A NOTE ON MIDDLE UNITARY SEMIGROUPS

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In this paper the author touches upon two problems, one of which is to determine what is the structure of \((\mathcal{M})\)-inversible semigroups and the other is of the general theory of special middle unitary semigroups. The proof of every theorem or lemma, however, is to be stated in another paper. In this paper, we use symbols \(\oplus\) or \(\otimes\) for the direct sum, i.e., the disjoint sum of sets. And moreover let \(A, B\) be two of any subsets of a semigroup, and use \(AB\) for the set \(\{xy | x \in A, y \in B\}\).

1. In this paragraph, we define the relative inversibility of semigroups and completely determine the structure of semigroups having such a property in regard to their own middle units. Let \(S\) be a semigroup and \(N\) be a subset of \(S\), then we shall say that \(S\) is relatively invertible, relatively left invertible or relatively right invertible in regard to \(N\) if \(S\) satisfies each of the following conditions.

1. For any \(x \in S\), there exists \(x^\# \in S\) with \(xx^\# = x^\# x \in N\).
2. For any \(x \in S\), there exists \(x' \in S\) with \(x'x \in N\).
3. For any \(x \in S\), there exists \(x^\ddagger \in S\) with \(xx^\ddagger \in N\).

We use the initial signature \(\{\mathcal{N}\}\)-inversible semigroup for any semigroup relatively invertible in regard to \(N\), and especially use \(\{\mathcal{M}\}\)-inversible semigroup for any semigroup relatively invertible in regard to the set consisting of all middle units of its own. It is clear that any semigroup is relatively left (right) invertible in regard to itself and that, generally speaking, even if a semigroup is both relatively right invertible and relatively left invertible in regard to \(N\), it is not necessarily relatively invertible in regard to \(N\). However, if \(S\) is a middle unitary semigroup \([1]\) and \(N\) is a subset consisting of all middle units of its own, the following lemma is satisfied.

[Lemma 1.1] If \(S\) is both relatively right invertible and relatively left invertible in regard to \(N\), \(S\) is relatively invertible in regard to \(N\).

Now any group is a semigroup relatively invertible in regard to its unit and it is well known to the general public that the semigroups relatively invertible in regard to all their own left (right) units are nothing but the right (left) groups \([2]\).

Therefore, we may consider the \(\{\mathcal{M}\}\)-inversible semigroups to be semigroups more general than both the groups and the right (left) groups. Hereafter we use \(G\) for a \(\{\mathcal{M}\}\)-inversible semigroup, and \(M\) for the set consisting of all middle units of \(G\).

[Lemma 1.2] \(xx^\# = x^\# x \in M\), \(xx^\ddagger = x^\ddagger x \in M\) for some \(x, x^\#, x^\ddagger \in G\) implies
\(x^\# x = x^\ddagger x \implies x = x^\# = x^\ddagger\)

We give the following equivalent relation \(\sim\) to the elements of \(G\).

\(a \sim b\) when and only when there exists such an element \(x \in G\) as in \(ax \in M\) and \(bx \in M\).

In this case,
1. \(a \sim a\) for any \(a \in G\).
2. \(a \sim b\) implies \(b \sim a\).
3. \(a \sim b, b \sim c\) implies \(a \sim c\).
4. \(a \sim b, c \sim d\) implies \(a \sim d\).
are satisfied. Let \(\Omega\) be the factor algebraic system and \(\bar{a}\) be the residue class of \(G\) which contains the element \(a\) for any \(a \in G\), then the following lemma holds good.

[Lemma 1.3] \(\Omega\) is a group and its unit coincides with \(M\).

Now if \(\Gamma\) stands for the set \(\{x^\# | x \in M\}\), it coincides with the set consisting of all idempotent middle units of \(G\), and consequently it is isomorphic to some outer product semigroup \(R \times L\), where \(R\) is a right singular semigroup and \(L\) is a left singular semigroup \([3]\). Accordingly there exists an isomorphism \(\xi\) of \(\Gamma\) onto \(R \times L\).
Secondly the mapping \( \theta \), which is the correspondence \( \phi \), is a homomorphism \( \gamma \) onto \( \pi \), (where an element \( \alpha \) is such as \( \alpha \gamma = \alpha \in \pi \)).

\[
\Gamma \cong \mathbb{R} \times \mathbb{L}
\]

(1)

Consequently, by (1) and (2)

\[
\Gamma \cong \mathbb{R} \times \mathbb{L}
\]

(2)

(3)

(Remark)

In general, by a quasi- \( \gamma \)-group is meant an outer product semigroup of a group, a right singular semigroup and a left singular semigroup. Accordingly \( \mathbb{R} \times \mathbb{L} \) is a quasi- \( \gamma \)-group.

