}(q) = -q^{2n}\overline{C_n(q)} \qquad (n = 1, 2, \cdots)$$

and

$$C_n(q) = \frac{1}{1 - q^{2n}} C_n \qquad (n = 1, 2, \cdots).$$

This shows that $\Phi(z)$ and $\Phi(z; q)$ constitute a pair of corresponding functions in Theorem 1. In view of Lemma 3, the correspondence is one-to-one.

We notice here, by the way, that the variability-region of the Taylor coefficients for the class $\mathfrak{R}$ has been determined by Carathéodory [1, 2] and re-formulated by Rogosinski [10] and others. It is shown that this classical result can be briefly derived by means of the integral representation of the coefficients for $\mathfrak{R}$ which has been given in the proof of Theorem 2. The variability-region of the Laurent coefficients for the class $\mathfrak{R}_q$ can be also determined in a similar way and its actual determination has been carried out by Nishimiya [9]. Accordingly, Theorem 1 may be regarded as a statement expressing the correspondence between the coefficient-regions of $\{C_n\}_{n=1}^m$ for $\mathfrak{R}$ and $\{C_{-n}(q), C_n(q)\}_{n=1}^m$ for $\mathfrak{R}_q$ with any positive integer m. It further gives the correspondence between the respective functions bearing the corresponding coefficients in series form while Theorem 2 transforms this into the integral form of Herglotz type.

4. Equivalent functional equations defining the correspondence.

Now, the expressions determining the correspondence established in Theorem 1 or 2 can further be transformed in such a way that the resulting relations contain neither the coefficients of series expansions nor the functions $\rho(\varphi)$ in the integral representations. The transformed relations thus obtained will become a pair of functional equations connecting the corresponding functions of the respective classes. Each of the equations may be then regarded as the inversion formula of another. We shall give below two pairs of such functional equations, one in the present section and the other in the following.

THEOREM 3. *For any $\Phi(z)\in\mathfrak{R}$, the function defined by*

$$\Phi(z;\,q)=\Phi(z)+\sum_{\nu=1}^{\infty}(\Phi(q^{2\nu}z)-\overline{\Phi(q^{2\nu}\bar{z}^{-1})})$$

belongs to $\mathfrak{R}_q$. Conversely, for any $\Phi(z;\,q)\in\mathfrak{R}_q$, the function defined by

$$\Phi(z)=1+\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-z}-\frac{1}{t-q^2z}\right)dt$$

belongs to $\mathfrak{R}$, where Γ is any simple contour lying in thea nnulus $q<|t|<1$ and surrounding the origin as well as the points z and q^2z in the positive sense. Further, the correspondence thus determined is the same as that of Theorem 1 or 2 so that each of the equations represents the inversion of the another.

Proof. It suffices to justify the final part of the theorem. First, substituting the Taylor expansion of $\Phi(z)$ given in Theorem 1 into the first equation of the present theorem, we get

$$\begin{aligned}\Phi(z;\,q)&=1+\sum_{n=1}^{\infty}C_nz^n+\sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty}(C_nq^{2n\nu}z^n-\bar{C}_nq^{2n\nu}z^{-n})\\&=1+\sum_{n=1}^{\infty}\left(C_nz^n\sum_{\nu=0}^{\infty}q^{2n\nu}-\bar{C}_nz^{-n}\sum_{\nu=1}^{\infty}q^{2n\nu}\right)\\&=1+\sum_{n=1}^{\infty}\frac{1}{1-q^{2n}}(C_nz^n-q^{2n}\bar{C}_nz^{-n}).\end{aligned}$$

The last expression coincides with the Laurent expansion of $\Phi(z; q)$ given in Theorem 1. Next, substituting the Laurent expansion of $\Phi(z; q)$ given in the theorem and reproduced just above into the second equation of the present theorem, we find, by means of a simple calculation of residues,

$$\begin{aligned}\Phi(z) &= 1+\frac{1}{2\pi i}\int_{\Gamma}\Big(1+\sum_{n=-\infty}^{\infty}{}' C_n(q)t^n\Big)\Big(\frac{1}{t-z}-\frac{1}{t-q^2z}\Big)dt \\ &= 1+\sum_{n=1}^{\infty} C_n(q)(z^n-q^{2n}z^n) \\ &= 1+\sum_{n=1}^{\infty}(1-q^{2n})C_n(q)z^n.\end{aligned}$$

The last expression coincides with the Taylor expansion of $\Phi(z)$ given in Theorem 1.

