ON DECOMPOSITIONS OF A COMMUTATIVE SEMIGROUP

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If there exists a homomorphism of a semigroup S onto a semigroup S^* having special type, all elements of S are decomposed into the class sum of mutually disjoint subsets. Then we say that the decomposition of S to S^* is obtained. In particular the decomposition to a semilattice is of importance, i.e., $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ where $S_\alpha \cap S_\beta = \phi$ $(\alpha \neq \beta)$, every S_α is a restrictive subsemigroup, and for any α, β, there is a unique γ such that $S_\alpha S_\beta \subset S_\gamma$ as well as $S_\beta S_\alpha \subset S_\gamma$. In §1 we argue that there is greatest decomposition of a semigroup to a semilattice; particularly in §2 we show a decomposition of a commutative semigroup by method different from Mr. Numakura's, and in §3 our decomposition is proved to be greatest.

§1 Greatest decomposition

In this paragraph S is assumed to be a general semigroup. A decomposition of S to an idempotent semigroup gives an equivalence relation; and an equivalence relation $\sim$ in S raises a decomposition of S to an idempotent semigroup if and only if

(1) $a \sim b$, $c \sim d$ imply $ac \sim bd$,
(2) if $a \sim b$ then $a \sim ab$.

Lemma 1. (1) and (2) are equivalent to (1') and (2'),

(1') $a \sim b$ implies $ac \sim bc$ and $ca \sim cb$ for every c,
(2') $a \sim a^2$ for every a.

Proof. (1') $\rightarrow$ (1): For, from $a \sim b$, follows $ac \sim bc$; and from $c \sim d$, $bc \sim bd$. By transitivity, $ac \sim bd$. (1) $\rightarrow$ (1'): evident. (1') & (2') $\rightarrow$ (2): from $a \sim b$, it follows that $a \sim a^2 \sim ab$. (2) $\rightarrow$ (2'): evident.

We denote by $\mathfrak{D}$ the set of all decompositions φ of S to a semilattice, and by $\underset{\varphi}{\sim}$ the congruence relation which gives φ. The relation $\underset{\varphi}{\sim}$ and $\underset{\psi}{\sim}$ are equal if they give the same decomposition. Obviously $\mathfrak{D}$ is not empty, because it includes at least a trivial decomposition, a partition of all elements of S into one class.

Now we introduce the ordering into $\mathfrak{D}$: i.e. $\varphi \geqq \psi$ means that $x \underset{\psi}{\sim} y$ if $x \underset{\varphi}{\sim} y$. The ordering is clearly a partial ordering. Then we have the below lammas.

Lemma 2. $\mathfrak{D}$ forms a complete semilattice.

Proof. Since $\mathfrak{D}$ is a partly ordered set, we show that any subset $\mathfrak{D}_1$ of $\mathfrak{D}$ has a least upper bound. We define a relation $\underset{\xi}{\sim}$ as follows. $x \underset{\xi}{\sim} y$ means that $x \underset{\varphi}{\sim} y$ for every $\varphi \in \mathfrak{D}_1$. It is not hard to verify that $\underset{\xi}{\sim}$ is an equivalence relation and satisfies the condition (1') and (2') (in Lemma 1). Clearly $\xi \geqq \varphi$ for all $\varphi \in \mathfrak{D}_1$. Take up any $\xi' \geqq \varphi$ for all $\varphi \in \mathfrak{D}_1$, then from $x \underset{\xi'}{\sim} y$ follows $x \underset{\varphi}{\sim} y$ for all $\varphi \in \mathfrak{D}_1$, i.e., $x \underset{\xi}{\sim} y$; hence $\xi' \geqq \xi$, and so ξ is the least upper bound of $\mathfrak{D}_1$. Consequently

Theorem 1. There is a greatest element of $\mathfrak{D}$. In other words, there exists the greatest decomposition of a semigroup to a semilattice.

In another article we shall relate what is an equivalence relation giving the greatest decomposition of a general semigroup.

