ON HARMONIC DIMENSION, II

By Mitsuru OZAWA

One of the most elegant systematic investigations for structure of a family of positive harmonic functions was made by R.S.Martin. He introduced the so-called Martin topology to the ideal boundary in the spatial case. M.Heins established many deep results concerning the ideal boundary of Riemann surfaces, though his subjects are very special.

In the present paper we shall explain a notion of harmonic dimension introduced in our previous paper in a somewhat clearer form.

1. First exposition.

Basic domain in the sequel is supposed to be a C-end or an extended C-end defined in our previous paper.

Let $F \in O_G$ and Ω be a subsurface with analytic curves Γ as its relative boundary which are not compact. Let $\hat{\Omega}$ denote the doubled surface of Ω, symmetric with regard to Γ, then $\hat{\Omega}$ belongs to O_G. (See Z. Kuramochi [1])

Moreover we see easily the following fact: Let Ω be the same as above. Let γ be a compact part of Γ. Let $\hat{\Omega}$ denote the doubled surface of Ω, symmetric with regard to $\Gamma - \gamma$, then $\hat{\Omega}$ can be imbedded in a Riemann surface belonging to O_G.

In our case Ω is a C-end or an extended C-end, therefore $\hat{\Omega}$ is an end in the sense of Heins whose boundary consists of $\gamma + \tilde{\gamma}$, $\tilde{\gamma}$ being the symmetric image of γ with respect to $\Gamma - \gamma$.

Now we assume that $\hat{\Omega}$ has finite harmonic dimension in the sense of Heins. Let $P_{\hat{\Omega}}$ be a family of positive harmonic functions vanishing on $\gamma + \tilde{\gamma}$. Then all the minimal positive harmonic functions introduced by R.S.Martin can be obtained from the Green function of $\hat{\Omega}$ by a limiting process along a suitably selected non-compact point-sequence consisting of the logarithmic poles $p_n^{(i)}$ of the Green function of $\hat{\Omega}$, that is,

$$\lim_{n \to \infty} G_{\hat{\Omega}}(z, p_n^{(i)}) = v_i(z).$$

Among these limit functions there are all the generators $(v_1, \cdots, v_m)$ of $P_{\hat{\Omega}}$.

Let $\tilde{z}$ be a symmetric point of z with respect to $\Gamma - \gamma$. Then we can select a set of generators $(v_1, \ldots, v_m)$ of $P_{\hat{\Omega}}$ such that, for any i, either

$$v_i(z) = v_i(\tilde{z})$$

or

$$v_i(z) = v_j(z), \quad i \neq j$$

hold for a suitable j and this correspondence $\{i\} \to \{j\}$ is one-to-one and onto manner as a whole. Here we define $\tilde{v}_j(z) = \lim_{n \to \infty} G(z, \tilde{p}_n^{(j)})$.

To see this, we proceed as follows. If $(v_1(z), \ldots, v_m(z))$ is a set of generators of $P_{\hat{\Omega}}$, then $(\tilde{v}_1(z), \ldots, \tilde{v}_m(z))$ is also so. In fact, from the symmetry character $v_i(\tilde{z}) = \tilde{v}_i(z)$, we have that

$$\sum_{i=1}^{m} c_i \tilde{v}_i(z) = 0$$

implies

$$\sum_{i=1}^{m} c_i v_i(z) = 0.$$

Hence all the c_i vanish, and

$$\tilde{v}(z) \equiv v(\tilde{z}) = \sum_{i=1}^{m} a_i v_i(z)$$

holds by the assumption whence we see that

$$v(z) = \sum_{i=1}^{m} a_i v_i(\tilde{z}) = \sum_{i=1}^{m} a_i \tilde{v}_i(z)$$

remains valid also. This shows that $(\tilde{v}_1(z), \ldots, \tilde{v}_m(z))$ is also a set of generators of $P_{\hat{\Omega}}$.

According to a result of R.S.Martin, — that is, if ($u_1, \ldots, u_n$) and ($v_1, \ldots, v_n$) are two sets of generators of $P_{\hat\Omega}$, then $u_i = c_k v_{k(i)}$, $c_k > 0$, ($k = 1, \ldots, n$) —

$$v_i(z) = k_{j(i)} \widetilde{v}_{j(i)}(z), \quad k_{j(i)} > 0$$

holds for any i and a suitable $j(i)$ and this correspondence $\{i\} \to \{j\}$ is one-to-one and onto in total.

