NOTES ON TRANSLATIONS OF A SEMIGROUP

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In the present paper we shall correct mistakes and incompleteness in the previous paper [1], and shall discuss the results again and develop them. Further the main purpose of this paper is to investigate the structure of a semigroup whose right translation semigroup is a group or a semilattice. We express many thanks to Prof. G. B. Preston and Mr. Kaichiro Fujiwara for their kind advice and their pointing out our mistakes. We shall retain the notations in the previous paper.

1. Corrections and Addenda.

1. Corrections to the theorems of the previous paper.

In Theorems 4, 4′ of the paper [1] pp. 68–69, *we assume* S *to satisfy* $S^2=S$. Read "*isomorphic*" for "*homomorphic*", line 15, right, from the bottom, p. 68, and line 4, left, p. 69.

We devide the theorems into two cases and describe them again.

THEOREM $\bar{4}$. ($\bar{4}'$.) *The conditions* (2) and (3) *are equivalent, and* (2′) *and* (3′) *are equivalent.* (*We may provide no condition for* S.)

(2) S *contains a right unit,* (3) $\Phi=R$;

(2′) S *contains a left unit,* (3′) $\Psi=L$.

THEOREM 4. (4′.) *Let* S *be a semigroup which satisfies* $S^2=S$. *The conditions* (1) *and* (2) *are equivalent,* (1′) *and* (2′) *are equivalent.*

(1) Φ *is isomorphic to* R, (2) S *contains a right unit;*

(1′) Ψ *is isomorphic to* L, (2′) S *contains a left unit.*

Theorem 5 is valid for S having no condition. We shall rewrite the theorem:

THEOREM 5. $\Phi(\Psi)$ *is dually isomorphic* (*isomorphic*) *to* S *if and only if* S *has a two-sided unit.*

In its proof, line 14, left, p. 69, read "*dually isomorphic*" for "*isomorphic*".

The converse of Theorem 9 is not generally true. In Theorem 9, line 7,

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left, p. 70, read "*Furthermore, if S has a right zero* 0 *and if* $xy=y$ *implies* $y=0$, *then the converse of the theorem is true*" for "*Furthermore the converse is true*". We shall describe Theorem 9 again:

THEOREM 9. *If* S *is a semigroup with multiplication* $xy=0$ *for every* x *and* $y\in S$, *then* $\Phi=\Psi$, *and* Φ *is composed of all mappings* φ *of* S *into* S *satisfying* $\varphi(0)=0$. *Conversely, let* S *be a semigroup which has a right zero* 0 *and in which* $xy=y$ *implies* $y=0$. *If* Φ *is composed of all the mappings* φ *of* S *into* S *satisfying* $\varphi(0)=0$, *then* S *is a semigroup defined as* $xy=0$ *for every* x *and* $y\in S$.

2. Addenda and remark.

Addendum to the proof of Theorem 4. Since Φ contains a two-sided unit i.e. an identical mapping ε, R also contains the isomorphic image f_e of ε: $f_ef_x=f_xf_e=f_x$ for every $x\in S$. This f_e is proved to be an identical mapping of S in the following way. Since $S^2=S$, any element z is written as $z=xy$ or $z=f_y(x)$ for some x and y. Then we have

$$f_e(z)=f_ef_y(x)=f_y(x)=z \text{ for any } z\in S,$$

in other words $ze=z$; e is a right unit of S.

Addendum to the proof of the converse of Theorem 9. At first we easily see that 0 is a unique right zero. Moreover 0 is proved to be a left zero at the same time in the following manner. Let $u=0y$. Then $0u=0(0y)=(00)y=0y=u$, concluding $u=0$ by the assumption. Hence $0y=0$ for all $y\in S$. Now let us prove that $xy=0$ for every x, $y\in S$. Without loss of generality we may assume that S contains three elements at least. For, if S is of order 2, then S is nothing but $S=\{0, a\}$ where $0^2=0a=a0=a^2=0$, and so the theorem holds. Suppose that the proposition is not true, then there are x_0, y_0 such that $x_0y_0=z_0\neq 0$ where $x_0\neq 0$, $y_0\neq 0$, and $y_0\neq z_0$ are concluded. Utilizing the mapping φ of S into S satisfying

$$\varphi(0)=0, \qquad \varphi(y_0)=y_0, \qquad \varphi(z_0)=0,$$

we can prove the proposition as the previous paper shows.

THEOREM $\bar{\bar{4}}$. *Theorems* 4, 4′ *hold even if* S *is finite.*

Proof. (1) → (2). Considering the construction method of all the translations [2], it follows that if S is finite, Φ is also so. Since Φ is isomorphic to the subsemigroup R of Φ, Φ must coincide with R: $\Phi=R$. By Theorem $\bar{4}$, we obtain (2). (2) → (1) is clear by Theorem $\bar{4}$.

REMARK. If S is neither finite nor $S^2=S$, then Theorems 4 and 4′ are not necessarily true as the following counter-example shows.

EXAMPLE. S is the set of all non-negative integers with an element p adjoined: $S=S_0\cup\{p\}$ where $S_0=\{0, 1, \cdots\}$. The multiplication of S is defined as

$$\begin{cases} ij=\max(i, j), & i, j\in S_0, \\ pi=ip=i, & \\ pp=0. & \end{cases}$$

S is proved to be a commutative semigroup, $S\neq S^2$, and so it has no unit. Since the right base of S is $\{p\}$, Φ is composed of the identical mappings φ_p and φ_i, $i\in S_0$, as follows (cf. [2]):

$$\varphi_i(x)=\begin{cases} \max(i, x) & \text{for} \quad x\in S_0, \\ i & \text{for} \quad x=p, \end{cases} \qquad i=0, 1, \cdots,$$

$$\varphi_p(x)=x \qquad \text{for} \quad x\in S.$$

Obviously all φ_i form R: $R=\{\varphi_i;\ i\in S_0\}$, where the correspondence $\varphi_i\leftrightarrow i$ between R and S_0 is one-to-one and satisfies $\varphi_i\varphi_j=\varphi_{ji}=\varphi_{ij}$. In fact,

$$\varphi_i\varphi_j(x)=\begin{cases} \varphi_i(\max(j, x))=\max(i, j, x) & \text{for} \quad x\in S_0, \\ \varphi_i(j)=\max(i, j) & \text{for} \quad x=p, \end{cases}$$

and

$$\varphi_{ji}(x)=\varphi_{ij}(x)=\begin{cases} \max(ji, x)=\max(i, j, x) & \text{for} \quad x\in S_0, \\ ji=\max(i, j) & \text{for} \quad x=p. \end{cases}$$

Hence the correspondence $\varphi_i\leftrightarrow i$ is an isomorphism.

Consider a mapping of Φ to R as the following manner:

$$\varphi\to\varphi_0, \qquad \varphi_i\to\varphi_{i+1}, \qquad i\in S_0. \tag{1.1}$$

Then, for $i, j\in S_0$, $\varphi_i\varphi_j=\varphi_{ji}=\varphi_{\max(i, j)}$ is mapped to $\varphi_{i+1}\varphi_{j+1}=\varphi_{\max(i+1, j+1)}=\varphi_{\max(i,j)+1}$; and $\varphi_i\varphi=\varphi\varphi_i=\varphi_i$ is mapped to $\varphi_{i+1}\varphi_0=\varphi_0\varphi_{i+1}=\varphi_{\max(i+1, 0)}=\varphi_{i+1}$; moreover $\varphi^2=\varphi$ is mapped to $\varphi_0^2=\varphi_{\max(0, 0)}=\varphi_0$. Hence the mapping (1.1) is an isomorphism.

