FOURIER SERIES XI: GIBBS' PHENOMENON

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1. Concerning the Gibbs phenomenon of Fourier series, H. Cramér [1] proved the following

THEOREM 1. *There exists a number* r_0, $0 < r_0 < 1$, *with the following property: If* $f(x)$ *is simply discontinuous at a point* ξ, *the* (C, r) *means* $\sigma_n^r(x)$ *of the Fourier series of* $f(x)$ *present Gibbs' phenomenon at* ξ *for* $r < r_0$, *but not for* $r \geqq r_0$.

We shall extend this theorem to the discontinuity of the second kind. In this direction S. Izumi and M. Satô [2] proved the the following

THEOREM 2. *Suppose that*

$$f(x) = a\psi(x - \xi) + g(x), \tag{1}$$

where $\psi(x)$ *is a periodic function with period* 2π *such that*

$$\psi(x) = (\pi - x)/2 \qquad (0 < x < 2\pi) \tag{2}$$

and where

$$\begin{aligned} &\limsup_{x \downarrow \xi} g(x) = 0, && \liminf_{x \uparrow \xi} g(x) = 0, \\ &\liminf_{x \downarrow \xi} g(x) \geqq -a\pi, && \limsup_{x \uparrow \xi} g(x) \leqq a\pi, \end{aligned} \tag{3}$$

$$\int_0^x |g(\xi + u)|\, du = o(|x|), \tag{4}$$

then the Gibbs' phenomenon of the Fourier series of $f(x)$ *appears at* $x = \xi$.

We shall prove that Theorem 1 holds even when ξ is the discontinuity point of the second kind, satisfying the condition in Theorem 2. More precisely,

THEOREM 3. *Suppose that*

$$f(x) = a\psi(x - \xi) + g(x), \tag{1}$$

where $\psi(x)$ *is a periodic function with period* 2π *such that*

$$\psi(x) = (\pi - x)/2 \qquad (0 < x < 2\pi) \tag{2}$$

and where

$$\begin{aligned} &\limsup_{x \downarrow \xi} g(x) = 0, && \liminf_{x \uparrow \xi} g(x) = 0, \\ &\liminf_{x \downarrow \xi} g(x) \geqq -a\pi, && \limsup_{x \uparrow \xi} g(x) \leqq a\pi, \end{aligned} \tag{3}$$

$$\int_0^x |g(\xi + u)|\, du = o(|x|). \tag{4}$$

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Then there exists a number r_0, $0 < r_0 < 1$, with the following property: the (C, r) means of the Fourier series of $f(x)$ present Gibbs' phenomenon at ξ for $r < r_0$, but not for $r \geqq r_0$, r_0 being the Cramér number in Theorem 1.

Further we prove the following

THEOREM 4. *Let $f(x)$ be an odd function about ξ such that*

$$\limsup_{x \downarrow \xi} f(x) = a\pi/2, \qquad \liminf_{x \uparrow \xi} f(x) = -a\pi/2.$$

Let the Fourier series of $f(x)$ be

$$f(x) \sim \sum_{n=1}^{\infty} \alpha_n \sin n(x - \xi), \tag{5}$$

where

$$\alpha_n = \frac{a}{n}(1 + o(1)), \qquad \sum \Delta\alpha_n < \infty, \qquad \sum |\Delta^2 \alpha_n| < \infty.$$

Then there exists a number r_0, $0 < r_0 < 1$, with the following property: the (C, r) means of the Fourier series of $f(x)$ present Gibbs' phenomenon at ξ for $r < r_0$, but not for $r \geqq r_0$. r_0 is the Cramér number in Theorem 1.

In order to prove Theorems 3 and 4 we use the methods of S. Izumi and M. Satô [2] and of H. Cramér [1], respectively.

