FURTHER SUPPLEMENT TO "ON TRANSFERENCE OF BOUNDARY VALUE PROBLEMS"

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In a previous paper [1] it has been shown that for some simple domains two main kinds of boundary value problems in potential theory are transferable each other by means of elementary operations. And then in another paper [2] it has been supplemented that the relations for transference can be readily verified also by deriving separately the explicit formulas for both problems.

However, in these papers explicit use has been made of well-known but special formulas for solving two kinds of boundary value problems with respect to a circle. In the present Note, considering the same topics once again, we shall derive the results without any reference to these special formulas. The present method may probably suggest a way to obtain analogous results for more general types of basic domains.

1. Rectilinear slit domain.

Theorem A. *Let D be a basic domain laid on the $z=x+iy$-plane whose boundary C is a segment defined by*

$$C: \qquad x=0, \qquad -1 \leqq y \leqq +1.$$

Let $u(z)=\Re f(z)$ and $v(z)=\Re g(z)$, $f(z)$ and $g(z)$ being analytic, be the solutions of Dirichlet and Neumann problems, respectively, with related boundary conditions

$$u=\pm V^{\pm}(y), \quad \frac{\partial v}{\partial \nu}=V^{\pm}(y) \qquad \text{for} \quad z=\pm 0+iy \quad (-1<y<+1),$$

$\partial/\partial\nu$ denoting the differentiation along inward normal; the condition for solvability of the latter problem, i.e.

$$\int_{-1}^{1}(V^{+}(y)+V^{-}(y))\,dy=0,$$

is, of course, supposed to be valid. Then there holds a connecting relation

$$f(z)=g'(z)+i\rho+\frac{\sigma z+i\tau}{\sqrt{1+z^2}}$$

or

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$$g(z)=g(\infty)+\int_{\infty}^{z}\left(f(\zeta)-i\rho-\frac{\sigma\zeta+i\tau}{\sqrt{1+\zeta^2}}\right)d\zeta.$$

Here $\sqrt{1+z^2}$ *denotes such a branch that* $\sqrt{1+z^2}\sim z$ *as* $z\to\infty$, *and* ρ, σ, τ *are real constants given by*

$$\rho=\Im f(\infty) \quad (arbitrary),$$

$$\sigma=\Re f(\infty)=\frac{i}{2}\left([\sqrt{1+z^2}g'(z)]^{z=+i}-[\sqrt{1+z^2}g'(z)]^{z=-i}\right),$$

$$\tau=-i[z(f(z)-f(\infty))]^{z=\infty}=\frac{i}{2}\left([\sqrt{1+z^2}g'(z)]^{z=+i}+[\sqrt{1+z^2}g'(z)]^{z=-i}\right)$$

or alternatively, in terms of boundary functions, also by

$$\sigma=\frac{1}{2\pi}\int_{-1}^{1}(V^+(y)-V^-(y))\frac{1}{\sqrt{1-y^2}}dy,$$

$$\tau=\frac{1}{2\pi}\int_{-1}^{1}(V^+(y)-V^-(y))\frac{y}{\sqrt{1-y^2}}dy.$$

Proof. Consider an analytic function defined by

$$F(z)=\sqrt{1+z^2}(g'(z)-f(z)+i\Im f(\infty))+z\Re f(\infty).$$

Since $g(z)$ remains regular at $z=\infty$ and hence $g'(z)$ vanishes there (at least in second order), it is evident that $F(z)$ remains regular also throughout D. Based on the assigned boundary conditions, there holds

$$\Re F(\pm 0+iy)=\pm\sqrt{1-y^2}\left(\pm\frac{\partial v}{\partial\nu}-u\right)=0 \qquad \text{along } C.$$

Hence, by reflection principle, $F(z)$ can be prolonged analytically across C into another sheet of Riemann surface associated to $\sqrt{1+z^2}$, in view of the equation $F(-\bar{z})=-\overline{F(z)}$. Thus, $F(z)$ becomes a function regular and single-valued throughout the closed two-sheeted Riemann surface and hence it must reduce to a purely imaginary constant. The last-mentioned fact is also an immediate consequence of the boundedness of $F(z)$ in D. Consequently, we may put

$$g'(z)=f(z)-i\rho-\frac{\sigma z+i\tau}{\sqrt{1+z^2}}$$

or

$$g(z)=g(\infty)+\int_{\infty}^{z}\left(f(\zeta)-i\rho-\frac{\sigma\zeta+i\tau}{\sqrt{1+\zeta^2}}\right)d\zeta,$$

where τ is a real constant and

$$\rho=\Im f(\infty), \qquad \sigma=\Re f(\infty).$$

Since $g'(z)=O(|z|^{-2})$ as $z\to\infty$, we have

$$\tau = -i \lim_{z\to\infty} (\sqrt{1+z^2}(f(z) - i\rho) - \sigma z)$$
$$= -i \lim_{z\to\infty} z(f(z) - \sigma - i\rho) = -i[z(f(z) - f(\infty))]^{z=\infty}.$$