§2. Extension of Theorem 9.

Let S_0 be a proper subset of a semigroup S, Δ be certain set of some mappings of S_0 into S_0, and Γ be the set of all the mappings of $S-S_0$ into S. For $\delta\in\Delta$ and $\gamma\in\Gamma$, we define φ as follows:

$$\varphi(x)=\begin{cases} \delta(x), & x\in S_0, \\ \gamma(x), & x\in S-S_0; \end{cases}$$

this φ is denoted by $\delta*\gamma$. We denote by $\Delta*\Gamma$ the set of all $\delta*\gamma$: $\Delta*\Gamma=\{\delta*\gamma;\ \delta\in\Delta,\ \gamma\in\Gamma\}$.

As the extension of Theorem 9 in [1], we shall determine the structure of a semigroup S whose right translation semigroup Φ satisfies each of the following conditions.

Condition I. *There are S_0 and Δ such that $\Phi=\Delta*\Gamma$.*

Condition II. *Φ is composed of all the mappings φ of S into S which fix S_0 elementwise.*

Condition III. *Φ is composed of all the mappings φ of S into S such that $\varphi(S_0)\subset S_0$.*

Condition IV. *Φ is composed of all the mappings φ of S into S which fix an element a.*

Firstly we shall investigate the structure of a semigroup S which satisfies Condition I.

LEMMA 2.1. *$S_0S\subset S_0$ and hence S_0 is a subsemigroup of S.*

Proof. Consider every inner right translation $f_x\in\Phi$. For $a\in S_0$, $x\in S$, we get $ax=f_x(a)\in f_x(S_0)\subset S_0$.

LEMMA 2.2. *Δ is a subsemigroup of the right translation semigroup of S_0,* and *Δ contains an identical mapping ε_0.*

Proof. Δ is considered as the set of all the contractions of all $\varphi\in\Phi$ to S_0. If δ_1 and δ_2 are contractions of φ_1 and φ_2 respectively, then it is easily shown that $\delta_1\delta_2$ is a contraction of $\varphi_1\varphi_2$, and hence $\delta_1\in\Delta$ and $\delta_2\in\Delta$ imply $\delta_1\delta_2\in\Delta$. Since the identical mapping ε of S belongs to Δ, Δ contains the identical mapping ε_0 of S_0.

Denote by T_0 the set of all x such that $xS\subset S_0$: $T_0=\{x;\ xS\subset S_0\}$. By Lemma 2.1, $S_0\subset T_0$.

LEMMA 2.3. *If $x\in T_0$, then xz is independent of z.*

Proof. Let y_0 be a fixed element of $S-S_0$. For $z\in S$, we can find a right translation φ_z such that

$$\begin{cases}\varphi_z(y)=y & \text{if}\quad y\in S_0,\\ \varphi_z(y_0)=z.\end{cases}$$

Then we have $xz=x\varphi_z(y_0)=\varphi_z(xy_0)=xy_0$ since $xy_0\in S_0$.

In virtue of this lemma, we can denote

$$xz=\alpha(x)\qquad\text{for any}\quad x\in T_0, \tag{2.1}$$

where α is a mapping of T_0 into S_0 and we derive

$$(2.2)\qquad \alpha^2(x)=\alpha(x)$$

from the associative law $(xy)z=x(yz)$ for $x\in T_0$.

LEMMA 2.4. *$xy\overline{\in}S_0$ implies $y\overline{\in}S_0$.*

Proof. Consider $\varphi_1\in\Phi$ such that

$$\varphi_1(u)=u \qquad \text{if} \quad u\in S_0,$$
$$\varphi_1(u)\in S_0 \qquad \text{if} \quad u\overline{\in}S_0.$$

If we suppose that $y\in S_0$, then $xy=x\varphi_1(y)=\varphi_1(xy)\in S_0$, contradicting $xy\overline{\in}S_0$. q.e.d.

LEMMA 2.5. *$x\overline{\in}T_0$ implies $xz=z$.*

Proof. Since $x\overline{\in}T_0$, there exists y_0 such that $xy_0\overline{\in}S_0$ and so $y_0\overline{\in}S_0$ by Lemma 2.4. For any $z\in S$, we consider $\varphi_2\in\Phi$ such that $\varphi_2(xy_0)=\varphi_2(y_0)=z$. Then we have $xz=x\varphi_2(y_0)=\varphi_2(xy_0)=z$.

Denote by S_1 the set of all $\alpha(x)$, for $x\in T_0$, i.e. $S_1=T_0S$. Clearly $S_1\subset S_0$. Combining the above lemmas, it follows that S has the following multiplication:

$$(2.3)\qquad xy=\begin{cases}\alpha(x) & x\in T_0,\\ y & x\overline{\in}T_0\end{cases}$$

where α is a mapping of T_0 onto S_1 such that $\alpha^2=\alpha$ i.e. $\alpha(a)=a$ for $a\in S_1$.

Conversely, the multiplication defined by (2.3) is associative, for

$$(xy)z=\alpha(x)z=\alpha^2(x)=\alpha(x)=x(yz) \qquad \text{if} \quad x\in T_0,$$
$$(xy)z=yz=x(yz) \qquad \text{if} \quad x\overline{\in}T_0.$$

LEMMA 2.6. *In a semigroup S with multiplication (2.3), the right translation semigroup Φ of S is composed of all the mappings of S into S which fix S_1 elementwise.*

Proof. At first, we shall show that $\varphi\in\Phi$ implies $\varphi(u)=u$ for $u\in S_1$. Since $u=\alpha(x)=xz$, $\varphi(u)=\varphi(\alpha(x))=\varphi(xz)=x\varphi(z)=\alpha(x)=u$. Next, we shall prove that a mapping of S into S, which fixes S_1 elementwise, is a right translation of S. If $x\in T_0$, then $\varphi(xy)=\varphi(\alpha(x))=\alpha(x)=x\varphi(y)$; and if $x\overline{\in}T_0$ then $\varphi(xy)=\varphi(y)=x\varphi(y)$. q.e.d.

LEMMA 2.7. *If Φ satisfies Condition I, then $S_1=S_0$; hence $\alpha(x)=x$ for $x\in S_0$ in other words, Δ is composed of only an identical mapping.*

Proof. Suppose $S_0-S_1\neq\phi$. Since Φ satisfies Condition I, S has a multiplication (2.3), and so Lemma 2.6 makes us to see that Φ contains a mapping which maps S_0-S_1 into $S-S_0$. This contradicts Condition I. Hence $S_1=S_0$. q.e.d.

At last (2.3) is reformed as

$$xy=\begin{cases}\alpha(x), & x\in T_0,\\ y, & x\overline{\in} T_0\end{cases} \tag{2.3'}$$

where α is a mapping of T_0 onto S_0 such that $\alpha^2=\alpha$ i.e. α fixes S_0 elementwise.