2. Proof of Theorem 3. Without loss of generality, we can suppose that $\xi = 0$ and $a = 1$. We have

$$\sigma_n^r(x, f) = \sigma_n^r(x, \psi) + \sigma_n^r(x, g).$$

By Theorem 1 $\sigma_n^r(\pi/n, \psi)$ tends to a constant which is greater than $\pi/2$ if $r < r_0$ but less than $\pi/2$ if $r \geqq r_0$. We can see that $\sigma_n^r(k\pi/n, \psi)$ is near to $\pi/2$ for large k, and if $r < r_0$, then there is a k such that

$$\frac{1}{2}(\sigma_n^r(\pi/n, \psi) + \sigma_n^r(k\pi/n, \psi)) \tag{6}$$

tends to a constant, greater than $\pi/2$, and if $r \geqq r_0$, then (6) tends to $\pi/2$. Hence it is sufficicient to prove that

$$\sigma_n^r(\pi/n, g) + \sigma_n^r(k\pi/n, g) \tag{7}$$

tends to zero as $n \to \infty$, for any r, $0 < r < 1$, and any k. Now

$$\sigma_n^r(x, g) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t + x) K_n^r(t)\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) K_n^r(t - x)\, dt,$$

where $K_n^r(t)$ is the Fejér kernel of order r. It is known that[1)]

$$|K_n^r(t)| \leqq An \tag{8}$$

1) A denotes a numerical constant which may by different in each occurence.

and

$$K_n^r(t) = \frac{1}{A_n^r} \frac{\sin\left[\left(n+\frac{1}{2}+\frac{1}{2}r\right)t - \frac{1}{2}\pi r\right]}{\left(2\sin\frac{1}{2}t\right)^{r+1}}$$

(9) $$+ \frac{r}{n+1}\frac{1}{\left(2\sin\frac{1}{2}t\right)^2} + \frac{\theta}{n^2}\frac{8r(1-r)}{\left(2\sin\frac{1}{2}t\right)^3}$$

$$= K_{n1}^r + K_{n2}^r + K_{n3}^r,$$

where $A_n^r = \binom{n+r}{n}$ and $|\theta| \leqq 1$. We write

$$\sigma_n^r(x, g) = \frac{1}{\pi}\left(\int_0^\pi + \int_{-\pi}^0\right) g(t) K_n^r(t-x)\,dt = \frac{1}{\pi}(I_1(x) + I_2(x)).$$

We shall estimate I_1 only, since I_2 may be estimated quite similarly. We write now

$$I_1(x) = \int_0^\pi = \int_0^{2k\pi/n} + \int_{2k\pi/n}^\pi = I_{11}(x) + I_{12}(x).$$

Then by (8) and (4) we have

$$|I_{11}| \leqq n\int_0^{2k\pi/n} |g(t)|\,dt = o(1).$$

Thus it is sufficient to prove that

(10) $$I_{12}(\pi/n) + I_{12}(k\pi/n) = o(1).$$

The left side is

(11) $$J = \int_{2k\pi/n}^\pi g(t)\,(K_n^r(t-\pi/n) + K_n^r(t-k\pi/n))\,dt.$$

We denote by J_i $(i = 1, 2, 3)$ the J, replaced K_n^r by K_{ni}^r. Then

$$J_1 = \frac{1}{A_n^r}\int_{2k\pi/n}^\pi g(t)\left\{\frac{\sin[(n+\alpha)(t-\pi/n)-\beta]}{(t-\pi/n)^{r+1}} - \frac{\sin[(n+\alpha)(t-\pi/n)-\beta]}{(t-k\pi/n)^{r+1}}\right\}dt$$

if we take k such that $(n+\alpha)(k-1)/n$ is an odd integer. Thus we have

$$J_1 = \frac{1}{A_n^r}\int_{2k\pi/n}^\pi g(t)\frac{(t-k\pi/n)^{r+1} - (t-\pi/n)^{r+1}}{(t-\pi/n)^{r+1}(t-k\pi/n)^{r+1}}\sin[(n+\alpha)(t-\pi/n)-\beta]\,dt.$$

Since

$$(t-k\pi/n)^{r+1} - (t-\pi/n)^{r+1} \leqq A(t-\pi/n)^r/n,$$

we have

(12) $$J_1 \leqq \frac{A}{n^{r+1}}\int_{2k\pi/n}^\pi |g(t)|\frac{dt}{t^{r+2}} = \frac{A}{n^{r+1}}\left[\frac{1}{t^{r+2}}\int_0^t |g(u)|\,du\right]_{2k\pi/n}^\pi + \frac{A}{n^{r+1}}\int_{2k\pi/n}^\pi \frac{dt}{t^{r+3}}\int_0^t |g(u)|\,du,$$

which is $o(1)$ by the condition (4). On the other hand we have

$$J_2 = \int_{2k\pi/n}^{\pi} g(t)\,(K_{n2}^r(t - \pi/n) + K_{n2}^r(t - k\pi/n))\,dt = J_{21} + J_{22}.$$