If we use \( \mathcal{G} \) for the inverse image of \( \gamma \), \( \mathcal{G}(\gamma, \gamma, \gamma) \) is the set consisting of only one element for any \( \gamma \). From the above observations the following lemma is concluded.

[Lemma 1.4] If \( \mathcal{G} \) is a \( (\mathcal{M}) \)-inversible semigroup, there are sets \{ \( \mathcal{G}_\alpha \) \} having as each coefficient an element of a quasi- \( \gamma \)-group \( \mathcal{Q} \), such as

\[
\mathcal{G}_\beta \mathcal{G}_\gamma = \text{single element} \in \mathcal{Q}
\]

for any \( \beta, \gamma \in \mathcal{Q} \).

Moreover, if \( \mathcal{G}(\gamma, \gamma, \gamma) \) stands for \( \mathcal{G}(\gamma, \gamma, \gamma, \gamma) \) for any \( \gamma \) in \( \mathcal{Q} \),

\[
\mathcal{G}(\gamma, \gamma, \gamma) = \mathcal{G}(\gamma, \gamma, \gamma, \gamma)
\]

for any \( \gamma \) in \( \mathcal{Q} \).

Therefore, the following theorem is satisfied by the above observations and the [Lemma 1.4]

[Theorem 1.1] If \( \mathcal{G} \) is a \( (\mathcal{M}) \)-inversible semigroup, there are sets \{ \( \mathcal{G}_\alpha \) \} having as each coefficient an element of a quasi- \( \gamma \)-group \( \mathcal{Q} \), and elements \{ \( \mathcal{G}_\alpha \) \} \( \mathcal{G}_\beta \mathcal{G}_\gamma = \text{single element} \in \mathcal{Q} \), such as

\[
\mathcal{G}_\beta \mathcal{G}_\gamma = \text{single element} \in \mathcal{Q}
\]

(2)

(3)

(Remark)

In this sense, the semigroups containing no idempotents except their own middle units are more generalized semigroups than the \( (\mathcal{M}) \)-inversible semigroups. Now, by a special middle unitary semigroup we mean a semigroup which is middle unitary and contains no idempotents except its own middle units. Then the purpose of the author in this paragraph is to expatiate on the general theory of the special middle unitary semigroups and to show that the problem determining the structure of such semigroups consequently reduced to that of the structure of the nonpotent semigroups. Hereafter \( \mathcal{S} \) stands for a middle unitary semigroup and \( \mathcal{M}_\mathcal{S} \) for the set consisting of all middle units of \( \mathcal{S} \) unless otherwise provided. Moreover, two symbols \( \exists \), \( \forall \) stand for 'existence', 'non existence' respectively. Accordingly, for example, if we write \( \exists x \), \( \forall x \), we mean that there exists an element \( x \) satisfying the condition ——.

Any element of \( \mathcal{S} \) is logically contained in one of the following \( \mathcal{S}_1 \), \( \mathcal{S}_2 \), \( \mathcal{S}_3 \) and \( \mathcal{S}_4 \).

\[
\mathcal{S}_1 = \{ x | \exists x : x \neq e \mathcal{M}_5, \exists x : x \neq e \mathcal{M}_3 \}
\]

\[
\mathcal{S}_2 = \{ x | \exists x : x \neq e \mathcal{M}_5, \exists x : x \neq e \mathcal{M}_3 \}
\]

\[
\mathcal{S}_3 = \{ x | \exists x : x \neq e \mathcal{M}_5, \exists x : x \neq e \mathcal{M}_3 \}
\]

\[
\mathcal{S}_4 = \{ x | \exists x : x \neq e \mathcal{M}_5, \exists x : x \neq e \mathcal{M}_3 \}
\]

Therefore,

\[
\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4
\]

(4)

(Remark)

The relation \( \exists x : x \neq e \mathcal{M}_5, \exists x : x \neq e \mathcal{M}_3 \) is equivalent to the relation \( \exists x : x \neq e \mathcal{M}_5 \). Accordingly, as \( \mathcal{S}_1 \) is \( (\mathcal{M}) \)-inversible, it is \( (\mathcal{M}) \)-inversible.