By Lemma 2.6, it is easily seen that a semigroup with a multiplication (2.3') satisfies Condition I. Thus we have

THEOREM 2.1. *The right translation semigroup Φ satisfies Condition I if and only if S has the following structure: There exist a subset T_0 containing S_0 and a mapping α of T_0 onto S_0 which fixes S_0 elementwise, and the multiplication in S is given by*

$$xy=\begin{cases}\alpha(x), & x\in T_0,\\ y, & x\in S-T_0.\end{cases} \tag{2.3'}$$

Therefore we have

COROLLARY 1. *Conditions I and II are equivalent.*

Next, let us investigate the structure of a semigroup whose Φ satisfies Condition III.

LEMMA 2.8. *S_0 is composed of only one element.*

Proof. We shall prove that $x\in S_0$, $x'\in S_0$, and $x\neq x'$ imply $\alpha(x)=\alpha(x')$. Let us consider $\varphi\in\Phi$ such that $\varphi(xy)=\varphi(x'y)\in S_0$. Then $\alpha(x)=x\varphi(y)=\varphi(xy)=\varphi(x'y)=x'\varphi(y)=\alpha(x')$.

COROLLARY 2. *Condition III is equivalent to Condition IV.*

As a special case of Theorem 2.1, we obtain

THEOREM 2.2. *A semigroup S whose Φ satisfies Condition IV has the structure: There exists a subset T_0 containing a, and the multiplication in S is given by*

$$xy=\begin{cases}a, & x\in T_0,\\ y, & x\in S-T_0.\end{cases} \tag{2.4}$$

Now Ψ denotes the left translation semigroup of S. We shall determine the structure of a semigroup S whose Φ and Ψ satisfy each of the following conditions.

Condition $\mathrm{I_a}$. *There are S_0 and Δ such that $\Phi=\Psi=\Delta*\Gamma$.*

Condition $\mathrm{II_a}$. *$\Phi=\Psi$ and it is composed of all the mappings of S into S which fix S_0 elementwise.*

Condition $\mathrm{III_a}$. *$\Phi=\Psi$ and it is composed of all the mappings of S into S such that S_0 is mapped into S_0.*

Condition $\mathrm{IV_a}$. *$\Phi=\Psi$ and it is composed of all the mappings of S into S which fix an element a.*

Let us consider the case of $\mathrm{I_a}$. S satisfies not only Theorem 2.1 but also the dual form of it: There exist a subset Q_0 containing S_0 and a mapping β of Q_0 onto S_0 which fixes S_0 elementwise and

$$xy=\begin{cases}\beta(y), & y\in Q_0,\\ x, & y\in S-Q_0.\end{cases} \tag{2.5}$$

Immediately we have

THEOREM 2.2. *The right translation semigroup Φ and the left translation semigroup Ψ satisfy Condition I_a if and only if S is either a semilattice of order 2 or a semigroup defined as $xy=0$ for all x, $y\in S$.*

Proof. If neither $S-T_0$ nor $S-Q_0$ is empty, let x_1 and y_1 be elements of $S-T_0$ and $S-Q_0$, respectively. For any $x_0\in T_0$, $y_0\in Q_0$, we get $x_0=x_0y_1=x_0y_0=x_1y_0=y_0$ by (2.3′) and (2.5), hence $T_0=Q_0$ and it is composed of only one element a. On the other hand, for any x_2 and $y_2\in S-T_0=S-Q_0$, we get $x_2=x_2y_2=y_2$ by (2.3′) and (2.5); hence $S-T_0$ is also composed of only one element b. Consequently $S=\{a, b\}$ where $a^2=ab=ba=a$, $b^2=b$. Next, if either $S-T_0$ or $S-Q_0$ is empty, for example, if $S-T_0=\phi$ then $S=T_0$ and, for every x, $y\in S$ and the fixed x', $y'\in Q_0$, we have $xy=xy'=x'y'$ which is independent of x and y; therefore S is a semigroup defined as $xy=0$ for every x, $y\in S$.

Hence Condition $\mathrm{I_a}$ is equivalent to Condition $\mathrm{IV_a}$ and at last we see

COROLLARY 3. *Conditions I_a, II_a, III_a and IV_a are equivalent.*

§3. The Case where Φ is a Group.

In this paragraph we shall investigate the structure of a semigroup S whose right translation semigroup Φ is a group.

THEOREM 3.1. *Φ is a group if and only if S is a left group.*

Proof. Assume that Φ is a group. Φ contains the identical mapping ε as a left unit of the group Φ. Then, for any $a\in S$, there exists $\varphi_a\in\Phi$ such that $f_a\varphi_a=\varphi_af_a=\varepsilon$. Now, for any $x\in S$ we have

$$x\varphi_a(a)=\varphi_a(xa)=\varphi_af_a(x)=\varepsilon(x)=x$$

and for any a, $x\in S$,

$$x=\varepsilon(x)=f_a\varphi_a(x)=\varphi_a(x)a.$$

Therefore S is a left group. Next, we shall prove the converse. Assume that S is a left group. Then by Theorem $\bar{4}$ in §1, we have $\Phi=R$. Denote by e a right unit of S. Evidently f_e is the two-sided unit of R. For $f_a\in R$, we have $f_af_{a'}=f_e$ where a' is an element such that $a'a=e$. Thus $\Phi=R$ is a group.

Easily we have

THEOREM 3.2. *Both the right translation semigroup Φ and the left translation semigroup Ψ are groups if and only if S is a group.*

Proof. According to Theorem 3.1, Φ and Ψ are groups if and only if S is a left group and a right group at the same time, namely S is a group.

§4. The Cases where Φ is a Semilattice.

In this paragraph we shall investigate the structure of a semigroup whose right translation semigroup Φ is a semilattice. If Φ is a semilattice, then the inner right tronslation semigroup $R=\{f_a;\ a\in S\}$ is a semilattice because it is a subsemigroup of Φ.

1. The Structure of a Semigroup S whose R is a Semilattice.

Suppose that the inner right translation semigroup R of S is a semilattice. We get easily

LEMMA 4.1. *R is a semilattice if and only if $xy=xy^2$ and $xyz=xzy$ for any $x,\ y,\ z\in S$.*

The homomorphism $S\ni a\to f_a\in R$ gives a semilattice-decomposition of S: $S=\sum_{\sigma\in R}S_\sigma$ where, for simplicity, we redenote by σ etc. the elements of R, and the multiplication in S_σ is given by

$$x_\sigma y_\sigma=\alpha_\sigma(x_\sigma), \tag{4.1}$$

α_σ being an idempotent mapping of S_σ into S_σ: $\alpha_\sigma^2=\alpha_\sigma$. We denote $x\sim y$ if $x\in S_\sigma$, $y\in S_\sigma$.

LEMMA 4.2. *By the homomorphism $a\to f_a$, the semigroup S is decomposed into the union of s-indecomposable semigroups S_σ defined by* (4.1).

