AN EXTENSION OF KINTCHINE-OSTROWSKI'S THEOREM AND ITS APPLICATIONS

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1. Introdution. As an extension of Vitali's theorem, A. Kintchine and A. Ostrowski have proved

THEOREM ([1],[2],[3] p. 157). *Let $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ be a sequence of functions regular and uniformly bounded in $|z| < 1$. If the sequence of boundary functions $\{f_n(e^{i\theta})\}$ $(n = 1, 2, \cdots)$ converges on a set E of θ of positive measure, then the sequence of $\{f_n(z)\}$ converges uniformly in the wider sense in $|z| < 1$.*

We shall first generalize this theorem as follows.

THEOREM 1. *Let $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ be a sequence of functions regular in $|z| < 1$ and of uniformly bounded characteristic, i.e.*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f_n(re^{i\theta})|\, d\theta < M < +\infty \qquad (0 \leqq r < 1,\ n = 1, 2, \cdots), \tag{1.1}$$

where M is a constant independent of n. If the sequence of boundary functions $\{f_n(e^{i\theta})\}$ converges on a set E of θ of positive measure, then the sequence $\{f_n(z)\}$ converges uniformly in the wider sense in $|z| < 1$.

P. Montel has proved.

THEOREM ([4] p. 170). *Let $f(s)$ $(s = \sigma + it)$ be regular except at $s = \infty$ and bounded in the strip: $\alpha \leqq \sigma \leqq \beta$. If $\lim_{t\to+\infty} f(\alpha + it) = a$, then $f(s)$ tends uniformly to a as $t \to +\infty$ in the strip: $\alpha \leqq \sigma \leqq \beta - \varepsilon$, ε being any given positive constant.*

Before we establish an extension of Montel's theorem by theorem 1, we begin with

DEFINITION 1. *Let $f(s)$ be regular in the domain D. If*

$$\log^+ |f(s)| \leqq h(s) \qquad \textit{for} \quad s \in D, \tag{1.2}$$

where $h(s)$ is a harmonic function in D, then we say that $f(s)$ belongs to the

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class H_0 *in D. For brevity we denote it by*

$$f(s) \in H_0(D, h(s)).^{1)}$$

DEFINITION 2. *If we replace* $\log^+|f(s)|$ *by* $|f(s)|^p\ (p > 0)$ *in definition* 1, *i.e.*

$$|f(s)|^p \leqq h(s) \qquad \text{for} \quad s \in D, \tag{1.3}$$

we call that $f(s)$ *belongs to the class* H_p *in D. We denote it by*

$$f(s) \in H_p(D, h(s)).$$

Our extension of Montel's theorem is

THEOREM 2. *Let* $f(s)\ (s = \sigma + it)$ *be regular and* $f(s) \in H_0(S, h(s))$ *in the strip* $S: \alpha < \sigma < \beta$. *Let E be the set of points on* $\sigma = \alpha$ *such that*

(1.4)
- (i) $E = \sum_{n=0}^{+\infty} E_n,$
- (ii) E_0: *the set of positive measure contained in the segment*: $\sigma = \alpha$, $|t| < t_0$,
- (iii) E_n: *the set obtained by the parallel translation of* E_0 *by* int_0 $(n = 1, 2, \cdots)$.

Suppose that

(1.5)
- (1°) $h(s_n)\ (n = 0, 1, 2, \cdots)$ *is bounded, where* $s_n = \sigma + int_0$, $\alpha < \sigma_0 < \beta$,
- (2°) $\lim_{\substack{s\to\infty \\ s\in E}} f(s) = a \qquad (\neq \infty).^{2)}$

Under these conditions, $f(s)$ *tends uniformly to a as* $t \to +\infty$ *in the strip* S^*: $\alpha + \varepsilon \leqq \sigma \leqq \beta - \varepsilon$, ε *being any given positive constant.*

As its corollary, we easily obtain the following theorem, which is an analogue of E. Lindelöf's theorem.

