ON A NON-NEGATIVE SUBHARMONIC FUNCTION IN A HALF-PLANE

BY MASATSUGU TSUJI

1. We shall prove

THEOREM 1. *Let* $u(z) = u(x+iy) \not\equiv 0$ *be a non-negative subharmonic function in a half-plane* $x > 0$, *which vanishes continuously on the imaginary axis.*
Let

$$m(r) = m(r, u) = \int_{-\pi/2}^{\pi/2} u(re^{i\theta}) \cos\theta \, d\theta, \qquad 0 < r < \infty,$$

then
(i) $m(r)/r$ *is a continuous non-decreasing function of* r *and is a convex function of* $1/r^2$. *Hence*

$$\lim_{r\to\infty} \frac{m(r)}{r} = c, \qquad 0 < c \leqq \infty,$$

exists.
If $0 < c < \infty$, *then*

(ii) $$u(z) = kx - \int_{\Re(a)>0} \log\left|\frac{z+\bar{a}}{z-a}\right| d\mu(a), \qquad k = \frac{2c}{\pi},$$

where μ *is a positive mass distribution in* $x > 0$, *such that*

$$\int_{\Re(a)>0} \frac{\Re(a)}{|a|^2} d\mu(a) < \infty.$$

(iii) *Except a set of* θ *of logarithmic capacity zero,*

$$\lim_{r\to\infty} \frac{u(re^{i\theta})}{r} = k\cos\theta$$

exists.

That $m(r)/r$ is a non-decreasing function of r is proved by Ahlfors [1] and the proof is simplified by Dinghas [2]. (iii) is proved by Ahlfors and Heins [3].
As a special case, we have

THEOREM 2. *Let* $f(z)$ *be regular in* $x > 0$ *and continuous and* $|f(z)| \leqq 1$ *on the imaginary axis. Suppose that* $\log^+|f(z)| \not\equiv 0$ *and let*

Received October 20, 1956.

$$m(r) = \int_{-\pi/2}^{\pi/2} \log^+ |f(re^{i\theta})| \cos\theta \, d\theta, \qquad 0 < r < \infty,$$

then $m(r)/r$ is a continuous non-decreasing function of r and is a convex function of $1/r^2$.

2. First we shall prove some lemmas.

LEMMA 1. *Let $u(z)$ be subharmonic in a domain D_0 and D be a subdomain of D_0, such that $\bar{D} \subset D_0$, whose boundary Γ consists of a finite number of analytic Jordan curves and $G(z, a)$ be the Green's function of D. Then*

$$u(z) = \frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta| - \int_D G(z, a) \, d\mu(a), \qquad z \in D,$$

where ν is the inner normal of Γ at ζ and μ is a positive mass distribution in D_0.

Proof. By F. Riesz' theorem, there exists a positive mass distribution μ in D_0, such that for any subdomain D_1 of D_0, such that $\bar{D}_1 \subset D_0$,

$$u(z) = v_1(z) - \int_{D_1} G_1(z, a) \, d\mu(a), \qquad z \in D_1,$$

where $v_1(z)$ is harmonic in D_1 and $G_1(z, a)$ is the Green's function of D_1. If we choose D_1, such that $\bar{D} \subset D_1 \subset D_0$, then if $z \in D$,

$$\begin{aligned} &\frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta| \\ &= \frac{1}{2\pi} \int_\Gamma v_1(\zeta) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta| - \frac{1}{2\pi} \int_{D_1} d\mu(a) \int_\Gamma G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta| \\ &= v_1(z) - \frac{1}{2\pi} \int_{D_1} d\mu(a) \int_\Gamma G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta| \\ &= u(z) + \int_{D_1} G_1(z, a) \, d\mu(a) - \frac{1}{2\pi} \int_{D_1} d\mu(a) \int_\Gamma G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta|. \end{aligned} \tag{1}$$

If $a \in D_1 - D$, then since $G_1(z, a)$ is harmonic in D,

$$\frac{1}{2\pi} \int_\Gamma G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} |d\zeta| = G_1(z, a), \quad a \in D_1 - D. \tag{2}$$