Proof. Let $x\approx y$ be a congruence relation by which S_σ is decomposed to a semilattice. Let a be a fixed element of S_σ. Then, for any $x\in S_\sigma$, $x\approx x^2=xa\approx ax=\alpha_\sigma(a)$. Hence any s-decomposition of S_σ gathers all the elements into a class, i.e. S_σ is s-indecomposable.

THEOREM 4.1. *There are given a semilattice T and a system of s-indecomposable semigroups S_σ ($\sigma\in T$) defined by* (4.1). *If we construct a composition S of S_σ by T such that*

$$a,\ b\in S_\sigma \text{ implies } xa=xb \text{ for every } x\in S, \tag{4.2}$$

then the inner right translation semigroup R of S is a semilattice which is isomorphic to T. Any semigroup whose R is a semilattice is obtained as the above.

Proof. Let R be the inner right translation semigroup of S. We consider a decomposition δ_1: $S=\sum_{\tau\in R} S_\tau$, beside the decomposition δ_0: $S=\sum_{\sigma\in T} S_\sigma$. By the assumption, if a, $b\in S_\sigma$, then $f_a=f_b$ and so we get $\delta_1\leqq\delta_0$ [1], in other words, T is homomorphic to R, whence R is a semilattice. On the other hand, since each S_σ ($\sigma\in T$) is s-indecomposable by the assumption, and each S_τ ($\tau\in R$) is also s-indecomposable by Lemma 4.2, the decompositions δ_1 and δ_0 are greatest. Therefore $\delta_1=\delta_0$, and hence R is isomorphic to T. q.e.d.

In order to construct such a composition of S_σ by T, it is sufficient and necessary to find a system of mappings $\{f_\sigma^\tau;\ \sigma,\ \tau\in T\}$ where f_σ^τ is a mapping of S_σ into $S_{\sigma\tau}$ and satisfies

$$f_\sigma^\sigma=\alpha_\sigma, \tag{4.3}$$

$$f_\sigma^{\tau\lambda}=f_{\sigma\tau}^\lambda f_\sigma^\tau \tag{4.4}$$

and the multiplication in S is defined as

$$x_\sigma x_\tau=f_\sigma^\tau(x_\sigma) \qquad \text{for} \quad x_\sigma\in S_\sigma,\quad x_\tau\in S_\tau.$$

There exists in reality such a system $\{f_\sigma^\tau\}$ as the following examples show, but in this paper we cannot solve the problem how all the systems are determined.

EXAMPLE 1. Let μ be a non-zero-element of T, i.e. $\mu\in T^2$. Take an element p_μ in any $S_\mu^2\subset S_\mu$. For the system $\{p_\mu;\ \mu\in T^2\}$, and for any σ, $\tau\in T$, f_σ^τ is defined as

$$f_\sigma^\tau(x_\sigma)=\begin{cases}\alpha_\sigma(x_\sigma), & \sigma\geqq\tau,\\ p_{\sigma\tau}, & \sigma\not\geqq\tau.\end{cases}$$

Clearly $f_\sigma^\tau(p_\sigma)=p_{\sigma\tau}$. We shall prove that (4.4) is fulfilled. If $\sigma\geqq\tau\lambda$, then $\sigma\geqq\lambda$, $\sigma\geqq\tau$, and

$$f_\sigma^\lambda f_\sigma^\tau(x_\sigma)=f_\sigma^\lambda(\alpha_\sigma(x_\sigma))=\alpha_\sigma^2(x_\sigma)=\alpha_\sigma(x_\sigma)=f_\sigma^{\tau\lambda}(x_\sigma).$$

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1) $\delta_1\leqq\delta_0$ means that the decomposition δ_0 is a refinement of the decomposition δ_1. See [3].

If $\sigma \not\geqq \tau\lambda$ and $\sigma \geqq \tau$, then $\sigma \not\geqq \lambda$ and

$$f^{\lambda}_{\sigma} f^{\tau}_{\sigma}(x_\sigma) = f^{\lambda}_{\sigma}(f^{\tau}_{\sigma}(x_\sigma)) = p_{\sigma\lambda} = p_{\sigma(\tau\lambda)} = f^{\tau\lambda}_{\sigma}(x_\sigma).$$

If $\sigma \not\geqq \tau\lambda$ and $\sigma \not\geqq \tau$, then

$$f^{\lambda}_{\sigma\tau} f^{\tau}_{\sigma}(x_\sigma) = f^{\lambda}_{\sigma\tau}(p_{\sigma\tau}) = p_{(\sigma\tau)\lambda} = p_{\sigma(\tau\lambda)} = f^{\tau\lambda}_{\sigma}(x_\sigma).$$

EXAMPLE 2. For any $\mu \in T^2$, we take $p_\mu \in S^2_\mu$ and select β_μ of T such that $\beta_\mu < \mu$. For the systems $\{\beta_\mu;\ \mu \in T^2\}$ and $\{p_\mu;\ \mu \in T^2\}$, and for any $\sigma, \tau \in T$, we define f^{τ}_{σ} as follows:

$$f^{\tau}_{\sigma}(x_\sigma) = \begin{cases} x_\sigma & \text{if } \sigma > \beta_\sigma \geqq \tau, \\ \alpha_\sigma(x_\sigma) & \text{if } \sigma \geqq \tau, \quad \beta_\sigma \not\geqq \tau, \\ p_{\sigma\tau} & \text{otherwise.} \end{cases}$$

If $\sigma > \beta_\sigma \geqq \tau\lambda$, then $\sigma > \tau$, $\sigma\tau = \sigma > \lambda$ and so

$$f^{\lambda}_{\sigma} f^{\tau}_{\sigma}(x_\sigma) = f^{\lambda}_{\sigma}(x_\sigma) = x_\sigma = f^{\tau\lambda}_{\sigma}(x_\sigma).$$

If $\sigma \geqq \tau\lambda$, $\beta_\sigma \not\geqq \tau\lambda$, and either $\beta_\sigma \geqq \lambda$ or $\beta_\sigma \geqq \tau$, then

$$f^{\lambda}_{\sigma} f^{\tau}_{\sigma}(x_\sigma) = \begin{cases} f^{\lambda}_{\sigma}(\alpha_\sigma(x_\sigma)) = \alpha_\sigma(x_\sigma) = f^{\tau\lambda}_{\sigma}(x_\sigma) & \text{if } \beta_\sigma \geqq \lambda, \quad \beta_\sigma \not\geqq \tau, \\ f^{\lambda}_{\sigma}(x^\sigma) = \alpha_\sigma(x_0) = f^{\tau\lambda}_{\sigma}(x_\sigma) & \text{if } \beta_\sigma \geqq \tau, \quad \beta_\sigma \not\geqq \lambda. \end{cases}$$

If $\sigma \geqq \tau\lambda$, $\beta_\sigma \not\geqq \tau\lambda$, $\beta_\sigma \not\geqq \lambda$, and $\beta_\sigma \not\geqq \tau$, then

$$f^{\lambda}_{\sigma} f^{\tau}_{\sigma}(x_\sigma) = f^{\lambda}_{\sigma}(\alpha_\sigma(x_\sigma)) = \alpha^2_\sigma(x_\sigma) = \alpha_\sigma(x_\sigma) = f^{\tau\lambda}_{\sigma}(x_\sigma).$$