Then

$$|J_{21}| = \left| \int_{2k\pi/n}^{\pi} g(t) \frac{r\,dt}{(n+1)\,(2\sin(t - \pi/n)/2)^2} \right| \leqq \frac{A}{n} \int_{2k\pi/n}^{\pi} |g(t)| \frac{dt}{t^2},$$

which is the case $r = 0$ in (12), and then tends to zero. Similarly we get $J_{22} = o(1)$. Finally

$$J_3 = \frac{8r(1-r)\theta}{n^2} \int_{2k\pi/n}^{\pi} g(t) \Big(\frac{1}{(2\sin(t - \pi/n)/2)^3} + \frac{1}{(2\sin(t - k\pi/n)/2)^3} \Big) dt,$$

which is majorated by

$$\frac{A}{n^2} \int_{2k\pi/n}^{\pi} |g(t)| \frac{dt}{t^3}.$$

This is the case $r = 1$ in (12), and then tends also to zero. Thus we have proved that $J = J_1 + J_2 + J_3 = o(1)$, and hence (10) is proved, which is the required.

3. Proof of Theorem 4. For the froof, we need a lemma, which is an extension of the Cramér's.

LEMMA. *Let $g(x)$ be an odd integrable function. Then, for any ε, there are an η and an N such that*

$$(13) \qquad |\sigma_n^r(x, g) - \varphi_r(n, nx, g)| < \varepsilon \qquad \text{for } |x| < \eta, \quad n > N,$$

where

$$\varphi_r(n, x, g) = n \int_0^1 (1-t)^r a(nt) \sin xt\,dt,$$

$$a(t) = \frac{2}{\pi} \int_0^{\pi} g(u) \sin tu\,du.$$

Proof. We denote by $\tau_n^r(x, g)$ the Riesz mean of the Fourier series of $g(x)$ of order r, that is

$$\tau_n^r(x, g) = \sum_{\nu=1}^{n} \Big(1 - \frac{\nu}{n}\Big)^r a_\nu \sin \nu x.$$

It is well known that

$$\sigma_n^r(x, g) - \tau_n^r(x, g) \to 0 \qquad (n \to \infty).$$

Hence (13) is equivalent to

$$(14) \qquad |\tau_n^r(x, g) - \varphi_r(n, nx, g)| < \varepsilon.$$

Let us now use Euler's summation formula which reads as follows:

$$(15) \qquad \sum_{\nu=0}^{n} h(\nu) = \int_0^n h(t)\,dt + \int_0^n \Big(t - [t] - \frac{1}{2}\Big) h'(t)\,dt - \frac{1}{2} h(0) + \frac{1}{2} h(n),$$

where $h(t)$ is continuously differentiable. In (15) we put

$$h(t) = \left(1 - \frac{t}{n}\right)^r a(t) \sin tx,$$

then $h(t)$ is continuously differentiable and $h(0) = h(n) = 0$. Furthermore

$$h'(t) = -\frac{r}{n}\left(1 - \frac{t}{n}\right)^{r-1} a(t) \sin tx$$
$$+ \left(1 - \frac{t}{n}\right)^r a'(t) \sin tx + \left(1 - \frac{t}{n}\right)^r a(t) x \cos xt.$$

Thus we get

$$\Delta_n = \tau_n^r(x, g) - \varphi_r(n, nx, g) = \int_0^n P(t) h'(t)\, dt$$
$$= -\frac{k}{n}\int_0^n P(t)\left(1 - \frac{t}{n}\right)^{r-1} a(t) \sin tx\, dx + \int_0^n P(t)\left(1 - \frac{t}{n}\right)^r a'(t) \sin tx\, dt$$
$$+ x\int_0^n P(t)\left(1 - \frac{t}{n}\right)^r a(t) \cos tx\, dt = I_1 + I_2 + I_3$$

where

$$P(t) = t - [t] - \frac{1}{2}.$$

We have first

$$I_1 = -\frac{r}{n^r}\int_0^n P(t)\,(n-t)^{r-1} a(t) \sin tx\, dt$$

and hence

$$|I_1| \leqq \frac{A}{n^r}\int_0^n \frac{|a(t)|}{(n-t)^{1-r}}\, dt = \frac{A}{n^r}\left(\int_0^m + \int_m^n\right).$$