If $a \in D$, let $\Gamma(a): |\zeta - a| = r$ be a circle about a and $\Gamma(z): |\zeta - z| = r$ be that about z, such that $\Gamma(a)$ and $\Gamma(z)$ are contained in D. By the Green's formula,

$$\begin{aligned} &\int_\Gamma \left(G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} - G(\zeta, z) \frac{\partial G_1(\zeta, a)}{\partial \nu} \right) |d\zeta| \\ &+ \int_{\Gamma(z)} \left(G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} - G(\zeta, z) \frac{\partial G_1(\zeta, a)}{\partial \nu} \right) |d\zeta| \\ &+ \int_{\Gamma(a)} \left(G_1(\zeta, a) \frac{\partial G(\zeta, z)}{\partial \nu} - G(\zeta, z) \frac{\partial G_1(\zeta, a)}{\partial \nu} \right) |d\zeta| = 0. \end{aligned}$$

If we make $r \to 0$, then since $G(\zeta, z) = 0$ on Γ, we have

$$(3) \qquad \frac{1}{2\pi}\int_{\Gamma} G_1(\zeta, a)\frac{\partial G(\zeta, z)}{\partial \nu}|d\zeta| = G_1(z, a) - G(z, a), \qquad a \in D.$$

Hence

$$\int_{D_1} G_1(z, a)\,d\mu(a) - \frac{1}{2\pi}\int_{D_1} d\mu(a)\int_{\Gamma} G_1(\zeta, a)\frac{\partial G(\zeta, z)}{\partial \nu}|d\zeta|$$

$$= \int_{D_1} G_1(z, a)\,d\mu(a) - \int_{D_1 - D} G_1(z, a)\,d\mu(a) - \int_{D} G_1(z, a)\,d\mu(a) + \int_{D} G(z, a)\,d\mu(a)$$

$$= \int_{D} G(z, a)\,d\mu(a),$$

so that by (1),

$$u(z) = \frac{1}{2\pi}\int_{\Gamma} u(\zeta)\frac{\partial G(\zeta, z)}{\partial \nu}|d\zeta| - \int_{D} G(z, a)\,d\mu(a), \qquad z \in D.$$

LEMMA 2. *Let Δ_ρ: $|z| < \rho$, $\Re(z) > 0$ be a half-disc and C_ρ: $|z| = \rho$, $\Re(z) \geqq 0$ be a semi-circle and*

$$G_\rho(z, a) = \log\left|\frac{z + \bar{a}}{z - a}\cdot\frac{\rho^2 - z\bar{a}}{\rho^2 + za}\right|$$

be the Green's function of Δ_ρ. Then if $r < \rho$,

$$(\text{i}) \quad \rho\frac{\partial G_\rho(\rho e^{i\varphi}, re^{i\theta})}{\partial \nu} = 4(\lambda\cos\varphi\cos\theta + \lambda^2\sin 2\varphi\sin 2\theta + \lambda^3\cos 3\varphi\cos 3\theta + \cdots), \quad \lambda = r/\rho < 1,$$

where ν is the inner normal of C_ρ at $z = \rho e^{i\varphi}$.

$$(\text{ii}) \qquad \int_{-\pi/2}^{\pi/2} \frac{\partial G_\rho(\rho e^{i\varphi}, re^{i\theta})}{\partial \nu}\rho\cos\theta\,d\theta = \frac{2\pi r}{\rho}\cos\varphi.$$

Proof. If $z = \rho e^{i\varphi}$, $a = re^{i\theta}$, then

$$\rho\frac{\partial G(z, a)}{\partial \nu} = -\Re\left(z\frac{d}{dz}\log\left(\frac{z + \bar{a}}{z - a}\cdot\frac{\rho^2 - z\bar{a}}{\rho^2 + za}\right)\right)$$

$$= \Re\left(\frac{-z}{z + \bar{a}} + \frac{z}{z - a} + \frac{z\bar{a}}{\rho^2 - z\bar{a}} + \frac{za}{\rho^2 + za}\right)$$

$$(1) \qquad = \Re\left(\frac{-z}{z + \bar{a}} + \frac{z}{z - a} + \frac{\rho^2}{\rho^2 - z\bar{a}} - \frac{\rho^2}{\rho^2 + za}\right)$$

$$= 2\Re\left(\frac{1}{1 - \lambda e^{i(\theta - \varphi)}} - \frac{1}{1 + \lambda e^{i(\theta + \varphi)}}\right)$$

$$= 4(\lambda\cos\varphi\cos\theta + \lambda^2\sin 2\varphi\sin 2\theta + \lambda^3\cos 3\varphi\cos 3\theta + \cdots).$$