If $i = j(i)$, then

$$v_i(z) = k_i \widetilde{v}_i(z)$$

leads to a relation

$$\widetilde{v}_i(z) = k_i v_i(z)$$

and hence

$$k_i = 1.$$

Since

$$\frac{1}{2\pi} \int_{\gamma + \widetilde{\gamma}} \frac{\partial}{\partial \nu} v_i(z)\, ds = 1$$

and

$$\frac{1}{2\pi} \int_{\gamma + \widetilde{\gamma}} \frac{\partial}{\partial \nu} \widetilde{v}_j(z)\, ds = 1$$

hold, $k_{j(i)} = 1$ remains valid even if $i \neq j(i)$.

Therefore we can reorder a set of generators by change of indices such that

$$(v_1, \ldots, v_n, v_{n+1}, \widetilde{v}_{n+1}, \ldots, v_{n+p}, \widetilde{v}_{n+p})$$

satisfies the relations

$$v_i(z) = v_i(\widetilde{z}), \quad 1 \leq i \leq n,$$

and $m = n + 2p$.

To be proved is that the harmonic dimension of Ω in our sense is equal to p.

Let $V(z)$ be an arbitrary function on $\hat\Omega$. We put

$$V^s(z) = \frac{1}{2}(V(z) + \widetilde{V}(z)), \qquad \widetilde{V}(z) \equiv V(\widetilde{z}).$$
$$V^a(z) = \frac{1}{2}(V(z) - \widetilde{V}(z)).$$

Then $(v_1, \ldots, v_n, v^s_{n+1}, v^a_{n+1}, \ldots, v^s_{n+p}, v^a_{n+p})$ is also a set of linearly independent functions and any function of $P_{\hat\Omega}$ can be obtained by a uniquely determined linear combination of these functions, provided that ($v_1, \ldots, v_n, v_{n+1}, \widetilde{v}_{n+1}, \ldots, v_{n+p}, \widetilde{v}_{n+p}$) is a set of generators of $P_{\hat\Omega}$. This leads to the linear independency of

$$(v^a_{n+j}(z)), \quad 1 \leq j \leq p.$$

Next we prove a fact that $(v^a_{n+j}(z))$, $1 \leq j \leq p$ is a set of generators, that is, any limit function $\lim_{n\to\infty} G_\Omega(z, p_n)$ ($\neq 0$) can be represented as a linear combination of the above set. Here $G_\Omega(z,p)$ is the Green function of Ω with pole at $p \in \Omega$.

In fact by the symmetry character of $\hat\Omega$ we have

$$G_\Omega(z, p_n) = G_{\hat\Omega}(z, p_n) - G_{\hat\Omega}(z, \widetilde{p}_n).$$

Extracting a suitable subsequence (p_{n_t}) of (p_n), we have that

$$\lim_{t\to\infty} G_\Omega(z, p_{n_t}) = \lim_{t\to\infty} G_{\hat\Omega}(z, p_{n_t}) - \lim_{t\to\infty} G_{\hat\Omega}(z, \widetilde{p}_{n_t})$$

is positive on Ω. Moreover each member of the right-hand side in the last relation belongs to $P_{\hat\Omega}$ and hence

$$\lim_{n\to\infty} G_\Omega(z, p_n) = \lim_{t\to\infty} G_\Omega(z, p_{n_t})$$
$$= \left(\sum_{i=1}^{n} c_i v_i(z) + \sum_{j=1}^{p} d_j v_{n+j} + \sum_{j=1}^{p} e_j \widetilde{v}_{n+j} \right)$$
$$- \left(\sum_{i=1}^{n} c_i \widetilde{v}_i(z) + \sum_{j=1}^{p} d_j \widetilde{v}_{n+j} + \sum_{j=1}^{p} e_j v_{n+j} \right)$$
$$= 2 \sum_{j=1}^{p} (d_j - e_j)\, v^a_{n+j}(z).$$

Next every $v^a_{n+j}(z)$ is a limit function of the Green function $G_\Omega(z, p_t^{(j)})$ along a suitable non-compact point-sequence ($p_t^{(j)}$) on Ω. Therefore we have the desired result which leads to a fact that

$$\dim \hat\Omega = 2 \dim \Omega + n$$

n being a non-negative integer, for an extended C-end or a C-end Ω.