We take m such that

$$|a(t)| < \varepsilon \qquad \text{for} \quad t > m,$$

which is possible by $a(t) = o(1)\ (t \to \infty)$. Thus we get

$$|I_1| \leqq \frac{A}{n^r}\int_0^m \frac{dt}{(n-t)^{1-r}} + \frac{A\varepsilon}{n^r}\int_m^n \frac{dt}{(n-t)^{1-r}}$$
$$\leqq A(n^r - (n-m)^r)/n^r + A\varepsilon$$
$$\leqq A\varepsilon,$$

for sufficiently large n. Secondly, we have

$$I_2 = \int_0^n P(t)\left(1 - \frac{t}{n}\right)^r a'(t) \sin tx\, dt.$$

By the second mean value theorem

$$I_2 = \int_0^{\theta n} P(t)\, a'(t) \sin tx\, dt \qquad (0 \leqq \theta_n \leqq n),$$

and then by integration by parts

$$I_2 = \left[a'(t)\int_0^t P(u)\sin ux\,du\right]_0^{\theta n} - \int_0^{\theta n} a''(t)\,dt\int_0^t P(u)\sin ux\,du.$$

Since $P(u) = \sum_{\nu=1}^{n}(\sin 2\pi\nu u)/\pi\nu$, we have

$$\int_0^t P(u)\sin ux\,du = \int_0^t \sin ux\Big(\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu u}{\pi\nu}\Big)du = \sum_{\nu=1}^{\infty}\frac{1}{\pi\nu}\int_0^t \sin ux\sin 2\pi\nu u\,du,$$

where the change of order of summation and integration is legitimate since the series $\sum \sin\nu u/\nu$ is boundedly convergent. The last sum is

$$\sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\int_0^t(\cos(x-2\pi\nu)u - \cos(x+2\pi\nu)u)\,du$$

$$= \sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\Big(\frac{\sin(x-2\pi\nu)t}{x-2\pi\nu} - \frac{\sin(x+2\pi\nu)t}{x+2\pi\nu}\Big)$$

$$= \sum_{\nu=1}^{\infty}\frac{1}{2\pi\nu}\,\frac{(x+2\pi\nu)\sin(x-2\pi\nu)t - (x-2\pi\nu)\sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)}$$

$$= \sum_{\nu=1}^{\infty}\Big(\frac{x}{2\pi\nu}\,\frac{\sin(x-2\pi\nu)t - \sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)} + \frac{\sin(x-2\pi\nu)t + \sin(x+2\pi\nu)t}{(x-2\pi\nu)(x+2\pi\nu)}\Big).$$

Accordingly we get

$$I_2 = \int_0^{\theta n} a''(t)\,dt\int_0^t P(u)\sin ux\,du$$

$$= x\sum_{\nu=1}^{\infty}\int_0^{\theta n} a''(t)\frac{\sin(x-2\pi\nu)t - \sin(x+2\pi\nu)t}{2\pi\nu(x-2\pi\nu)(x+2\pi\nu)}\,dt$$

$$+ \sum_{\nu=1}^{\infty}\int_0^{\theta n}\frac{a''(t)\{\sin(x-2\pi\nu)t + \sin(x+2\pi\nu)t\}}{(x-2\pi\nu)(x+2\pi\nu)}\,dt = J_1 + J_2.$$

Since $\int_0^{\infty}|a''(t)|\,dt < \infty$, $\int_0^{\theta n} a''(t)\sin ut\,dt$ is bounded, and hence

$$|J_1| \leqq Ax\sum\nu^{-3}$$

which is less than ε for small x. Concerning J_2 we write

$$J_2 = \sum_{\nu=1}^{\infty} = \sum_{\nu=1}^{N} + \sum_{\nu=N+1}^{\infty} = J_{21} + J_{22},$$

where N is taken such that $\sum_{\nu=N+1}^{\infty}\nu^{-2} < \varepsilon$. Then

$$|J_{22}| < A\varepsilon.$$

Since $a''(t)$ is absolutely integrable,

$$\Big|\int_0^{\theta n} a''(t)\,[\sin(2\pi\nu + x)t - \sin(2\pi\nu - x)t]\,dt\Big|$$

$$\leqq 2\Big|\int_0^{M} a''(t)\sin xt\cos 2\pi\nu t\,dt\Big| + 2\int_M^{\theta n}|a''(t)|\,dt.$$