Since

$$\int_{-\pi/2}^{\pi/2}\cos^2\theta d\theta = \frac{\pi}{2}, \qquad \int_{-\pi/2}^{\pi/2}\sin 2n\theta \cos\theta d\theta = 0,$$

$$\int_{-\pi/2}^{\pi/2}\cos(2n+1)\theta\cos\theta d\theta = 0, \qquad (n = 1, 2, \cdots),$$

$$(2) \qquad \int_{-\pi/2}^{\pi/2}\frac{\partial G_\rho(\rho e^{i\varphi}, re^{i\theta})}{\partial\nu}\rho\cos\theta d\theta = \frac{2\pi r}{\rho}\cos\varphi.$$

LEMMA 3 [3]. *Let*

$$G(z, a) = \log\left|\frac{z+\bar{a}}{z-a}\right|, \qquad z = re^{i\theta}, \qquad a = \tau e^{i\varphi},$$

be the Green's function of $x > 0$, *then*

(i) $$G(z, a) \leqq G(e^{i\theta}, e^{i\varphi}),$$

(ii) *If* $|\theta| \leqq \theta_0 < \pi/2$, *then*

$$G(z, a) \leqq K(\theta_0)\frac{r\Re(a)}{|a|^2}(G(e^{i\theta}, e^{i\varphi}) + 1), \qquad K(\theta_0) = \frac{8}{\cos^2\theta_0}.$$

Proof. Since (i) can be proved easily, we shall prove (ii). Let $\Re(z) = x$, $\Re(a) = \xi$, then $x \geqq r\cos\theta_0$.

$$(1) \qquad G(z, a) = \frac{1}{2}\log\left(1 + \frac{4x\xi}{|z-a|^2}\right) \leqq \frac{2x\xi}{|z-a|^2} \leqq \frac{2r\xi}{|z-a|^2}.$$

If $|z-a| \geqq \frac{|a|\cos\theta_0}{2}$, then

$$(2) \qquad G(z, a) \leqq \frac{8r\Re(a)}{\cos^2\theta_0|a|^2} \leqq \frac{8r\Re(a)}{\cos^2\theta_0|a|^2}(G(e^{i\theta}, e^{i\varphi}) + 1).$$

If $|z-a| \leqq \frac{|a|\cos\theta_0}{2}$, then $\left|\frac{1}{a} - \frac{1}{z}\right| \leqq \frac{\cos\theta_0}{2|z|}$, so that

$$\frac{\xi}{|a|^2} \geqq \frac{x}{|z|^2} - \frac{\cos\theta_0}{2|z|} \geqq \frac{\cos\theta_0}{|z|} - \frac{\cos\theta_0}{2|z|} = \frac{\cos\theta_0}{2|z|},$$

$$\frac{2r\Re(a)}{\cos\theta_0|a|^2} \geqq 1,$$

hence

$$(3) \qquad G(z, a) \leqq G(e^{i\theta}, e^{i\varphi}) \leqq \frac{2r\Re(a)}{\cos\theta_0|a|^2}(G(e^{i\theta}, e^{i\varphi}) + 1).$$

Hence by (2),(3),

$$G(z, a) \leqq K(\theta_0)\frac{r\Re(a)}{|a|^2}(G(e^{i\theta}, e^{i\varphi}) + 1), \qquad K(\theta_0) = \frac{8}{\cos^2\theta_0}.$$