If $\sigma \not\geqq \tau\lambda$, $\sigma \geqq \tau$, then we get $\sigma \not\geqq \lambda$ and

$$f^{\lambda}_{\sigma} f^{\tau}_{\sigma}(x_\sigma) = f^{\lambda}_{\sigma}(f^{\tau}_{\sigma}(x_\sigma)) = p_{\sigma\lambda} = p_{\sigma(\tau\lambda)} = f^{\tau\lambda}_{\sigma}(x_\sigma).$$

If $\sigma \not\geqq \tau\lambda$ and $\sigma \not\geqq \tau$, then

$$f^{\lambda}_{\sigma\tau} f^{\tau}_{\sigma}(x_\sigma) = f^{\lambda}_{\sigma\tau}(p_{\sigma\tau}) = p_{(\sigma\tau)\lambda} = p_{\sigma(\tau\lambda)} = f^{\tau\lambda}_{\sigma}(x_\sigma).$$

Now the translation semigroup $\bar{\Phi}$ of the semilattice R is a semilattice. (See [4].) We shall relate to the relation between Φ and $\bar{\Phi}$ when R is a semilattice.

LEMMA 4.3. *$x \sim y$ implies $\varphi(x) \sim \varphi(y)$ for every $\varphi \in \Phi$.*

Proof. From $zx = zy$, we get

$$z\varphi(x) = \varphi(zx) = \varphi(zy) = z\varphi(y) \qquad \text{for all } z \in S,$$

and hence $\varphi(x) \sim \varphi(y)$.

For any $\varphi \in \Phi$, we define a mapping $\bar{\varphi}$ of R into R as follows:

$$\bar{\varphi}(\alpha)=\gamma \quad \text{means} \quad \sigma(\varphi(x))=\gamma \qquad \text{for} \quad x\in S_\alpha,$$

in which γ is determined independently from an element x by Lemma 4. 3. This $\bar{\varphi}$ is a translation of R, because we have

$$\bar{\varphi}(\alpha\beta)=\sigma(\varphi(yz))=\sigma(y\varphi(z))=\sigma(y)\sigma(\varphi(z))=\alpha\bar{\varphi}(\beta)$$

where $y\in S_\alpha$, $z\in S_\beta$, $yz\in S_{\alpha\beta}$. Next we shall show that $\varphi\rightarrow\bar{\varphi}$ is a homomorphism of Φ into $\bar{\Phi}$. For φ, $\psi\in\Phi$, $x\in S_\alpha$,

$$\bar{\psi}\bar{\varphi}(\alpha)=\sigma(\psi\varphi(x))=\sigma(\psi(\varphi(x)))=\bar{\psi}(\sigma\varphi(x))=\bar{\psi}\bar{\varphi}(\alpha).$$

Therefore we have

THEOREM 4. 2. *If R is a semilattice, then Φ is homomorphic into $\bar{\Phi}$ by the correspondence $\varphi\rightarrow\bar{\varphi}$.*

2. The Structure of a Semigroup S in Case that $S=S^2$ and Φ is a Semilattice.

THEOREM 4. 3. *Let $S=S^2$. The right translation semigroup Φ of S is a semilattice if and only if the inner right translation semigroup R is a semilattice.*

Proof. We shall prove only that Φ is a semilattice if R is a semilattice. For any $x\in S$, $x=yz$, we have, by Lemma 4. 1,

$$\varphi^2(x)=\varphi^2(yz)=\varphi^2(yz^2)=\varphi(yz\varphi(z))=\varphi(y\varphi(z)z)=y\varphi(z)\varphi(z)=y\varphi(z)=\varphi(yz)=\varphi(x)$$

whence $\varphi^2=\varphi$.

$$(\varphi\psi)(x)=\varphi\psi(yz)=\varphi\psi(yz^2)=\varphi(yz\psi(z))=\varphi(y\psi(z)z)=y\psi(z)\varphi(z).$$

Similarly $(\psi\varphi)(x)=\psi\varphi(yz)=y\varphi(z)\psi(z)=y\psi(z)\varphi(z)$. Hence $\varphi\psi=\psi\varphi$. q.e.d.

Consider the decomposition of S: $S=\sum_{\sigma\in R}S_\sigma$. Since $S=S^2$, we see that $S_\sigma=S_\sigma^2$ if σ is minimal in T. We are arbitrarily given semilattice T and a system of s-indecomposable semigroups S_σ $(\sigma\in T)$ defined by (4. 1) such that $S_\sigma=S_\sigma^2$ if σ is minimal. Then a semigroup S whose Φ is a semilattice is constructed as a composition of S_σ by T such that $S=S^2$ in addition to (4. 2). In order to construct S, it is necessary and sufficient to find $\{f_\sigma^\tau;\ \sigma,\ \tau\in T\}$ where f_σ^τ is a mapping of S_σ into $S_{\sigma\tau}$ such that in addition to (4. 3) and (4. 4),

$$\sum_{\sigma\geqq\tau\in T} f_\sigma^\tau(S_\sigma)=S_\sigma \qquad \text{for any} \quad \sigma\in T. \tag{4. 5}$$

For example, Example 2 shows that $\{f_\sigma^\tau\}$ exists.

By the way, we shall relate to the relation between Φ and $\bar{\Phi}$ in the case where R is a semilattice and $S=S^2$.

THEOREM 4.4. *If $S=S^2$ and R is a semilattice, then Φ is isomorphic onto the subsemilattice of $\bar{\Phi}$ by the correspondence $\varphi \rightarrow \bar{\varphi}$ (described in Theorem 4.2).*

Proof. In order to prove the isomorphism, we may show only that $\varphi \rightarrow \bar{\varphi}$ is one-to-one, or $\sigma(\varphi(x))=\sigma(\psi(x))$ for all $x \in S$ implies $\varphi(x)=\psi(x)$ for all $x \in S$. Since $S=S^2$, any x is decomposed into the product $x=yz$ for some y and z, and so $\varphi(x)=\varphi(yz)=y\varphi(z)=y\psi(z)=\psi(yz)=\psi(x)$. Since the isomorphic image of Φ is a subsemigroup of the semilattice $\bar{\Phi}$, it is a semilattice.

3. The Structure of a Semigroup S in Case that $S \neq S^2$ and Φ is a Semilattice.

LEMMA 4.4. *We assume $S \neq S^2$ and let $p \in S-S^2$. The class S_α which contains p is a semigroup of order 2 defined as $S_\alpha=\{p, q\}$ where $p^2=pq=qp=q^2=q$.*

Proof. Suppose that S_α contains two distinct elements q, r at least besides p. Consider two mappings φ_1 and φ_2 of S into S defined as follows:

$$\begin{cases} \varphi_1(p)=q, & \\ \varphi_1(x)=x & \text{if} \quad x \neq p; \end{cases} \qquad \begin{cases} \varphi_2(p)=r, & \\ \varphi_2(x)=x & \text{if} \quad x \neq p. \end{cases}$$

Since $p \in S-S^2$ and $p \sim q$, we get

$$\varphi_1(xy)=xy=x\varphi_1(y) \qquad \text{for } y \neq p \text{ and every } x,$$

$$\varphi_1(xp)=xp=xq=x\varphi_1(p) \qquad \text{for every } x.$$

Hence φ_1 is a right translation of S; and φ_2 is similarly proved to be also so. Then $\varphi_1\varphi_2 \neq \varphi_2\varphi_1$ is proved as follows:

$$\varphi_1\varphi_2(p)=\varphi_1(r)=r, \qquad \varphi_2\varphi_1(p)=\varphi_2(q)=q.$$

This contradicts the assumption that Φ is a semilattice. Therefore S_α contains one element at most besides p. Since $p \in (S-S^2)\frown S_\alpha$, we see $S_\alpha \neq S_\alpha^2$. Consequently it follows that S_α is nothing but a semigroup $S_\alpha=\{p, q\}$, where $p^2=pq=qp=q^2=q$.