If M is taken such that $\int_M^\infty |a''(t)|\,dt < \varepsilon$, then, for such fixed M

$$\left|\int_0^M a''(t)\sin xt\cos 2\pi\nu t\,dt\right| \leqq xM\int_0^M |a''(t)|\,dt \leqq Ax,$$

which is less than ε for sufficiently small x. Thus we have proved that

$$|I_2| = \left|\int_0^{\theta n} a''(t)\,dt\int_0^t P(u)\sin ux\,du\right| < A\varepsilon.$$

Finally

$$I_3 = x\int_0^n P(t)\left(1-\frac{t}{n}\right)^r a(t)\cos tx\,dt = \frac{x}{n^r}\int_0^n P(t)\,(n-t)^r a(t)\cos tx\,dt,$$

and then by the second mean value theorem

$$I_3 = x\int_0^{\xi_n} P(t)\,a(t)\cos tx\,dt \qquad (0\leqq \xi_n \leqq n)$$

$$= x\left[a(t)\int_0^t P(u)\cos ux\,du\right]_0^{\xi_n} - x\int_0^{\xi_n} a'(t)\,dt\int_0^t P(u)\cos ux\,du.$$

We have now

$$\int_0^t P(u)\cos ux\,du = \int_0^t \cos ux\left(\sum_{\nu=1}^\infty \frac{\sin 2\pi\nu u}{\pi\nu}\right)du$$

$$= \sum_{\nu=1}^\infty \frac{1}{\pi\nu}\int_0^t \cos ux\sin 2\pi\nu u\,du$$

$$= \sum_{\nu=1}^\infty \frac{1}{2\pi\nu}\int_0^t \{\sin(x+2\pi\nu)u - \sin(x-2\pi\nu)u\}du$$

$$= \sum_{\nu=1}^\infty \frac{1}{2\pi\nu}\int_0^t \{\sin(x+2\pi\nu)u + \sin(2\pi\nu-x)u\}du$$

$$= \sum_{\nu=1}^\infty \frac{1}{2\pi\nu}\left(\frac{1-\cos(x+2\pi\nu)t}{2\pi\nu+x} - \frac{1-\cos(2\pi\nu-x)t}{2\pi\nu-x}\right)$$

$$= O(\textstyle\sum \nu^{-2}) = O(1).$$

Therefore

$$|I_3| \leqq Ax + Ax\int_0^{\xi_n} |a'(t)|\,dt \leqq Ax$$

which is less than ε for sufficiently small x. Summing up above estimations, we get

$$|\tau_n^r(x,g) - \varphi_r(n,nx,g)| < A\varepsilon,$$

which prove (7). Thus the lemma is proved.

We shall now prove Theorem 4. Since

$$f(x) = \psi(x) + g(x), \qquad (a=1,\ \xi=0)$$

we have

$$\sigma_n^r(x,f) = \sigma_n^r(x,\psi) + \sigma_n^r(x,g).$$

By Theorem 1, $\sigma_n^r(x,\psi)$ presents Gibbs' phenomenon for $r < r_0$ but not for $r \geqq r_0$, and hence it is sufficient to prove that

$$\sigma_n^r(x,g) \to 0 \qquad (n \to \infty) \tag{16}$$

for all r and for all x. Let us put

$$g(x) \sim \sum_{n=1}^{\infty} a_n \sin nx,$$

then, by Lemma, (16) is equivalent to

$$n\int_0^1 (1-t)^r a(nt) \sin xt\, dt \to 0. \tag{17}$$

Let us take m such that

$$|ta(t)| < \varepsilon \qquad (t > m),$$

and write

$$n\int_0^1 (1-t)^r a(nt) \sin tx\, dt = n\int_0^{m/n} + n\int_{m/n}^1,$$

and then its absolute value is less than

$$A_n\int_0^{m/n} (1-t)^r \frac{\sin tx}{t} dt + \varepsilon \int_{m/n}^1 (1-t)^r |\sin tx| dt$$
$$\leqq Axm + \varepsilon < A\varepsilon$$

for sufficiently small x. Thus (11) and then (10) is proved. Thus the theorem is proved.

Finally I wish to express here my hearty thanks to Professor S. Izumi and Miss M. Satô for their kind advices.

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