3. Now we shall prove the theorem. We extend the definition of u for $x < 0$ by putting $u = 0$ for $x < 0$, then since $u = 0$ on the imaginary axis, u becomes a non-nagative subharmonic function for $|z| < \infty$. Let μ be the positive mass distribution, defined by u, then since $u = 0$ for $x < 0$, $\mu = 0$

for $x < 0$. Let Δ_ρ: $|z| < \rho$, $\Re(z) > 0$ be a half-disc and C_ρ: $|z| = \rho$, $\Re(z) \geqq 0$ be a semi-circle and L_ρ: $z = iy\,(-\rho \leqq y \leqq \rho)$ be a segment on the imaginary axis, then $\Gamma_\rho = C_\rho + L_\rho$ is the boundary of Δ_ρ and

$$(1) \qquad G_\rho(z, a) = \log\left|\frac{z+\bar{a}}{z-a}\cdot\frac{\rho^2 - z\bar{a}}{\rho^2 + za}\right|$$

is the Green's function of Δ_ρ. Since $u = 0$ on L_ρ, we have by Lemma 1,

$$(2) \qquad u(z) = v_\rho(z) - w_\rho(z) \quad \text{in} \quad \Delta_\rho,$$

where

$$(3) \qquad v_\rho(z) = \frac{1}{2\pi}\int_{-2/\pi}^{\pi/2} u(\rho e^{i\varphi})\frac{\partial G_\rho(\rho e^{i\varphi}, z)}{\partial \nu}\rho\, d\varphi,$$

$$(4) \qquad w_\rho(z) = \int_{\Delta_\rho} G_\rho(z, a)\, d\mu(a).$$

Let

$$m(r, v_\rho) = \int_{-\pi/2}^{\pi/2} v_\rho(re^{i\theta})\cos\theta d\theta, \qquad m(r, w_\rho) = \int_{-\pi/2}^{\pi/2} w_\rho(re^{i\theta})\cos\theta d\theta,$$

then

$$(5) \qquad m(r, u) = m(r, v_\rho) - m(r, w_\rho).$$

By Lemma 2,

$$m(r, v_\rho) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} u(\rho e^{i\varphi})\, d\varphi \int_{-\pi/2}^{\pi/2} \frac{\partial G_\rho(\rho e^{i\varphi}, re^{i\theta})}{\partial \nu}\rho\cos\theta d\theta$$

$$= \frac{r}{\rho}\int_{-\pi/2}^{\pi/2} u(\rho e^{i\varphi})\cos\varphi d\varphi = \frac{rm(\rho, u)}{\rho},$$

so that

$$(6) \qquad \frac{m(r, v_\rho)}{r} = \frac{m(\rho, u)}{\rho}, \qquad (0 < r \leqq \rho).$$

Now

$$w_\rho(z) = \int_{\Delta_\rho}\log\left|\frac{z+\bar{a}}{z-a}\right| d\mu(a) - \int_{\Delta_\rho}\log\left|\frac{\rho^2 + za}{\rho^2 - z\bar{a}}\right| d\mu(a) = w'_\rho(z) - w''_\rho(z).$$

If $z = re^{i\theta}$, $a = \tau e^{i\varphi}$, then we can prove similarly as Lemma 2, that if $\tau \leqq r \leqq \rho$,

$$\int_{-\pi/2}^{\pi/2}\log\left|\frac{z+\bar{a}}{z-a}\right|\cos\theta d\theta = \frac{\pi\tau\cos\varphi}{r} = \frac{\pi\Re(a)}{r},$$

and if $r \leqq \tau \leqq \rho$,

$$\int_{-\pi/2}^{\pi/2}\log\left|\frac{z+\bar{a}}{z-a}\right|\cos\theta d\theta = \frac{\pi r\cos\varphi}{\tau} = \frac{\pi r\Re(a)}{|a|^2},$$

hence

$$(7) \qquad \frac{m(r, w'_\rho)}{r} = \frac{\pi}{r^2}\int_{|a|<r}\Re(a)\, d\mu(a) + \pi\int_{r\leqq|a|<\rho}\frac{\Re(a)\, d\mu(a)}{|a|^2}.$$