LEMMA 4.5. *Let $S \neq S^2$ and let $p \in S-S^2$ and so $p \in S_\alpha=\{p, q\}$. (Cf. Lemma 4.4.) Furthermore suppose that $\alpha<\beta$ which shows the ordering in the semilattice R. Then S_β is composed of only one element.*

Proof. Suppose that S_β contains two elements a, b at least. We see that there is $a \in S_\beta$ such that $pa=a$. For, if $pa=b \neq a$, then $pb=pa$ because $a \sim b$; and then we can find b such that $pb=b$. Hence we may assume

$$f_a(p)=a \tag{4.6}$$

without loss of generality. Then since $f_a^2=f_a$ by the initial assumption,

(4. 7) $$f_a(a)=f_af_a(p)=f_a(p)=a.$$

Now, let us consider a mapping φ defined as

$$\varphi(x)=\begin{cases} f_a(x), & x\neq p, \\ b, & x=p. \end{cases}$$

This φ is proved to be a right translation of S:

$$\varphi(xy)=f_a(xy)=xf_a(y)=x\varphi(y) \qquad \text{for } y\neq p \text{ and any } x,$$

$$\varphi(xp)=f_a(xp)=xf_a(p)=xa=xb=x\varphi(p) \qquad \text{for any } x,$$

because we use $a\sim b$ and (4. 6). Since $\varphi^2=\varphi$ must hold,

(4. 8) $$f_a(b)=\varphi(b)=\varphi^2(p)=\varphi(p)=b.$$

On the other hand, we see $\varphi f_a\neq f_a\varphi$. In fact,

$$\varphi f_a(p)=\varphi(a)=f_a(a)=a \qquad \text{(by (4. 6), (4. 7))},$$

$$f_a\varphi(p)=f_a(b)=b \qquad \text{(by (4. 8))}.$$

This result contradicts the assumption that Φ is a semilattice. Thus the lemma has been completely proved.

LEMMA 4. 6. *$p, r\in S-S^2$ and $p\neq r$ imply $p\nsim r$. If $p\in S_\alpha$ and $r\in S_\beta$, then $\alpha\nleqq\beta$.*

Proof. Since neither p nor r lies in S^2, Lemma 4. 4. makes it impossible that p and r belong to the same S_α. Also $\alpha\nleqq\beta$ is immediately shown by the result of Lemma 4. 5.

Thus we have known that if $S^2\neq S$ and Φ is a semilattice, then R is a semilattice, and the decomposition $S=\sum_{\sigma\in R} S_\sigma$ of S is obtained where the subsemigroups S_σ satisfy Lemmas 4. 4, 4. 5, and 4. 6. On the other hand, the converse will hold. Hereafter we shall prove that if R is a semilattice and the classes S_σ in the decomposition $S=\sum_{\sigma\in R} S_\sigma$ satisfy Lemmas 4. 4, 4. 5 (and consequently 4. 6), then Φ is a semilattice.

Let $S-S^2=\{b_\lambda;\ \lambda\in R'\subset R\}$, and S_λ be the class which contains b_λ. By Lemma 4. 4,

(4. 9) $$S_\lambda=\{a_\lambda, b_\lambda\} \quad \text{where} \quad a_\lambda^2=a_\lambda b_\lambda=b_\lambda a_\lambda=b_\lambda^2=a_\lambda,$$

and there is a one-to-one correspondence between b_λ and S_λ. As easily seen,

(4. 10) $$xa_\lambda=a_\lambda x=xb_\lambda=b_\lambda x=\begin{cases} a_\lambda & \text{if } \sigma(x)\leqq\lambda, \\ u=u^2 \text{ where } \sigma(u)=\lambda\cdot\sigma(x) & \text{if } \sigma(x)\nleqq\lambda. \end{cases}$$

LEMMA 4.7. *Letting $S^*=S^2$, we have $S^{*2}=S^*$. If R is a semilattice, then the inner right translation semigroup R^* of S^* is a semilattice.*

Proof. We may show only $S^*\subset S^{*2}$. Letting any $x\in S^*=S^2$, $x=yz$ for some y, $z\in S$, $x=yz=yz^3=(yz)z^2$ by Lemma 4.1. Hence $S^*\subset S^{*2}$. By the assumption, R is a semilattice and so $f_s=f_{s^2}$,

$$\overline{R}=\{f_s;\ s\in S^*\}=\{f_s;\ s\in S\}=R,$$

hence $\overline{R}$ is also a semilattice. On the other hand, since any inner right translation f_s^* of S^* is a contraction of $f_s\in\overline{R}$, $\overline{R}$ is homomorphic to R^*. Consequently R^* is a semilattice.

LEMMA 4.8. *Let φ be a right translation of a semigroup M, and let N be a subsemigroup of M. If $\varphi(N)\subset N$, then the contraction of φ to N is a right translation of N.*

Let φ be a right translation of S and let φ^* be the contraction of φ to $S^*=S^2$. Since $\varphi(S^2)\subset S^2$, we have

LEMMA 4.9. *The contraction φ^* of $\varphi\in\Phi$ to $S^*=S^2$ is a right translation of S^*.*

LEMMA 4.10. *Let S_λ be the subsemigroup defined in* (4.9). *If φ is a right translation of S, we have one of the following three cases* (4.11), (4.12), (4.13):

$$\varphi(a_\lambda)=\varphi(b_\lambda)=a_\lambda, \tag{4.11}$$

$$\varphi(a_\lambda)=a_\lambda, \qquad \varphi(b_\lambda)=b_\lambda, \tag{4.12}$$

$$\varphi(a_\lambda)=\varphi(b_\lambda)=c \quad \textit{where} \quad \sigma(c)>\lambda. \tag{4.13}$$

Proof. By Lemma 4.3, $\varphi(S_\lambda)\subset S_\mu$ for some μ. Reminding us of Theorem 4.2 and Lemma 19 in [4], we see $\lambda\leqq\mu$. In the case $\lambda=\mu$, since $\varphi(S_\lambda)\subset S_\lambda$, φ is considered as a right translation of S_λ, and consequently we have either (4.11) or (4.12). In the case $\lambda<\mu$, $S_\mu=\{c\}$ by Lemma 4.5, and so we have (4.13).