The proof of Theorem 3 given just above depends on the intermediation of Theorem 1 in order to verify the coincidence of the correspondence with that introduced there. However, if the assertion of this coincidence is excluded, the remaining part of Theorem 3 together with the one-to-one correspondence can be proved also independently of Theorem 1 or 2 and directly as in the proof of the theorem.

To verify this, we first observe the boundary behaviors of the referring functions. The first equation of Theorem 3 shows that for any $\Phi(z)\in\mathfrak{R}$ the function $\Phi(z; q)$ thus defined is regular and single-valued in $q<|z|<1$ (and moreover in $q^2<|z|<1$). We now observe the difference

$$\Phi(z; q)-\Phi(z)=\sum_{\nu=1}^{\infty}(\Phi(q^{2\nu}z)-\overline{\Phi(q^{2\nu}\bar{z}^{-1})}).$$

The last series converges in the annulus $q^2<|z|<q^{-2}$. Its value on $|z|=1$ is given by

$$\lim_{z\to e^{i\theta}}(\Phi(z; q)-\Phi(z))=2i\sum_{\nu=1}^{\infty}\mathfrak{J}\Phi(q^{2\nu}e^{i\theta}) \qquad (-\pi\leqq\theta<\pi)$$

which is purely imaginary everywhere along $|z|=1$. On the other hand, the boundary value of $\Phi(z; q)$ along $|z|=q$ is given by

$$\Phi(qe^{i\theta}; q)=1+2i\sum_{\nu=1}^{\infty}\mathfrak{J}\Phi(q^{2\nu-1}e^{i\theta}) \qquad (-\pi\leqq\theta<\pi)$$

and hence its real part is always equal to unity. Consequently, $\mathfrak{R}\Phi(z; q)$ remains positive throughout the annulus $q<|z|<1$. Finally, the last relation readily implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(qe^{i\theta}; q)d\theta=1+2i\sum_{\nu=1}^{\infty}\mathfrak{J}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(q^{2\nu-1}e^{i\theta})d\theta=1,$$

so that $\Phi(z; q)$ belongs to the class $\mathfrak{R}_q$.

Next, for any $\Phi(z; q)\in\mathfrak{R}_q$, the second equation of Theorem 3 defines a function $\Phi(z)$ regular in $|z|<1$. Since $\Phi(z; q)$ is regular and single-valued for $q\leqq|z|<1$, we have by Cauchy's integral formula

$$\Phi(z;q)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi(t;q)}{t-z}dt-\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-z}dt,$$

every integral being taken in the positive sense with respect to the interior of respective contour. Hence, we get

$$\Phi(z)-\Phi(z;q)=1-\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi(t;q)}{t-q^2z}dt+\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-z}dt.$$

It is evident that in the first integral of the last expression the contour Γ may be replaced by the circumference $|t|=q$. In view of the functional equation satisfied by $\Phi(z;q)$, the last integral is further transformed. In fact, remembering that along $|t|=q$ we have $t=q^2\bar{t}^{-1}$ and $dt=-q^2\bar{t}^{-2}d\bar{t}$, it becomes

$$\begin{aligned}\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-z}dt&=\frac{1}{2\pi i}\int_{|t|=q}\frac{2-\overline{\Phi(q^2/\bar{t};q)}}{t-z}dt\\&=\frac{1}{2\pi i}\int_{|t|=q}\frac{\overline{\Phi(t;q)}}{q^2\bar{t}^{-1}-z}q^2\bar{t}^{-2}d\bar{t}\\&=-\overline{\frac{1}{2\pi i}\int_{|t|=q}\Phi(t;q)\left(\frac{1}{t}-\frac{1}{t-q^2\bar{z}^{-1}}\right)dt}\\&=-1+\overline{\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-q^2\bar{z}^{-1}}dt}.\end{aligned}$$

Thus, we get the relation

$$\Phi(z)-\Phi(z;q)=-\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-q^2z}dt+\overline{\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-q^2\bar{z}^{-1}}dt}.$$

The right hand member of the last relation represents a function regular and single-valued in $q<|z|<q^{-1}$ (and further in $q^2<|z|<q^{-2}$). Its value on $|z|=1$ is given by

$$\lim_{z\to e^{i\theta}}(\Phi(z)-\Phi(z;q))=-2i\Im\left(\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;q)}{t-q^2e^{i\theta}}dt\right)$$

which is purely imaginary everywhere. It is evident that $\Phi(0)=1$. We thus conclude that the real part of $\Phi(z)$ remains positive in $|z|<1$ and hence this function belongs to $\mathfrak{R}$.