Since

$$\int_{-\pi/2}^{\pi/2} \log \left| \frac{1+(r\tau/\rho^2)e^{i(\theta+\varphi)}}{1-(r\tau/\rho^2)e^{i(\theta-\varphi)}} \right| \cos\theta d\theta = \frac{\pi r\tau}{\rho^2}\cos\varphi = \frac{\pi r \Re(a)}{\rho^2},$$

we have

$$(8) \qquad \frac{m(r, w_\rho'')}{r} = \frac{\pi}{\rho^2}\int_{|a|<\rho} \Re(a)\, d\mu(a),$$

so that

$$\frac{m(r, w_\rho)}{r} = \frac{\pi}{r^2}\int_{|a|<r} \Re(a)\, d\mu(a) + \pi\int_{r\leqq|a|<\rho} \frac{\Re(a)\, d\mu(a)}{|a|^2} - \frac{\pi}{\rho^2}\int_{|a|<\rho} \Re(a)\, d\mu(a).$$

Hence if we put

$$(9) \qquad \Omega(r) = \int_{|a|<r} \Re(a)\, d\mu(a),$$

then by the partial integration, we have easily

$$(10) \qquad \frac{m(r, w_\rho)}{r} = 2\pi\int_r^\rho \frac{\Omega(t)\,dt}{t^3},$$

so that by (5), (6), (10),

$$(11) \qquad \frac{m(r, u)}{r} = \frac{m(\rho, u)}{\rho} - 2\pi\int_r^\rho \frac{\Omega(t)\,dt}{t^3}, \qquad 0 < r \leqq \rho.$$

Hence $m(r, u)/r$ is a continuous non-decreasing function of r and since

$$\frac{d(m(r, u)/r)}{d(1/r^2)} = -\pi\Omega(r),$$

$m(r, u)/r$ is a convex function of $1/r^2$. From (11),

$$2\pi\int_r^\rho \frac{\Omega(t)\,dt}{t^3} \leqq \frac{m(\rho, u)}{\rho},$$

so that

$$(12) \qquad 2\pi\int_0^\rho \frac{\Omega(t)\,dt}{t^3} \leqq \frac{m(\rho, u)}{\rho}.$$

Since $m(r, u)/r$ is a non-decreasing function of r,

$$(13) \qquad \lim_{r\to\infty} \frac{m(r, u)}{r} = c, \qquad 0 < c \leqq \infty,$$

exists.

If $0 < c < \infty$, then by (11),

$$(14) \qquad \frac{m(r, u)}{r} = c - 2\pi\int_r^\infty \frac{\Omega(t)\,dt}{t^3},$$

and from (12),

$$(15) \qquad 2\pi\int_0^\infty \frac{\Omega(t)\,dt}{t^3} \leqq c.$$

Hence

$$\lim_{r\to 0}\frac{\Omega(r)}{r^2}=0,\qquad \lim_{r\to\infty}\frac{\Omega(r)}{r^2}=0, \tag{16}$$

and from this, we have

$$\int_{|a|<\infty}\frac{\Re(a)}{|a|^2}d\mu(a)=2\int_0^\infty\frac{\Omega(t)dt}{t^3}\leqq\frac{c}{\pi}<\infty. \tag{17}$$

By Lemma 2, if $z=re^{i\theta}$, $\lambda=r/\rho<1$, then

$$\begin{aligned} v_\rho(z)&=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}u(\rho e^{i\varphi})\frac{\partial G_\rho(\rho e^{i\varphi},re^{i\theta})}{\partial\nu}\rho d\varphi\\ &=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}u(\rho e^{i\varphi})\Big(\frac{r}{\rho}\cos\varphi\cos\theta+O\Big(\frac{r}{\rho}\Big)^2\Big)d\varphi,\end{aligned}$$

so that

$$\lim_{\rho\to\infty}v_\rho(z)=\frac{2r\cos\theta}{\pi}\lim_{\rho\to\infty}\int_{-\pi/2}^{\pi/2}\frac{u(\rho e^{i\varphi})}{\rho}\cos\varphi d\varphi=kx,\qquad k=\frac{2c}{\pi}. \tag{18}$$