According to Lemma 4.7 and Theorem 4.3,

LEMMA 4.11. *The translation semigroup Φ^* of S^* is a semilattice.*

For any $\varphi^*\in\Phi^*$, a mapping φ of S into S is defined as follows:

$$\varphi(x)=\begin{cases} \varphi^*(x) & \text{if} \quad x\in S^2, \\ a_\lambda \quad \textit{or} \quad b_\lambda & \text{if} \quad x=b_\lambda, \qquad \varphi^*(a_\lambda)=a_\lambda, \\ c & \text{if} \quad x=b_\lambda, \qquad \varphi^*(a_\lambda)=c, \qquad \sigma(c)>\lambda. \end{cases} \tag{4.14}$$

LEMMA 4.12. *This φ is a right translation of S.*

Proof. In order to prove $\varphi(xy)=x\varphi(y)$, we must consider several cases.

I. The Case $y\in S^2$.

If $x\in S^2$, $\varphi(xy)=\varphi^*(xy)=x\varphi^*(y)=x\varphi(y)$, and if $x\overline{\in} S^2$ i.e. $x=b_\lambda$,

$$\varphi(b_\lambda y)=\varphi(a_\lambda y)=\varphi^*(a_\lambda y)=a_\lambda\varphi^*(y)=b_\lambda\varphi^*(y)=b_\lambda\varphi(y),$$

because $b_\lambda z=a_\lambda z$ for all z by (4.10).

II. The Case $y\overline{\in} S^2$ i.e. $y=b_\lambda$.

$$\begin{aligned}\varphi(xb_\lambda)&=\varphi(xa_\lambda) && \text{by (4.10)},\\ &=x\varphi(a_\lambda) && \text{by the Case I},\end{aligned}$$

while, if $\varphi(a_\lambda)=\varphi(b_\lambda)$, then $x\varphi(a_\lambda)=x\varphi(b_\lambda)$, and if $\varphi(a_\lambda)\neq\varphi(b_\lambda)$ i.e. $\varphi(a_\lambda)=a_\lambda$, $\varphi(b_\lambda)=b_\lambda$, then

$$x\varphi(a_\lambda)=xa_\lambda=xb_\lambda=x\varphi(b_\lambda)$$

by (4.10). Therefore we obtain $\varphi(xy)=x\varphi(y)$ in all cases.

LEMMA 4.13. *Φ denotes the set of all translations defined by* (4.14). *Φ* is *a semilattice.*

Proof. The Proof of $\varphi^2=\varphi$.

In the case $x\in S^2$,

(4.15) $\varphi^2(x)=\varphi(\varphi(x))=\varphi(\varphi^*(x))=\varphi^*\varphi^*(x)=\varphi^*(x)=\varphi(x)$ by (4.14) and Lemma 4.11.

In the case $x\overline{\in} S^2$, i.e. $x=b_\lambda$,

if $\varphi(b_\lambda)=\varphi(a_\lambda)$, then $\varphi^2(b_\lambda)=\varphi(\varphi(b_\lambda))=\varphi(\varphi(a_\lambda))=\varphi(a_\lambda)=\varphi(b_\lambda)$ because $a_\lambda\in S^2$ and so (4.15) is used;

if $\varphi(b_\lambda)\neq\varphi(a_\lambda)$, then $\varphi(b_\lambda)=b_\lambda$ and $\varphi^2(b_\lambda)=\varphi(\varphi(b_\lambda))=\varphi(b_\lambda)$.

The Proof of $\varphi\psi=\psi\varphi$.

For $x\in S^2$, $\varphi\psi(x)=\varphi^*\psi^*(x)=\psi^*\varphi^*(x)=\psi\varphi(x)$ by Lemma 4.11.

For $x\overline{\in} S^2$, if $\varphi(b_\lambda)=\varphi(a_\lambda)$ and $\psi(b_\lambda)=\psi(a_\lambda)$, then

$$\varphi\psi(b_\lambda)=\varphi\psi(a_\lambda)=\varphi^*\psi^*(a_\lambda)=\psi^*\varphi^*(a_\lambda)=\psi\varphi(a_\lambda)=\psi\varphi(b_\lambda);$$

if $\varphi(b_\lambda)=\varphi(a_\lambda)$ and $\psi(b_\lambda)\neq\psi(a_\lambda)$ i.e. $\psi(a_\lambda)=a_\lambda$, $\psi(b_\lambda)=b_\lambda$, then

$$\varphi\psi_\lambda(b)=\varphi(b_\lambda)=\varphi(a_\lambda)=\varphi^*(a_\lambda)=\varphi^*\psi^*(a_\lambda)=\psi^*\varphi^*(a_\lambda)=\psi\varphi(a_\lambda)=\psi\varphi(b_\lambda);$$

if $\varphi(b_\lambda)\neq\varphi(a_\lambda)$ and $\psi(b_\lambda)\neq\psi(a_\lambda)$, then

$$\varphi\psi(b_\lambda)=\varphi(b_\lambda)=b_\lambda=\psi(b_\lambda)=\psi\varphi(b_\lambda).$$

Thus we conclude the following theorem:

THEOREM 4.5. *Let $S\neq S^2$. The right translation semigroup Φ of a semigroup S is a semilattice if and only if S satisfies the following conditions*:

(1) *The inner right translation semigroup R of S is a semilattice, and so $S=\sum_{\sigma\in R} S_\sigma$ where every S_σ is defined as $x_\sigma y_\sigma=\alpha_\sigma(x_\sigma)$.*

(2) *Letting $S-S^2=\{b_\lambda;\ \lambda\in R'\subset R\}$, the subsemigroup S_λ which contains b_λ is a semigroup of order 2 defined as*

$$S_\lambda=\{a_\lambda, b_\lambda\}, \qquad a_\lambda^2=a_\lambda b_\lambda=b_\lambda a_\lambda=b_\lambda^2=a_\lambda,$$

and if $\lambda<\beta$, then S_β is a semigroup composed of only one element.

Lastly we shall consider the relation between the structure of S and that of $S^*=S^2$ when $S\neq S^2$ and Φ is a semilattice. We denote by S_σ^* the intersection of S^* and S_σ: $S_\sigma^*=S^*\frown S_\sigma$. Then $S=\sum_{\sigma\in R} S_\sigma$ corresponds to $S^*=\sum_{\sigma\in R} S_\sigma^*$ where, letting $S_\lambda=\{a_\lambda, b_\lambda\}$ and $b_\lambda\in S-S^*$,

$$\text{if}\quad \sigma=\lambda, \quad \text{then}\quad S_\sigma^*=\{a_\lambda\}, \qquad \text{if}\quad \sigma\neq\lambda, \quad\text{then}\quad S_\sigma^*=S_\sigma;$$

especially if $\sigma\geqq\lambda$, S_σ consists of only one element. Lemma 4.6 reminds us of the fact that λ_1 and λ_2 are incomparable.

Now we shall construct S from S^*. Let R be a semilattice and Q be a subset of R, any two elements of which is incomparable:

$$Q=\{\lambda_1,\cdots,\lambda_m\}, \qquad \lambda_i\not\gtreqless\lambda_j, \qquad i\neq j.$$

Furthermore a system $\{S_\sigma^*;\ \sigma\in R\}$ of s-indecomposable semigroups S_σ^* is given such that

(4.15) the multiplication in S_σ^* is defined as $x_\sigma y_\sigma=\alpha_\sigma(x_\sigma)$.