COROLLARY. *Let* $f(s)\ (s = \sigma + it)$ *be regular and* $f(s) \in H_0(S, h(s))$ *in the strip* $S: \alpha < \sigma < \beta$. *Suppose that*

(1°) $h(s_n)\ (n = 0, 1, 2, \cdots)$ *is bounded, where* $s_n = \sigma_0 + int_0$, $t_0 > 0$, $\alpha < \sigma_0 < \beta$,

1) By remark of lemma 2, $f(s) \in H_0(D(|s| < 1), h(s))$ is equivalent to the boundedness of characteristic.

2) If $f(s) \in H_0(S, h(s))$, mapping S conformally onto the unit circle, by remark of lemma 2 and lemma 1, the boundary function $f(s)$ on $\sigma = \alpha$ exists almost everywhere.

(1.6) (2°) *$f(s)$ is continuous on $\sigma = \alpha$ and β, except at $s = \infty$,*

(3°) $$\lim_{t\to+\infty} f(\alpha + it) = a, \quad \lim_{t\to+\infty} f(\beta + it) = b, \qquad (a, b \neq \infty).$$

Under these conditions, $a = b$ and $f(s)$ tends uniformly to a as $t \to +\infty$ in the strip S^*: $\alpha + \varepsilon \leqq \sigma \leqq \beta - \varepsilon$, ε being any given positive constant.

If $f(s)$ belongs to the class H_p ($p > 0$) in the strip, another extension of Montel's theorem can be established.

THEOREM 3. *Let $f(s)$ $(s = \sigma + it)$ be regular and $f(s) \subset H_p(S, h(s))$ $(p > 0)$ in the strip S: $\alpha < \sigma < \beta$. Suppose that*

(1°) *$h(s_n)$ $(n = 0, 1, 2 \cdots)$ is bounded, where $s_n = \sigma_0 + int_0$, $t_0 > 0$, $\alpha < \sigma_0 < \beta$,*

(1.7) (2°) *$f(s)$ is continuous on $\sigma = \alpha$, except at $s = \infty$,*

(3°) $$\lim_{t\to+\infty} f(\alpha + it) = a \qquad (\neq \infty).$$

Under these conditions, $f(s)$ tends uniformly to a as $t \to +\infty$ in the strip S^{**}: $\alpha \leqq \sigma \leqq \beta - \varepsilon$, ε being any given positive constant.

2. Lemmas.

To establish our theorems, we need some lemmas.

LEMMA 1. *If $f(z)$ is regular and of bounded characteristic in $|z| < 1$, then following propositions hold:*

[A] $\lim_{r\to1} f(re^{i\theta}) = f(e^{i\theta})$ *for almost all θ,*
[B] $\log|f(e^{i\theta})|$ *is Lebesgue-integrable,*
[C] *if $f(z) \neq 0$ at a fixed point z, then*

$$\int_0^{2\pi} \log|f(e^{i\theta})|\,|\,d\theta \leqq 4M\pi((1+|z|)/(1-|z|))^2 - 2\pi(1+|z|)/(1-|z|)\cdot\log|f(z)|,$$

$$\text{where } (1/2\pi)\cdot\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta < M < +\infty.$$

Proof. Since $T(r, f) = (1/2\pi)\cdot\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta$ is bounded, by R. Nevanlinna's theorem ([5] p. 197) the proposition [A] holds.

If $f(z) \neq 0$, putting $P = (R^2 - r^2)/(R^2 - 2Rr\cos(\theta - \varphi) + r^2)$ $(0 \leqq r < R < 1)$, by Poisson-Jensen's formula we get

$$\log|f(z)| \leqq \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\cdot P\,d\varphi \qquad (z = re^{i\theta})$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\varphi})|\cdot P\,d\varphi - \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f(Re^{i\theta})}\right|\cdot P\,d\varphi$$

$$= \frac{1}{\pi}\int_0^{2\pi}\log^+|f(Re^{i\varphi})|\cdot P\,d\varphi - \frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\varphi})||\cdot P\,d\varphi.$$