The expressions for σ and τ in terms of $g(z)$ are readily derived by means of the connecting equation established. Namely, we have

$$[\sqrt{1+z^2}g'(z)]^{z=\pm i} = -(\pm i\sigma + i\tau) = \mp i\sigma - i\tau.$$

On the other hand, the alternative expressions for σ and τ in terms of the boundary functions are obtained as follows. By integrating both sides of the equation

$$\frac{g'(z)}{\sqrt{1+z^2}} = \frac{f(z) - i\rho}{\sqrt{1+z^2}} - \frac{\sigma z + i\tau}{1+z^2}$$

along both banks of the boundary slit, we get

$$i\int_{-1}^{1} g'(+0+iy)\frac{1}{\sqrt{1-y^2}}\,dy + i\int_{1}^{-1} g'(-0+iy)\frac{1}{-\sqrt{1-y^2}}\,dy$$
$$= \lim_{\delta\to+1}\left\{\int_{+0-(1-\delta)i}^{+0+(1-\delta)i} + \int_{-0+(1-\delta)i}^{-0-(1-\delta)i}\right\}\left(\frac{f(z) - i\rho}{\sqrt{1+z^2}} - \frac{\sigma z + i\tau}{1+z^2}\right)dz,$$

the right-hand member representing the sum of principal values of the improper integrals and *not* a contour integral. Here it is further to be noted that $\sqrt{1+z^2}$ denotes the branch which is equal to $\pm\sqrt{1-y^2}$ at $z = \pm 0 + iy\,(-1 < y < 1)$. Since two integrals of the single-valued (rational) function $(\sigma z + i\tau)/(1+z^2)$ cancel out, this leads to

$$i\int_{-1}^{1} (g'(+0+iy) + g'(-0+iy))\,\frac{1}{\sqrt{1-y^2}}\,dy$$
$$= \left\{\int_{+0-i}^{+0+i} + \int_{-0+i}^{-0-i}\right\}\frac{f(z) - i\rho}{\sqrt{1+z^2}}\,dz$$
$$= \oint_{|z|=R>1} \frac{f(z) - i\rho}{\sqrt{1+z^2}} = 2\pi i(f(\infty) - i\rho),$$

the contour integration being to be taken along any circle with center at 0 and radius greater than unity in the positive sense with respect to the origin. But, since, by definition, there holds

$$\Re g'(\pm 0 + iy) = \pm\frac{\partial}{\partial\nu}\Re g(\pm 0 + iy) = \pm V^{\pm}(y)$$

for $-1 < y < 1$, we finally have

$$\sigma = \Re f(\infty) = \frac{1}{2\pi}\int_{-1}^{1} (V^{+}(y) - V^{-}(y))\,\frac{1}{\sqrt{1-y^2}}\,dy.$$

Next, we get in a similar manner a relation

$$-\int_{-1}^{1}(g'(+0+iy)+g'(-0+iy))\frac{y}{\sqrt{1-y^2}}\,dy$$

$$=\lim_{\delta\to+1}\left\{\int_{+0-(1-\delta)i}^{+0+(1-\delta)i}+\int_{-0+(1-\delta)i}^{-0-(1-\delta)i}\right\}\left(\frac{z(f(z)-i\rho)}{\sqrt{1+z^2}}-\frac{z(\sigma z+i\tau)}{1+z^2}\right)dz$$

$$=\oint_{|z|=R>1}\frac{z(f(z)-i\rho)}{\sqrt{1+z^2}}\,dz$$

$$=\oint_{|z|=R>1}\left(\Re f(\infty)+\frac{i\tau}{z}+\cdots\right)\left(1-\frac{z}{2z^2}+\cdots\right)dz=-2\pi\tau,$$

whence readily follows

$$\tau=\frac{1}{2\pi}\int_{-1}^{1}(V^+(y)-V^-(y))\frac{y}{\sqrt{1-y^2}}\,dy.$$

2. Circular slit domain.

THEOREM B. *Let D be the whole $z=re^{i\theta}$–plane cut along a circular slit defined by*

$$C:\qquad r=1,\qquad \alpha\leqq\theta\leqq 2\pi-\alpha\qquad\qquad (0<\alpha<\pi).$$