[Lemma 2.1] If \( \mathcal{S} \) is a special middle unitary semigroup, it follows that \( \mathcal{S}_1 \) is a \( (\mathcal{M}) \)-inversible semi-
group, \( S_\phi \), and \( S_\phi^\phi \) is an ideal of \( S \).

Now the following theorem is concluded by \((\ast)\) and [Lemma 2.1], because \( S_\phi \) contains no idempotents. (We call such semigroup 'nonpotent semigroup')

[Theorem 2.1] If \( S \) is a special middle unitary semigroup, the relation

\[
S = V + T
\]

is satisfied, where \( V \) is a \((W)\)-inversible subsemigroup and \( T \) is a nonpotent ideal of \( S \).

(Remark)

In the above theorem, it is easy to see that \( V, T \) are uniquely determined by \( S \).

Hereafter, \( V \) stand for a \((W)\) -inversible semigroup and \( T \) for a nonpotent semigroup.

A middle unitary semigroup \( S \) is said to be an extension by \( T \) if \( S \) satisfies the following conditions.

1. \( S = V + T \); both \( V \) and \( T \) are subsemigroups of \( S \).
2. \( S \) is a special middle unitary semigroup.
3. \( V \) is a \((W)\)-inversible subsemigroup.

(A) \( S \in V + T \).

(B) \( S \) is a special middle unitary semigroup.

(C) \( V \) is a \((W)\)-inversible subsemigroup.

(D) Both \( \lambda \) and \( \rho \) are middle unitary for any \( \lambda \in V \), \( \beta \in T \).

(E) \( \lambda \) is a right unit of \( \lambda \), and \( \beta \) is a left unit of \( \beta \).

Conversely,

[Theorem 2.2] If \( S \) is an extension of \( V \) by \( T \), there exist homomorphisms \( \lambda, \rho \) of \( V \) into \( \lambda, \rho \), respectively such as, if \( \lambda_A, \rho_B \) stands for \( \lambda(A) \) and \( \rho_B \) for \( \rho(A) \).

(1) \( \lambda_A, \rho_B \) are linked for any \( \lambda \in V \).

(2) \( \lambda_A, \rho_B \) are commuted for any \( \lambda, \beta \in V \).

(3) Both \( \lambda_E \) and \( \rho_E \) are middle unitary for any \( E \in M_V \).

(4) \( \lambda_E \) is a right unit of \( \lambda(E) \) and \( \rho_E \) is a left unit of \( \rho(E) \) for any \( E \in M_V \).

are satisfied, and the product \( \circ \) in \( S \) is represented by

\[
\begin{bmatrix}
\alpha \circ \beta = \alpha \lambda \beta \\
= \alpha \lambda \beta \\
= \lambda \alpha \beta \\
= \alpha \rho \beta \end{bmatrix}
\]

for any \( \alpha, \beta \in V \).

Conversely,

[Theorem 2.3] If there are two homomorphisms \( \lambda, \rho \) of \( V \) into \( \lambda, \rho \), respectively such as, if \( \lambda_A, \rho_B \) stands for \( \lambda(A) \) and \( \rho_B \) for \( \rho(A) \),

(1) \( \lambda_A, \rho_B \) are linked for any \( \lambda \in V \).

(2) \( \lambda_A, \rho_B \) are commuted for any \( \lambda, \beta \in V \).

(3) Both \( \lambda_E \) and \( \rho_E \) are middle unitary for any \( E \in M_V \).