Since $(R-r)/(R+r)\leqq P\leqq (R+r)/(R-r)$, we have

$$\log|f(z)|\leqq 2(R+r)/(R-r)\cdot\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\varphi})|\,d\varphi$$

$$-(R-r)/(R+r)\cdot\frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\varphi})||\,d\varphi.$$

Hence,

$$\frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\varphi})||\,d\varphi$$

$$\leqq 2M((R+r)/(R-r))^2-(R+r)/(R-r)\cdot\log|f(z)|<+\infty.$$

By [A] and Fatou's lemma,

$$\frac{1}{2\pi}\int_0^{2\pi}|\log|f(e^{i\varphi})||\,d\varphi\leqq \varliminf_{R\to 1}\frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\varphi})||\,d\varphi$$

$$\leqq 2M((1+r)/(1-r))^2-(1+r)/(1-r)\cdot\log|f(z)|,$$

which proves [B] and [C].

LEMMA 2. *The necessary and sufficient condition for* $\{f_n(z)\}$ *to satisfy* (1.1) *is the existence of a sequence of positive harmonic functions* $\{u_n(z)\}$ *such that*

$$(2.1)\quad \begin{cases} \text{(i)} & \log^+|f_n(z)|\leqq u_n(z) \quad \text{in} \quad |z|<1, \\ \text{(ii)} & u_n(0)\leqq M<+\infty. \end{cases} \qquad (n=1,2,\cdots).$$

Its proof is essentially due to W. Rudin ([6] p. 47).

REMARK. By the entirely similar arguments as in lemma 2, we can prove that

(1°) *the boundedness of* $(1/2\pi)\cdot\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$ $(0\leqq r<1)$ *is equivalent to the existence of the harmonic function* $h(s)$ *satisfying* (1.2) *in* $|s|<1$.

(2°) *the boundedness of* $(1/2\pi)\cdot\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta$ $(p>0,\ 0\leqq r<1)$ *is equivalent to the existence of the harmonic function* $h(s)$ *satisfying* (1.3) *in* $|s|<1$.

Proof. (I) Sufficiency: By (2.1),

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f_n(re^{i\theta})|\,d\theta \leqq \frac{1}{2\pi}\int_0^{2\pi}u_n(re^{i\theta})\,d\theta = u_n(0) \leqq M < +\infty,$$

which proves (1.1).

(II) Necessity: Let us define a sequence of positive harmonic functions $u_n(z, r)$ $(n = 1, 2, \cdots; 0 \leqq r < 1)$ such that

$$(2.2)\qquad \begin{aligned} &u_n(z, r) = \log^+|f_n(re^{i\theta})| \quad \text{on} \quad |z| = r, \\ &u_n(z, r): \quad \text{harmonic in } |z| < r. \end{aligned}$$

Then,

$$u_n(0, r) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f_n(re^{i\theta})|\,d\theta < M < +\infty.$$

Hence

$$(2.3)\qquad u_n(0, r) < M < +\infty \qquad (n = 1, 2, \cdots; 0 \leqq r < 1).$$

Since $\log^+|f_n(z)|$ is subharmonic, by (2.2)

$$(2.4)\qquad \log^+|f_n(z)| \leqq u_n(z, r) \quad \text{in} \quad |z| \leqq r.$$

By (2.2) and (2.4)

$$u_n(z, r) = \log^+|f_n(z)| \leqq u_n(z, R) \quad \text{on} \quad |z| = r < R,$$

so that

$$u_n(z, r) \leqq u_n(z, R) \quad \text{in} \quad |z| \leqq r < R.$$

Therefore, $\{u_n(z, r)\}$ $(0 \leqq r < 1)$ is an increasing sequence of r. Hence, by (2.3) and Harnack's theorem, $u_n(z, r)$ $(n = 1, 2, \cdots; 0 \leqq r < 1)$ converges to $u_n(z)$ $(n = 1, 2, \cdots)$ uniformly in the wider sense in $|z| < 1$. Letting $r \to 1$ in (2.3) and (2.4), we get (2.1), which proves the necessity.