Finally, we shall verify that the equations given in Theorem 3 are mutually inverse. We first show that the right hand member of the second equation after substituting the first equation reproduces the function $\Phi(z)$. For this purpose, we note that by Cauchy's integral formula it follows the relation

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi(q^{2\nu}t)}{t-z}dt=\Phi(q^{2\nu}z)\qquad(\nu=0,1,\cdots),$$

and further, since $\overline{\Phi(q^{2\nu}\bar{t}^{-1})}$ $(\nu\geqq 1)$ is regular in $|t|<q^2$,

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{\Phi(q^{2\nu}\bar{t}^{-1})}}{t-z}dt=-[\overline{\Phi(q^{2\nu}\bar{t}^{-1})}]^{t=\infty}=-\overline{\Phi(0)}=-1\quad(\nu=1,2,\cdots).$$

Hence, we get for $|z|<1$

$$\frac{1}{2\pi i}\int_{\Gamma}\left(\Phi(t)+\sum_{\nu=1}^{\infty}(\Phi(q^{2\nu}t)-\overline{\Phi(q^{2\nu}\bar{t}^{-1})})\right)\frac{dt}{t-z}$$
$$=\Phi(z)+\sum_{\nu=1}^{\infty}(\Phi(q^{2\nu}z)-1).$$

Consequently, we obtain the equation

$$1+\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-z}-\frac{1}{t-q^2z}\right)dt$$
$$=1+\Phi(z)+\sum_{\nu=1}^{\infty}(\Phi(q^{2\nu}z)-1)-\left(\Phi(q^2z)+\sum_{\nu=1}^{\infty}(\Phi(q^{2\nu+2}z)-1)\right)$$
$$=\Phi(z),$$

which is the desired one. In order to show the converse, we substitute the second equation of Theorem 3 into the right hand member of the first equation. It then becomes

$$1+\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-z}-\frac{1}{t-q^2z}\right)dt$$
$$+\sum_{\nu=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-q^{2\nu}z}-\frac{1}{t-q^{2\nu+2}z}\right)dt\right.$$
$$\left.-\overline{\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-q^{2\nu}\bar{z}^{-1}}-\frac{1}{t-q^{2\nu+2}\bar{z}^{-1}}\right)dt}\right)$$
$$=1+\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-z}-\frac{1}{t-q^2z}\right)dt$$
$$+\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-q^2z}-\frac{1}{t}\right)dt-\overline{\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-q^2\bar{z}^{-1}}-\frac{1}{t}\right)dt}$$
$$=\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi(t;q)}{t-z}dt-\overline{\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-q^2\bar{z}^{-1}}-\frac{1}{t}\right)dt}.$$

For $q<|z|<1$ we have $q^2<|q^2\bar{z}^{-1}|<q$, and hence the contour Γ of the last integral may be replaced by the circumference $|t|=q$, from which it follows by remembering again the previous calculation the relation

$$\overline{\frac{1}{2\pi i}\int_{\Gamma}\Phi(t;\,q)\left(\frac{1}{t-q^2\bar{z}^{-1}}-\frac{1}{t}\right)dt}$$
$$=\overline{\frac{1}{2\pi i}\int_{|t|=q}\Phi(t;\,q)\left(\frac{1}{t-q^2\bar{z}^{-1}}-\frac{1}{t}\right)dt}=\frac{1}{2\pi i}\int_{|t|=q}\frac{\Phi(t;\,q)}{t-z}dt.$$

Consequently, in view of the Cauchy's integral formula, the above expression is really equal to $\Phi(z;\,q)$ what is to be proved.