This gives an exposition of harmonic dimension in our sense. An integer n shows a sort of degree of symmetry of $\hat\Omega$.

2. Bounded harmonic functions on an end.

In this section we shall prove a property of an H-end $\hat\Omega$ to have harmonic dimension one or two. Here $\hat\Omega$ means a doubled surface of a C-end or an extended C-end Ω with respect to

$\Gamma-\gamma$ as in §1. Let S_Ω denote a family of harmonic bounded functions on Ω satisfying $\frac{\partial}{\partial\nu}u(z)\equiv 0$ on $\Gamma-\gamma$. Let Q_Ω be a family of non-trivial positive harmonic functions on Ω which have the identically vanishing normal derivative along $\Gamma-\gamma$ and vanish continuously on γ and moreover satisfy a normalization

$$\int_\gamma \frac{\partial V}{\partial\nu}\,ds = 1 .$$

Let $N(z,p)$ denote a function harmonic on Ω except only at a point $p\in\Omega$ where $N(z,p)$ has a logarithmic pole with local expansion

$$N(z,p) = \log\frac{1}{|\lambda|}$$

$+$(bounded harmonic function around p),

λ being a local parameter at p; and further $N(z,p)\equiv 0$ on γ and $\frac{\partial}{\partial\nu}N(z,p)\equiv 0$ on $\Gamma-\gamma$. Evidently we see

$$N(z,p) = G_{\hat\Omega}(z,p) + G_{\hat\Omega}(z,\tilde p),$$

$G_{\hat\Omega}$ being the Green function of $\hat\Omega$. From the Green's formula and the O_G-property of $\hat\Omega$, we see that

$$u(p_n) = \frac{1}{2\pi}\int_\gamma u(q)\frac{\partial}{\partial\nu}N(q,p_n)\,ds$$

for any $u\in S_\Omega$. Since γ lies in a compact part of $\bar\Omega$ (or $\bar{\hat\Omega}$), $\lim_{n\to\infty}u(p_n)$ exists provided that $\{p_n\}$ is a non-compact point-sequence such that $\lim_{n\to\infty}N(q,p_n)$ exists. Evidently

$$\frac{1}{2\pi}\int_\gamma\frac{\partial}{\partial\nu}N(q,p_n)\,ds = 1 \quad\text{and hence}$$

$$\frac{1}{2\pi}\int_\gamma\frac{\partial}{\partial\nu}\lim_{n\to\infty}N(q,p_n)\,ds = 1 \quad\text{hold.}$$

Moreover $\lim_{n\to\infty}N(q,p_n)=\lim_{n\to\infty}N(\tilde q,\tilde p_n)$ $=\lim_{n\to\infty}N(q,\tilde p_n)>0$ holds on $\hat\Omega$. Therefore we have

$$\lim_{n\to\infty}N(q,p_n)=\sum_{i=1}^{n}c_i v_i(q)+\sum_{j=1}^{p}d_j v_{n+j}^{s}(q),$$

$$\sum_{i=1}^{n}c_i+\sum_{j=1}^{p}d_j = 2 .$$

where v_i, v_{n+j}^s are defined in §1. Let v belong to Q_Ω, then v can be expressed by a linear combination

$$v=\sum_{i=1}^{n}c_i v_i+\sum_{j=1}^{p}d_j v_{n+j}^{s},$$

$$\sum_{i=1}^{n}c_i+\sum_{j=1}^{p}d_j = \frac{1}{\pi} .$$

The set of limiting values of $u\in S_\Omega$ at the ideal boundary is $\mu(Q_\Omega)$, where

$$\mu(V)=\int_\gamma u\frac{\partial V}{\partial\nu}\,ds\ ,\quad V\in Q_\Omega .$$

This fact can be similarly deduced as a lemma 11.1 in Heins' paper [1]. From this we see immediately the following fact:

If Ω is a C-end or an extended C-end on which Q_Ω consists of only one member, then any $u\in S_\Omega$ possesses a limit at the ideal boundary.