Since

$$\log\left|\frac{\rho^2+z\bar{a}}{\rho^2-za}\right|=\frac{1}{2}\log\Big(1+\frac{4\rho^2x\xi}{|\rho^2-za|^2}\Big)\leqq\frac{2\rho^2x\xi}{|\rho^2-za|^2}\leqq\frac{2|z|\xi}{(\rho-|z|)^2},$$

$$x=\Re(z),\qquad \xi=\Re(a),$$

$$w_\rho''(z)\leqq\frac{2|z|}{(\rho-|z|)^2}\int_{|a|<\rho}\Re(a)d\mu(a)=\frac{2|z|\rho^2}{(\rho-|z|)^2}\cdot\frac{\Omega(\rho)}{\rho^2},$$

hence by (16),

$$\lim_{\rho\to\infty}w_\rho''(z)=0,$$

so that in $x>0$,

$$u(z)=kx-w(z), \tag{19}$$

where

$$w(z)=\int_{|a|<\infty}\log\left|\frac{z+\bar{a}}{z-a}\right|d\mu(a). \tag{20}$$

Next we shall prove that, except a set of θ of logarithmic capacity zero,

$$\lim_{r\to\infty}\frac{w(re^{i\theta})}{r}=0. \tag{21}$$

By Lemma 3, if $|\theta|\leqq\theta_0<\pi/2$ and $\arg a=\varphi$, we have for any $r_0>0$,

$$\begin{aligned}\frac{w(re^{i\theta})}{r}&=\frac{1}{r}\int_{|a|<r_0}G(re^{i\theta},a)d\mu(a)+\frac{1}{r}\int_{r_0\leqq|a|<\infty}G(re^{i\theta},a)d\mu(a)\\ &\leqq\frac{1}{r}\int_{|a|<r_0}G(re^{i\theta},a)d\mu(a)+K(\theta_0)\int_{r_0\leqq|a|<\infty}\frac{\Re(a)}{|a|^2}(G(e^{i\theta},e^{i\varphi})+1)d\mu(a).\end{aligned} \tag{22}$$

Hence if we put

$$\chi(\theta) = \overline{\lim_{r\to\infty}} \frac{w(re^{i\theta})}{r}, \tag{23}$$

then if $|\theta| \leqq \theta_0 < \pi/2$,

$$\chi(\theta) \leqq K(\theta_0) \int_{r_0 \leqq |a| < \infty} \frac{\Re(a)}{|a|^2} (G(e^{i\theta}, e^{i\varphi}) + 1) d\mu(a). \tag{24}$$

Suppose that $\chi(\theta) > 0$ on a set E of positive logarithmic capacity on $|z| = 1$, then by taking a suitable closed subset, we may assume that E is a closed set, contained in $|\arg z| \leqq \theta_0 < \pi/2$. Let ν be the mass of equilibrium distribution of E and

$$U(z) = \int_E \log \left| \frac{z + e^{-i\theta}}{z - e^{i\theta}} \right| d\nu(\theta), \qquad \nu(E) = 1$$

be the conductor potential of E, such that $U(z) \leqq V < \infty$ for any z. Then

$$\int_E \chi(\theta) d\nu(\theta) \leqq K(\theta_0)(V+1) \int_{r_0 \leqq |a| < \infty} \frac{\Re(a) d\mu(a)}{|a|^2}.$$

Since $\int_{|a|<\infty} \frac{\Re(a) d\mu(a)}{|a|^2} < \infty$, the right hand side tends to zero, if $r_0 \to \infty$, hence

$$\int_E \chi(\theta) d\nu(\theta) = 0,$$

which is absurd. Hence $\chi(\theta) = 0$, except a set of θ of logarithmic capacity zero, which is equivalent to (21). Hence

$$\lim_{r\to\infty} \frac{u(re^{i\theta})}{r} = k \cos\theta,$$

except a set of θ of logarithmic capacity zero.

References

[1] L. Ahlfors, On Phragmén-Lindelöf principle. Trans. Amer. Math. Soc. **41** (1933), 1—8.

[2] A. Dinghas, Über das Phragmén-Lindelöfsche Prinzip und den Julia-Carathéodryschen Satz. Sitzungsber. preuss. Akad. Wiss. (1938), 32—48.

[3] L. Ahlfors and M. Heins, Questions of regularity connected with the Phragmén-Lindelöf principle. Ann. Math. **50** (1949), 341—346.

Mathematical Institute,
Rikkyo University, Tokyo.