5. Another form of functional equations.

While the correspondence introduced in Theorem 1 or 2 has been transformed into the form as stated in Theorem 3, it is possible to transform it into another alternative form. We supplement here a form of this nature.

THEOREM 4. *For any $\Phi(z) \in \mathfrak{R}$, the function defined by*

$$\Phi(z;\, q) = \Phi(z) + \frac{1}{2\pi i}\int_{\Gamma} \Phi(t)A(t,\, z)\,dt - \overline{\frac{1}{2\pi i}\int_{\Gamma} \Phi(t)A(t,\, \bar{z}^{-1})\,dt}$$

belongs to the class $\mathfrak{R}_q$, where Γ is any simple contour lying in the annulus $q < |t| < 1$ and surrounding the origin in the positive sense, and $A(t, z)$ is defined by

$$A(t,\, z) = \sum_{n=1}^{\infty} \frac{q^{2n}z^n}{(1-q^{2n})t^{n+1}} \equiv \frac{1}{2t}\left(\frac{1}{2\pi i}\int_{\gamma} \frac{\Phi^*(s;\, q)}{s - q^2 z t^{-1}}\,ds - 1\right),$$

$$\Phi^*(s;\, q) = \frac{2}{i}\left(\zeta(i \lg s) - \frac{\eta_1}{\pi}\lg s\right) \equiv 1 + 2\sum_{n=-\infty}^{\infty}{}' \frac{s^n}{1-q^{2n}},$$

γ being any simple contour lying in $q < |s| < 1$ and surrounding the origin as well as the point $q^2 z t^{-1}$. Conversely, for any $\Phi(z;\, q) \in \mathfrak{R}_q$, the function defined by

$$\Phi(z) = \Phi(z;\, q) - \frac{1}{2\pi i}\int_{|t|=q} \frac{\Phi(t;\, q)}{t - q^2 z}\,dt + \overline{\frac{1}{2\pi i}\int_{|t|=q} \frac{\Phi(t;\, q)}{t - q^2 \bar{z}^{-1}}\,dt}$$

is analytically continuable throughout $|z| < 1$ and belongs to the class $\mathfrak{R}$. Further, the correspondence between two classes thus determined is the same as in either of Theorems 1, 2 and 3 so that each of the equations again represents the inversion of another.

Proof. It suffices to justify the final part of the theorem. For this purpose, we first show that the first equation of Theorem 3 can be transferred into the first equation of the present theorem. In fact, based on the Cauchy's integral formula, the former is written in the form

$$\Phi(z;\, q) = \Phi(z) + \sum_{\nu=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma} \frac{\Phi(t)}{t - q^{2\nu}z}\,dt - \overline{\frac{1}{2\pi i}\int_{\Gamma} \frac{\Phi(t)}{t - q^{2\nu}\bar{z}^{-1}}\,dt}\right).$$

Now, in view of $\Phi(0) = 1$, we have

$$\sum_{\nu=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma} \frac{\Phi(t)}{t - q^{2\nu}z}\,dt - 1\right) = \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{\Gamma} \Phi(t) \sum_{n=1}^{\infty} \frac{q^{2n\nu}z^n}{t^{n+1}}\,dt$$

$$= \frac{1}{2\pi i}\int_{\Gamma} \Phi(t) \sum_{n=1}^{\infty} \frac{q^{2n}z^n}{(1-q^{2n})t^{n+1}}\,dt = \frac{1}{2\pi i}\int_{\Gamma} \Phi(t)A(t,\, z)\,dt.$$

Putting $\bar{z}^{-1}$ instead of z in this relation, we get correspondingly

$$\sum_{\nu=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma}\frac{\Phi(t)}{t-q^{2\nu}\bar{z}^{-1}}dt-1\right)=\frac{1}{2\pi i}\int_{\Gamma}\Phi(t)A(t,\bar{z}^{-1})dt.$$

Combining these relations, it follows readily the desired equation.

Similarly, the second equation of Theorem 3 is transferred into the second equation of the present theorem. This fact has been, however, already shown subsequently to the proof of Theorem 3.

Of course, we may supplement here a remark similar to that stated subsequently to the proof of Theorem 3. But, since it proceeds quite similarly as before and really it is implicitly involved in the arguments until now, the details will be left to the reader.

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