LEMMA 3. *Under the condition* (1.1) *in theorem* 1, *the family* $\{f_n(z)\}$ $(n = 1, 2, \cdots)$ *is normal in* $|z| < 1$.

Proof. By lemma 2, there exists a sequence of positive harmonic function $\{u_n(z)\}$ such that

$$\begin{aligned} &\text{(i)} \qquad \log^+|f_n(z)| \leqq u_n(z) \quad \text{in} \quad |z| < 1, \\ &\text{(ii)} \qquad u_n(0) \leqq M < +\infty \end{aligned} \qquad (n = 1, 2, \cdots).$$

Since $u_n(z) > 0$ in $|z| < 1$, the family $\{u_n(z)\}$ is normal. Hence, by (ii) $\{u_n(z)\}$

is uniformly bounded in any closed domain D completely interior to the unit-circle, so that $\{f_n(z)\}$ is also uniformly bounded in D. Then, by the usual way, the normality of $\{f_n(z)\}$ is $|z|<1$ is easily established.

LEMMA 4 ([7] p. 47). *Let* $f(z)$ $(z=re^{i\theta})$ *be regular and* $(1/2\pi)\cdot\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta$ $(p>0,\ 0\leqq r<1)$ *be bounded in* $|z|<1$. *If* $\lim_{\varphi\to\theta+0}f(e^{i\varphi})=a$,[3] $\lim_{\varphi\to\theta-0}f(e^{i\varphi})=b$, *then* $a=b$ *and* $f(z)$ *tends uniformly to* a *as* $z\to e^{i\theta}$ *in* $|z|<1$.

3. Proofs of theorems.

Proof of theorem 1. We shall first prove that $\{f_n(z)\}$ converges at $z=\alpha$ $(|\alpha|<1)$. On the contrary, there would exist two sequences $\{f_{k_n}(z)\}$, $\{f_{m_n}(z)\}$ such that

$$\lim_{n\to\infty}f_{k_n}(\alpha)\neq\lim_{n\to\infty}f_{m_n}(\alpha). \tag{3.1}$$

Put $\varphi_n(z)=f_{k_n}(z)-f_{m_n}(z)$. Then, by the assumptions and Egoroff's theorem, there exists a sub-set E^* of E such that

$$\lim_{n\to\infty}\varphi_n(e^{i\theta})=0 \quad \text{uniformly for} \quad \theta\in E^* \qquad (mE^*>0). \tag{3.2}$$

By the inequality

$$\log^+|a+b|\leqq\log^+|a|+\log^+|b|+\log 2,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|\varphi_n(re^{i\theta})|\,d\theta\leqq 2M+\log 2=M^*<+\infty.$$

Hence, by lemma 1 [C]

$$\int_{E^*}|\log|\varphi_n(e^{i\varphi})||\,d\varphi\leqq 4\pi M^*\cdot((1+|\alpha|)/(1-|\alpha|))^2 - 2\pi(1+|\alpha|)/(1-|\alpha|)\cdot\log|\varphi_n(\alpha)|. \tag{3.3}$$

By (3.2) and (3.3), letting $n\to\infty$, we have $\lim_{n\to\infty}\varphi_n(\alpha)=0$, which is contrary with (3.1). Hence $\{f_n(z)\}$ converges at $z=\alpha$ $(|\alpha|<1)$. By lemma 3, $\{f_n(z)\}$ is normal in $|z|<1$, so that $\{f_n(z)\}$ converges uniformly in the wider sense in $|z|<1$, which proves our theorem.