§2 A decomposition of a commutative semigroup

Let S be a commutative semigroup. We define an ordering $a \geqq b$ between elements a and b of S to mean that a certain element $x \in S$ and a positive integer m are found such that

$$a^m = bx$$

The definition is obviously equivalent to the following:

$a^m = b^n y$ for some positive integers m, n, and an element $y \in S$.

Lemma 3. This ordering is a quasi-ordering.

Proof. (1) $a \geqq a$ for all a, because $a^m = a a^{m-1}$ for $m > 1$. (2) $a \geqq b$ and $b \geqq c$ imply $a \geqq c$. For, from $a^m = b x$ and $b^n = c y$, we get $a^{mn} = c z$ where $z = y x^n$.

Lemma 4. $a \geqq b$ implies $ac \geqq bc$ for every $c \in S$.

Proof. By the assumption $a^m = bx$ for some m and x. Multiply c^m by both sides of the equality, we get $(ac)^m = (bc)(c^{m-1}x)$ where m may be supposed to be greater than 1. This shows $ac \geqq bc$.

Now, if we define a relation as $a \geqq b$ and $b \geqq a$, the relation is an equivalence relation.

Lemma 5. $a \sim b$ implies that $ac \sim bc$.

Proof. Use Lemma 4.

Lemma 6. $a \sim a^2$ for every $a \in S$.

Proof. Obvious by the definition.

From Lemma 1, 5, and 6, we have

Theorem 2. We have a decomposition of a commutative semigroup S by introducing the equivalence relation $a \sim b$, or $a \geqq b$ and $b \geqq a$, into S.

Next, we investigate the property of the subsemigroup S_α whose class sum is S.

Lemma 7. Let e be an idempotent element of S. If $e \geqq a$, there exists x of S such that $e \geqq x \geqq a$ and $ax = e$.

Proof. By the definition of the ordering, $e = a^n y$ for some $y \in S$ where we may assume $n > 1$. Set $x = a^{n-1} y$ then $e = ax$ and $e \geqq x \geqq a$.

Lemma 8. If $a \sim e$ where e is an idempotent, there is x such that $ax = e$ and $e \sim x$.

Proof. Since $e \geqq a$ by Lemma 7, there is x such that $ax = e$ and $e \geqq x \geqq a$. On the other hand $a \geqq e$; hence $e \sim x$.

Now, let D be the set of all idempotents of a commutative semigroup S.

Lemma 9. D is not only a subsemigroup of S but a semilattice.

The partial ordering $\succ$ is introduced into D in usual way:

$$e \succ f \quad \text{if} \quad e = fe$$

Lemma 10. As far as elements of D are concerned, it holds that $e \succ f$ if and only if $e \geqq f$.

Proof. Suppose $e \geqq f$ i.e. $e = fx$ for some $x \in S$. Then $fe = f^2 x = fx = e$. Hence $e \succ f$. The converse is trivial.

Lemma 11. Let $e, f \in D$. $e \sim f$ implies $e = f$.

Proof. From $e \sim f$, we have $e \succ f$ and $f \succ e$ by Lemma 10. Since $\succ$ is a partial ordering, $e = f$ is concluded.

From Lemma 11 we have the interesting theorem.

Theorem 3. In the decomposition of a commutative semigroup as Theorem 2, S_α is a subsemigroup having at most one idempotent.

Furthermore, if S_α contains an idempotent, S_α is a unipotent inversible semigroup [1]. Then $S_\alpha e$, in which e is an idempotent of S_α, is the greatest group of S_α and S_α has the property that

For $x \in S_\alpha$ there is a positive integer n such that $x^n \in S_\alpha e$.

The structure of a commutative nonpotent semigroups such as S_α will be argued precisely in another paper.

§3 Two decompositions

Mr. K. Numakura obtained a decomposition of a commutative semigroup S by the following equivalence relation $\approx$ [2] as follows.

$a \approx b$ if and only if $\bigcap_{n=1}^{\infty}(Spa^n) = \bigcap_{n=1}^{\infty}(Spb^n)$ for all $p \in S$.

The decomposition due to $\sim$ (§2) and $\approx$ are denoted by φ_1 and φ_2 respectively. We shall discuss the relations between φ_1 and φ_2.