Let $u(z)=\Re f(z)$ and $v(z)=\Re g(z)$, $f(z)$ and $g(z)$ being analytic, be the solutions of Dirichlet and Neumann problems, respectively, with related boundary conditions

$$u=\pm V^{\pm}(\theta),\qquad \frac{\partial v}{\partial\nu}=V^{\pm}(\theta)\quad \textit{for}\quad z=(1\pm 0)e^{i\theta}\quad (\alpha<\theta<2\pi-\alpha),$$

$\partial/\partial\nu$ denoting the differentiation along inward normal; the condition for solvability of the latter problem, i. e.

$$\int_{\alpha}^{2\pi-\alpha}(V^+(\theta)+V^-(\theta))\,d\theta=0,$$

is, of course, supposed to be valid. Then there holds a connecting relation

$$f(z)=zg'(z)+i\rho+\frac{\sigma z+\tau}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}$$

or

$$g(z)=g(\infty)+\int_{\infty}^{z}\left(f(\zeta)-i\rho-\frac{\sigma\zeta+\tau}{\sqrt{(\zeta-e^{i\alpha})(\zeta-e^{-i\alpha})}}\right)\frac{d\zeta}{\zeta}.$$

Here $\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}$ denotes such a branch that $\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\sim z$ as $z\to\infty$, ρ is an unessential real constant depending on the arbitrariness of adding any purely imaginary quantity to $f(z)$, and σ and τ are constants given by

$$\sigma = f(\infty) - i\rho = \frac{i}{2\sin\alpha}\left[z\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,g'(z)\right]_{e^{-i\alpha}}^{e^{i\alpha}},$$

$$\tau = \overline{f(\infty)} + i\rho = -\frac{i}{2\sin\alpha}\left[\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,g'(z)\right]_{e^{-i\alpha}}^{e^{i\alpha}}$$

or alternatively, in terms of boundary functions, also by

$$\sigma = \bar{\tau} = \frac{i}{4\pi}\int_{\alpha}^{2\pi-\alpha}(V^{+}(\theta) - V^{-}(\theta))\frac{e^{i\theta/2}}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}\,d\theta.$$

Proof. Without loss of generality, it may be supposed that $\rho = 0$, since otherwise it is only necessary to replace $f(z)$ by $f(z) - i\rho$. Consider an analytic function defined by

$$F(z) = \sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,(zg'(z) - f(z)).$$

For $z = (1 \pm 0)e^{i\theta}$ with $\alpha < \theta < 2\pi - \alpha$, there holds

$$\Re\frac{F(z)}{z-1} = \Re\frac{\mp 2ie^{i\theta/2}\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}{2ie^{i\theta/2}\sin\frac{\theta}{2}}\left[zg'(z) - f(z)\right]^{z=e^{i\theta}}$$

$$= \pm\frac{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}{\sin\frac{\theta}{2}}(\pm v - u) = 0.$$

Thus $F(z)/(z-1)$, and hence $F(z)$ itself also, is analytically prolongeable across C into another sheet of Riemann surface associated to $\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}$, in view of the equation

$$\frac{F(1/\bar{z})}{1/\bar{z}-1} = -\frac{\overline{F(z)}}{\bar{z}-1} \quad\text{or}\quad \bar{z}F\left(\frac{1}{\bar{z}}\right) = \overline{F(z)}.$$

The function $F(z)$ thus prolonged becomes single–valued on the closed two–sheeted Riemann surface and is regular throughout except for the point at infinity lying on the original sheet, around which it behaves in such a manner that

$$F(z) = z\left(1 - \frac{e^{i\alpha}}{2z} + O\left(\frac{1}{|z|^2}\right)\right)\left(1 - \frac{e^{-i\alpha}}{2z} + O\left(\frac{1}{|z|^2}\right)\right)$$
$$\times\left(-f(\infty) - \frac{\lambda}{z} + O\left(\frac{1}{|z|^2}\right)\right)$$
$$= -f(\infty)z + (f(\infty)\cos\alpha - \lambda) + O\left(\frac{1}{|z|}\right),$$

where λ is a constant; though explicitly unnecessary, its value is really given by

$$\lambda = \lim_{z\to\infty} z(f(z) - f(\infty) + g(z) - g(\infty)).$$

Accordingly, the function defined by

$$F(z) + f(\infty)z \equiv \sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,(zg' - f(z)) + f(\infty)z$$

is regular throughout the closed Riemann surface and hence it reduces to a constant. Consequently, we obtain an identity

$$\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,(zg'(z) - f(z)) + f(\infty)z = -f(0),$$

which represents the connecting relation stated in the theorem, the values of the constants involved being

$$(\rho = 0), \qquad \sigma = f(\infty), \qquad \tau = f(0).$$

Now, since, as shown above, $\Re(F(z)/(z-1)) = 0$ for $z = (1\pm 0)e^{i\theta}$ with $\alpha < \theta < 2\pi - \alpha$, we get, for such z, an equation

$$\begin{aligned} 0 &= \Re((e^{-i\theta}-1)F((1\pm 0)e^{i\theta})) \\ &= \Re((e^{-i\theta}-1)(-f(\infty)e^{i\theta} - f(0))) \\ &= (\cos\theta - 1)\Re(f(\infty) - f(0)) - \sin\theta\,\Im(f(\infty) + f(0)), \end{aligned}$$

which implies

$$\Re(f(\infty) - f(0)) = \Im(f(\infty) + f(0)) = 0,$$

i. e.