Proof of theorem 2. We consider two function-families

$$\{f_n(s)\}\equiv\{f(s+int_0)\}, \qquad \{h_n(s)\}\equiv\{h(s+int_0)\}$$

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3) Apart from a set of measure zero, the boundary function $f(e^{i\varphi})$ exists.

in the domain D: $\alpha < \sigma < \beta$, $|t| < 2t_0$. By the assumption (1.5)(1°),

$$\text{(3.4)} \qquad \begin{aligned} &\log^+ |f_n(s)| \leqq h_n(s) \qquad \text{for} \quad s \in D, \\ &0 < h_n(\sigma_0) < M < +\infty. \end{aligned}$$

We map conformally D onto $|z| < 1$ by $s = g(z)$ $(\sigma_0 = g(0))$. Then, by well-known F. and M. Riesz's theorem ([8]), the set E_0 on the segment L: $\sigma = \alpha$, $|t| < 2t_0$ is mapped on a set E_0^* of positive linear measure on the circular arc corresponding to the segment L. By (3.4)

$$\text{(3.5)} \qquad \begin{aligned} &\log^+ |F_n(z)| \leqq H_n(z) \qquad \text{for} \quad |z| < 1, \\ &0 < H_n(0) < M < +\infty, \end{aligned}$$

where $F_n(z) = f_n(g(z))$, $H_n(z) = h_n(g(z))$. Since

$$\lim_{\substack{n\to\infty \\ z \in E_0^*}} F_n(z) = a \ (mE^* > 0) \qquad \text{by} \qquad (1.5)(2°),$$

taking account of (3.5), lemma 2 and theorem 1, $\{F_n(z)\}$ converges uniformly to a in the wider sense in $|z| < 1$. In particular, $\{F_n(z)\}$ converges uniformly to a in the closed domain corresponding to the domain: $|t| \leqq t_0 + \varepsilon$, $a + \varepsilon \leqq \sigma \leqq \beta - \varepsilon$, ε being any given positive constant, which proves our theorem.

Proof of theorem 3. Since, by the inequality

$$\log^+ x < (1/p)\cdot(x^p + 1) \qquad (p > 0, \ x \geqq 0),$$

$f(s) \in H_0(S, h^*(s))$ $(h^*(s) = (1/p)\cdot(h(s) + 1))$ follows from $f(s) \in H_p(S, h(S))$, by (1.7)(2°), (3°) and theorem 2, $f(s)$ tends uniformly to a as $t \to +\infty$ in the strip S^*: $\alpha + \varepsilon \leqq \sigma \leqq \beta - \varepsilon$, ε being any given positive constant. Hence $\lim_{t\to+\infty} f(\alpha + it) = \lim_{t\to+\infty} f(\alpha + \varepsilon + it) = a$, so that by remark of lemma 2 and lemma 4, $f(s)$ tends uniformly to a as $t \to +\infty$ in the strip: $\alpha \leqq \sigma \leqq \alpha + \varepsilon$. Therefore, in the strip S^{**}: $\alpha \leqq \sigma \leqq \beta - \varepsilon$, $f(s)$ tends uniformly to a as $t \to +\infty$, which proves theorem 3.

References

[1] A. Kintchine, Sur les suites de fonctions analytiques bornées dans leur ensemble. Fund. Math. **4** (1923), 72—75.
[2] A. Ostrowski, Über allgemeine Konvergenzsätze der komplexen Funktionentheorie. Jahresb. Deutsch. Math.-Vereinig. **32** (1923), 185—194.
[3] L. Bieberbach, Lehrbuch der Funktionentheorie II. (1927).
[4] E. C. Titchmarsh, The theory of functions. Second Edition. Oxford. (1939).
[5] R. Nevanlinna, Eindeutige analytische Funktionen. Berlin. (1936).
[6] W. Rudin, Analytic functions of class H_p. Trans. Amer. Math. Soc. **78**

(1955), 46—66.
[7] C. TANAKA, On the class H_p of functions analytic in the unit circle. Yokohama Math. Journ. 4 (1956), 47—53.
[8] F. AND M. RIESZ, Über die Randwerte einer analytischen Funktion. Quatrième congrès des mathématiciens scandinaves à Stockholm. 1925.

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