Theorem 4. $\varphi_1 \geqq \varphi_2$, in other words, if $a \sim b$ then $a \approx b$.

From $a \sim b$, i.e. $a^m = bx$, $b^n = ay$, for any $p \in S$,

$$\bigcap_{i=1}^{\infty}(Spa^i) = \bigcap_{k=1}^{\infty}(Spa^{km}) \subset \bigcap_{i=1}^{\infty}(Spb^i).$$

Similarly

$$\bigcap_{i=1}^{\infty}(Spb^i) \subset \bigcap_{i=1}^{\infty}(Spa^i).$$

Thus we have

$$\bigcap_{i=1}^{\infty}(Spa^i) = \bigcap_{i=1}^{\infty}(Spb^i), \text{ i.e., } a \approx b.$$

Let φ_0 be the greatest decomposition of S to a semilattice (for the existence of φ_0 is assured in §1), and let $\equiv$ be the equivalence relation determined by φ_0. Evidently $\varphi_2 \leqq \varphi_1 \leqq \varphi_0$.

Theorem 5. It holds that $\varphi_1 = \varphi_0$, in other words, φ_1 is the greatest decomposition of S to a semilattice.

Proof. It is sufficient to show that $a \sim b$, (or $a^m = bx$ and $b^n = ay$) implies $a \equiv b$. Since each class by φ_0 is a subsemigroup, it follows that $\bar{a} = \bar{a}^m$. Let $\bar{a}$ be a class to which a belongs, and $\bar{S}$ be the semilattice which is determined by φ_0. The multiplication in $\bar{S}$ is denoted by $\cup$.
From $a^m = bx$, we get $\bar{a} = \bar{b} \cup \bar{x}$, and consequently $\bar{a} \succ \bar{b}$ where $\succ$ is a partial ordering in $\bar{S}$. Similarly, from $b^n = ay$, we have $\bar{a} \prec \bar{b}$. Thus it has proved that $a \sim b$ implies $\bar{a} = \bar{b}$ or $a \equiv b$.

Now, if a semigroup S is decomposed to a semilattice composed of only one element, S is called an s-indecomposable semigroup.
We have immediately from Theorem 5 the below theorem.

Theorem 6. A commutative semigroup S is s-indecomposable, if and only if, for every pair a, b of elements of S, there exist a positive in integer m and an element $x \in S$ such that $a^m = bx$.

Finally we show $\varphi_2 < \varphi_0$ by an example. Let S be the set of all pairs (i, j) of non-negative integers except one $(0, 0)$, and the multiplication is defined as

$$(i_1, j_1)\ (i_2, j_2) = (i_1 + i_2,\ j_1 + j_2)$$

where $i_1 + i_2$, $j_1 + j_2$ are usual additions.
S is a commutative semigroup. Now let

$$A = \{(i, 0);\ i \geqq 1\},$$

$$B = \{(0, j);\ j \geqq 1\},$$

$$C = \{(i, j);\ i \geqq 1,\ j \geqq 1\},$$

A, B and C are mutually disjoint subsemigroups and

$$S = A \cup B \cup C$$

It is easily seen that this is a decomposition, written by φ', of S to a semilattice. Of course $\varphi' \leqq \varphi_0$. On the other hand, we consider the mapping f of S on the additive semigroup I of all natural numbers as follows.

$$(i, j) \xrightarrow{f} i + j$$

f is a homomorphism of S on I. Setting $a = (i, j)$,

$$\begin{aligned} f(xpa^n) &= f(x) + f(p) + n f(a) \\ &\geqq 1 + 1 + n(i + j) \\ &> n \end{aligned}$$

Let $I_n = \{i;\ i > n\}$. Then $f(Spa^n) \subset I_n$. Since $\bigcap_{n=1}^{\infty} I_n = \phi$, $\bigcap_{n=1}^{\infty}(Spa^n) = \phi$ for every p, $a \in S$. It follows that φ_2 decomposes all elements of S into one class. Clearly $\varphi_2 < \varphi'$. At last we arrived at $\varphi_2 < \varphi_0$.

References

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