$$\tau = f(0) = \overline{f(\infty)}.$$

The expressions for σ and τ in terms of $g(z)$ can be readily derived by means of the connecting equation established. In fact, we have only to notice the relations

$$\begin{aligned} \sigma e^{\pm i\alpha} + \tau &= -\left[z\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,g'(z)\right]_{e^{\pm i\alpha}} \\ &= -e^{\pm i\alpha}\left[\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}\,g'(z)\right]_{e^{\pm i\alpha}}. \end{aligned}$$

On the other hand, the alternative expressions for σ and τ in terms of the boundary functions can be obtained as follows. By integrating both sides of the equation

$$\frac{(z-1)g'(z)}{i\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}} = \frac{(z-1)f(z)}{iz\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}} - \frac{(z-1)(f(\infty)z + \overline{f(\infty)})}{iz}$$

along both banks of the boundary slit and remembering that the last term is a single-valued (rational) function, we get

$$\int_{\alpha}^{2\pi-\alpha} e^{i\theta}(g'((1+0)e^{i\theta}) + g'((1-0)e^{i\theta}))\frac{\sin\frac{\theta}{2}}{-\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}\,d\theta$$

$$= \lim_{\delta\to+0}\left\{\int_{(1+0)e^{i\alpha}}^{(1-0)e^{i(2\pi-\alpha)}} + \int_{(1-0)e^{i(2\pi-\alpha)}}^{(1-0)e^{i\alpha}}\right\}\frac{(z-1)f(z)}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\frac{dz}{iz}$$

$$= \oint_{|z|=R>1}\frac{(z-1)f(z)}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\frac{dz}{iz}$$

$$- 2\pi i \operatorname*{Res}_{z=0}\left[\frac{(z-1)f(z)}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\frac{1}{iz}\right]$$

$$= 2\pi f(\infty) + 2\pi f(0) = 4\pi\Re f(\infty).$$

since, as shown above, $\overline{f(0)} = f(\infty)$; here $\operatorname*{Res}_{z=0}[\quad]$ designates the residue of the function involved within the brackets at $z=0$. It is further to be noted that $\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}$ denotes the branch which is equal to $+1$ at $z=0$ and to $\mp 2ie^{i\theta/2}\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}$ at $z=(1\pm 0)e^{i\theta}$ $(\alpha<\theta<2\pi-\alpha)$. But, since, by definition, there holds

$$\Re(e^{i\theta}g'((1\pm 0)e^{i\theta})) = \pm\frac{\partial}{\partial\nu}\Re g((1\pm 0)e^{i\theta}) = \pm V^{\pm}(\theta)$$

for $\alpha<\theta<2\pi-\alpha$, the above relation finally implies

$$\Re f(\infty) = \frac{1}{4\pi}\int_{\alpha}^{2\pi-\alpha}(V^{+}(\theta)-V^{-}(\theta))\frac{-\sin\frac{\theta}{2}}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta.$$

Quite similarly, if we replace in the above argument the factor $z-1$ by $z+1$, there results a relation

$$\int_{\alpha}^{2\pi-\alpha}e^{i\theta}(g'((1+0)e^{i\theta})+g'((1-0)e^{i\theta}))\frac{\cos\frac{\theta}{2}}{-i\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta$$

$$= 2\pi f(\infty) - 2\pi f(0) = 4\pi i\Im f(\infty),$$

whence readily follows

$$\Im f(\infty) = \frac{1}{4\pi}\int_{\alpha}^{2\pi-\alpha}(V^{+}(\theta)-V^{-}(\theta))\frac{\cos\frac{\theta}{2}}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta.$$

Consequently, there holds an equation

$$f(\infty) = \frac{i}{4\pi}\int_{\alpha}^{2\pi-\alpha}\frac{e^{i\theta/2}}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta,$$

which establishes the desired result.

3. Radial slit domain.

By the same reason as stated at the end of the previous paper [2], the case of radial slit domain which is a sort of rectilinear slit domain may be omitted.

REFERENCES

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