(4.16) $\sigma\geqq\lambda_i$ (for certain $i\in Q$) implies that S_σ^* consists of only one element a_σ.

(4.17) if $\sigma\bar{\in} Q$ and σ is minimal, then $S_\sigma=S_\sigma^2$.

At first, construct S^* as a composition of S_σ by R such that $S^*=S^{*2}$ and the inner right translation semigroup R^* of S^* is a semilattice. (See Theorems 4.1 and 4.3.) By Theorem 4.1, R^* is isomorphic to R and so R^* is a semilattice. Compose S by adding new elements $b_{\lambda_1},\cdots,b_{\lambda_m}$ to S^* such that $S=\sum_{\sigma\in R} S_\sigma$, where $S_{\lambda_i}=\{a_{\lambda_i}, b_{\lambda_i}\}$, $i=1,\cdots,m$, and $\sigma\bar{\in} Q$ implies $S_\sigma=S_\sigma^*$ and the multplication $x\cdot y$ in S is defined as follows.

Let $x\in S_\sigma$, $y\in S_\tau$. Clearly $\sigma\tau\not\geqq\lambda_i$ (for all i) implies $\sigma\not\geqq\lambda_i$ and $\tau\not\geqq\lambda_i$ for all i, and hence $S_\sigma=S_\sigma^*$, $S_\tau=S_\tau^*$.

$$x\cdot y=\begin{cases} a_{\sigma\tau} & \text{if } \sigma\tau\geqq\lambda_i \text{ for some } \lambda_i\in Q, \\ xy & \text{otherwise,}\end{cases} \tag{4.18}$$

where xy means the product of elements x and y of S^*. Then S is a semigroup. In fact, we shall prove $(x_\sigma\cdot y_\tau)\cdot z_\mu=x_\sigma\cdot(y_\tau\cdot z_\mu)$.

If $(\sigma\tau)\mu=\sigma(\tau\mu)\geqq\lambda_i$ for some $\lambda_i\in Q$,

$$(x_\sigma\cdot y_\tau)\cdot z_\mu=a_{(\sigma\tau)\mu}=a_{\sigma(\tau\mu)}=x_\sigma\cdot(y_\tau\cdot z_\mu),$$

if $(\sigma\tau)\mu=\sigma(\tau\mu)\neq\lambda_i$ for all i, then $\sigma\tau\neq\lambda_i$, $\mu\neq\lambda_i$, $\sigma\neq\lambda_i$, $\tau\mu\neq\lambda_i$ for all i,

$$(x_\sigma\cdot y_\tau)\cdot z_\mu=(x_\sigma\cdot y_\tau)z_\mu=(x_\sigma y_\tau)z_\mu=x_\sigma(y_\tau z_\mu)=x_\sigma\cdot(y_\tau z_\mu)=x_\sigma\cdot(y_\tau\cdot z_\mu).$$

Clearly $S^2=S^*$ and $Q=S-S^2$. From the definition (4.18),

$$x\cdot a_{\lambda_i}=x\cdot b_{\lambda_i}=a_{\lambda_i}\cdot x=b_{\lambda_i}\cdot x \qquad \text{for} \quad x\in S,$$

in particular $a_{\lambda_i}\cdot b_{\lambda_i}=b_{\lambda_i}\cdot a_{\lambda_i}=a_{\lambda_i}^2=b_{\lambda_i}^2=a_{\lambda_i}$ $(i=1, \cdots, m)$ and so it is easily shown that S satisfies (4.2) and every S_σ is s-indecomposable. By Theorem 4.1, the inner right translation semigroup of S is isomorphic to the semilattice R. Thus we see that S fulfils all the conditions of Theorem 4.5. Therefore it is assured that Φ of S is a semilattice.

THEOREM 4.6. *Let $Q=\{\lambda_1, \cdots, \lambda_m\}$ be a subset of a semilattice R, where $\lambda_1, \cdots, \lambda_m$ are mutually incomparable. Let $S^*=\sum_{\sigma\in R}S_\sigma^*$ be a semigroup whose inner right translation semigroup is isomorphic to R such that* (4.15), (4.16), *and* (4.17) *are fulfilled. For S^*, a semigroup S:*

$$S=S^*\cup\{b_{\lambda_1}, \cdots, b_{\lambda_m}\}=\sum_{\sigma\in R}S_\sigma, \quad \text{where} \quad S_{\lambda_i}=\{a_{\lambda_i}, b_{\lambda_i}\}, \quad S_\sigma=S_\sigma^* \quad (\sigma\notin Q)$$

is defined by the multiplication (4.18). *Then the right translation semigroup Φ of S is a semilattice. Conversely a semigroup S with $S\neq S^2$ whose Φ is a semilattice, is obtained in such a manner.*

4. The Case whose R and L are Semilattices.

Ψ denotes the left translation semigroup of S, and L denotes the inner left translation semigroup of S.

LEMMA 4.14. *If R and L are semilattices, then S is commutative.*

Proof. By Lemma 4.1 and its dual form, we have

$$xy=xyy=yxy=yyx=yx.$$

LEMMA 4.15. *If and only if R and L are semilattices and $S=S^2$, then S is a semilattice.*

Proof. If R and L are semilattices and $S=S^2$, then $x=yz$,

$$x^2=yzyz=(yzy)z=(yyz)z=(yz)z=y(zz)=yz=x.$$

Combining it with Lemma 4.14, it follows that S is a semilattice. The proof of the converse is already shown in [4].

Using Theorem 4.6 and Lemma 4.15, we shall investigate the structure of a semigroup S whose R and L are semilattices and in which $S\neq S^2$.

THEOREM 4.7. *Let S^* be a semilattice and $Q=\{\lambda_1, \cdots, \lambda_m\}$ be a subset of S^*, $\lambda_1, \cdots, \lambda_m$ being mutually incomparable. We construct a semigroup S*

by adding new elements $b_{\lambda_1}, \cdots, b_{\lambda_m}$ *to* S^* *such that* $S=\sum_{\sigma\in S^*} S_\sigma$ *where* $S_{\lambda_i}=\{\lambda_i, b_{\lambda_i}\}$ and $S_\sigma=\{\sigma\}$, *if* $\sigma\overline{\in}Q$, *and the multiplication in* S *is given as follows*: *For* $x\in S_\sigma$ *and* $y\in S_\tau$

$$x\cdot y=\begin{cases}\lambda_i & \text{if } \sigma\tau=\lambda_i\in Q,\\ xy & \text{if } \sigma\tau\overline{\in}Q, \text{ where } xy \text{ is the product of } x\in S^* \text{ and } y\in S^*.\end{cases} \tag{4.19}$$

References

[1] T. Tamura, On translations of a semigroup. Kōdai Math. Sem. Rep. **7** (1955), 67–70.

[2] ——, One-sided bases and translations of a semigroup. Math. Jap. **3** (1955), 137–141.

[3] ——, The theory of construction of finite semigroups I (Greatest decomposition of a semigroup). Osaka Math. J. 8 (1956), 243–261.

[4] ——, The theory of construction of finite semigroups II (Compositions of semigroups and finite s-decomposable semigroups). Osaka Math. J. **9** (1957), 1–42.

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