(4) \( \lambda_E \) is a right unit of \( \lambda(E) \) and \( \rho_E \) is a left unit of \( \rho(E) \) for any \( E \in M_V \).

are satisfied, the direct sum \( V + T \) becomes an extension of \( V \) by \( T \) if the product \( \circ \) in \( V + T \) is defined as follows;

\[
\begin{bmatrix}
\alpha \circ \beta = \alpha \lambda \beta \\
= \alpha \lambda \beta \\
= \lambda \alpha \beta \\
= \alpha \rho \beta \end{bmatrix}
\]

for any \( \alpha, \beta \in V \).

(Remarks)

1. The set \( M_V \) in the two theorems above-mentioned consists of all idempotent middle units of \( V \).
2. By the [Theorem 2.3], it is easy to see that the following product \( \alpha \cdot \beta = \alpha_0 \beta \) for any \( \alpha, \beta \in V \)
\( = \alpha \alpha_0 \beta \) for any \( \alpha, \beta \in T \).
\( = \alpha \beta \) for any \( \alpha \in V, \beta \in T \).
\( = \beta \) for any \( \alpha \in V, \beta \in T \).

Accordingly, there exists at least one extension of \( V \) by \( T \).

Any middle unitary semigroup \( S \) is uniquely decomposed to the direct sum of two sets \( V \) and \( T \), where \( V \) is a \((K,K^-)\)-inversible subsemigroup of \( S \) and \( T \) is a nonpotent ideal of \( S \). And from the [Theorem 2.2], it is obvious that there exist some homomorphisms \( \Phi, \Psi \) of \( V \) into the left translation semigroup of \( T \) and the right translation semigroup of \( T \) respectively, such that the product \( \phi \) of \( S \) is represented by the following relation.

\[
\phi = \Phi \circ \Psi \quad \text{for any} \quad \alpha \in V, \beta \in T.
\]

where \( \Phi \), stands for \( \Phi(\alpha) \), \( \Psi(\beta) \) for \( \Psi(\beta) \) and \( \alpha, \beta \) are the products in \( V, T \), respectively. \( S \) is therefore determined by \( V, T, \Phi \) and \( \Psi \). We shall describe this sense as \( S = (V, T; \Phi, \Psi) \).

Now, in what cases are two special middle unitary semigroups isomorphic to each other? The following theorem is the answer of this question.

[Theorem 2.4] In order to two special middle unitary semigroups \( S = (V, T; \Phi, \Psi), S = (V', T'; \Phi', \Psi') \) are isomorphic to each other, it is necessary and sufficient that there exist two homomorphisms \( \xi, \xi' \) of \( V \) onto \( V' \) and \( \xi' \) of \( T \) onto \( T' \) such that

\[
\lambda_A = \xi^{-1} T \cdot \lambda'_A \xi_V T
\]

\[
\sigma_A = \xi^{-1} T \cdot \sigma_V \xi T
\]

are satisfied for any \( A \in T \), where \( \lambda_A, \sigma_A, \lambda'_A, \sigma'_V \) and \( \xi, \xi' \) stand for \( \Phi(A), \Psi(A), \Phi'(A), \Psi'(A) \) respectively.

By the above observations, the structure of the special middle unitary semigroups have become clear on the whole. But, we have to determine the structure of the nonpotent semigroups in order to completely determine the structure of the special middle unitary semigroups. The author will touch upon this problem in another opportunity.

REFERENCES

[1] By a middle unitary semigroup we mean a semigroup having at least one middle unit of its own.
By a right (left) group we mean a semigroup which is left regular and right simple (right regular and left simple). Such a semigroup is isomorphic to an outer product semigroup \( G \times R (G \times L) \), where \( G \) is a group and \( R(\omega) \) is a right (left) singular semigroup.
[4] By a completely non-commutative semigroup we mean a semigroup which has the following property.

\( x y = y x \) implies \( x = y \) for any two elements \( x, y \).

A semigroup \( C \) is a completely non-commutative when and only when it is isomorphic to some \( L \times R \), where \( L \) is a left singular semigroup and \( K \) is a right singular semigroup. In other words, a completely non-commutative semigroup is a semigroup whose elements are all idempotent middle units of its own. This fact is proved in [6]. See [2] or [3] about the right (left) simple semigroups, the right (left) regular semigroups and the regular semigroups.

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*) Received June 6, 1955.