Let $u(z)$ be a harmonic bounded function on Ω having the boundary value $\frac{\partial}{\partial\nu}(v_1-v_2)$; $v_1, v_2\in Q_\Omega$. If any $u\in S_\Omega$ has a limit at the ideal boundary, then there holds

$$\int_\gamma u\frac{\partial v_1}{\partial\nu}\,ds=\int_\gamma u\frac{\partial v_2}{\partial\nu}\,ds$$

and hence

$$\int_\gamma u^2\,ds = 0 .$$

This implies that $u\equiv 0$ on γ which leads to $v_1\equiv v_2$ on Ω.

Thus we have a result:

Let Ω be a C-end or an extended C-end on which any function $u\in S_\Omega$ possesses a limit at the ideal boundary, then Q_Ω consists of only one member.

If Q_Ω consists of only one member, then $\hat\Omega$ is either of harmonic dimension one or two. Moreover in the former case Ω is of harmonic dimension zero and in the latter case Ω is of harmonic dimension one.

This fact is seen in the following manner. Since

$$\dim\hat\Omega = 2\dim\Omega + n,$$

n being a non-negative integer, we have $\dim\hat\Omega=$ 1 or 2 provided that $\dim\Omega+n=1$ remains valid, a fact which is equivalent to the assumption.

Theorem. Let Ω be a C-end or an extended C-end on which any function $u \in S_\Omega$ possesses a limit at the ideal boundary, then $\dim \hat{\Omega} = 1$ or 2 and hence $\dim \Omega = 0$ or 1.

3. Second exposition.

In this section we shall give another exposition of our harmonic dimension in relation to a notion "harmonically simple or multiple" due to B.Kjellberg. Kjellberg's notion is also a local invariant. We can give another "harmonic dimension" more extensive than his. In our domain, we can distinguish two sorts of minimal functions. Every minimal function can be obtained by a limiting process

$$\lim_{n\to\infty} \frac{G_\Omega(z, p_n^{(i)})}{G_\Omega(z_0, p_n^{(i)})} = G_i(z),$$

z_0 a fixed point in Ω,

for a suitable non-compact point-sequence ($p_n^{(i)}$) tending to the ideal boundary. According to either $\lim_{t\to\infty} G_\Omega(z_0, p_{n_t}^{(i)}) \neq 0$ for a suitable subsequence ($p_{n_t}^{(i)}$) or not, we call that $G_i(z)$ belongs to either G-class or X-class.

If $G_i(z) \in$ X-class, then $\lim_{n\to\infty} G_\Omega(z, p_n^{(i)}) \equiv 0$ on Ω. Therefore, if $u \in P_\Omega$, then

$$u(z) = \sum_{i=1}^{p} a_i G_i(z) + \sum_{j=1}^{q} b_j X_j(z),$$

where $G_i(z) \in$ G-class and $X_j(z) \in$ X-class. The set of all the G's and X's constitutes, as a whole, the set of generators of P_Ω. P_Ω means a class of positive harmonic functions with vanishing boundary values on Ω. Here p is equal to $\dim \Omega$ in our sense and q is also a local invariant of the ideal boundary. $p + q$ is the harmonic dimension in the sense of B.Kjellberg. If we introduce a metric induced by the inner product in P_Ω as in the ordinary vector space, then P_Ω can be decomposed into two mutually orthogonal and complementary subspaces which are spanned by G-class and X-class, respectively.

References

M.Heins [1] : Riemann surfaces of infinite genus. Ann. of Math. 55 (1952).

B.Kjellberg [1] : On certain integral and harmonic functions. A study in minimum modulus. Uppsala Thesis (1948).
[2] : On the growth of minimal positive harmonic functions in a plane region. Arkiv för Mat. Bd. 1 Nr. 25 (1950), 347-351.

Z.Kuramochi [1] : On covering surfaces. Osaka Math. Journ. 5 (1953), 155-201.

R.S.Martin [1] : Minimal positive harmonic functions. Trans. Amer. Math. Soc. 49 (1941), 137-172.

M.Ozawa [1] : On harmonic dimension. These Reports.

Department of Mathematics,
Tokyo Institute of Technology.

(*) Received June 6, 1954.