ON HARMONIC DIMENSION

By Mitsuru OZAWA

An important notion of harmonic dimension of an end has recently been introduced by M. Heins. He has exhaustively investigated the relation between harmonic dimension and bounded harmonic or analytic functions on an end.

For completeness we shall explain the concept of harmonic dimension of an end in the sense of Heins.

An admitted Riemann surface of finite or infinite genus subjects to the following two conditions:

(i) the surface has precisely one ideal boundary element,
(ii) the ideal boundary is null in the sense of R. Nevanlinna.

An end is a subregion of an admitted surface whose complement is compact. Without loss of generality we may assume that the relative boundary of an end consists of a finite number of compact smooth Jordan curves. We shall abbreviate such an end by H-end. Let Ω be an H-end and P_{Ω} denote the family of non-trivial functions non-negative, one-valued and harmonic on Ω which vanish continuously on the relative boundary.

Harmonic dimension of an H-end is then the minimum number of elements of P_{Ω} which generate P_{Ω} with non-negative coefficients provided that such a finite set exists, otherwise it is ∞. We shall abbreviate this integer or ∞ by $H(\Omega)$. Heins has also given an example of an end of an arbitrarily assigned finite $H(\Omega)$. He has stated further that the existence of an end of infinite $H(\Omega)$ seems plausible.

In the present paper we shall introduce another sort of harmonic dimension of a domain. For this purpose, we first state the known results concerning the behavior of Green function at a neighborhood of the ideal boundary. Because of various difficulties we cannot yet succeed to construct the theory in its fully general form. We shall finally offer various unsolved problems which would seem very important for the investigation of the structure of an ideal boundary as well as the classification theory.

1. C-end and Green function.

Admitted Riemann surfaces are the same as in Heins' case. Let Ω be a subsurface of an admitted surface satisfying the following conditions:

(i) Ω has a finite number of non-compact piecewise smooth Jordan curves as its boundary Γ,

(ii) $\Omega \cap (F - F_n)$ has only one component for any n, where F and F_n denote the original admitted surface and its n th exhausting domain, respectively.

This Ω is abbreviated as a C-end. Let $\Omega_n = \Omega \cap F_n$ being assumed to be connected, and $\Gamma_n = \Gamma \cap F_n$ and let γ_n be the remaining boundary of Ω_n.

Let $g(z,t)$ be the Green function of Ω with the pole at $t \in \Omega_1$. Then $g(z,t)$ is a harmonic function bounded on $\Omega - \Omega_n$, $n \geqq 3$. By maximum principle $\underset{\Omega - \Omega_n}{\mathrm{Max}}\, g(z,t) = \underset{\gamma_n}{\mathrm{Max}}\, g(z,t) \equiv M(t,n)$. Moreover $M(t,n) \geqq M(t,m)$ holds for $n < m$. Thus there happen two possibilities, that is, either $\lim_{n\to\infty} M(t,n) = 0$ or > 0. In the former case the ideal boundary element is called regular and in the latter case irregular with respect to Ω.

In the regular case $\lim_{m\to\infty} g(z_m, t)$ exists and vanishes for any non-compact point-sequence $\{z_m\}$. Moreover, by Harnack's inequality, $\lim_{m\to\infty} g(z_m, t) = 0$ holds for t lying in any compact part of Ω. Hence by symmetry character of Green function, $\lim_{m\to\infty} g(t, z_m)$ also vanishes identically.

In the irregular case $\lim_{m\to\infty} g(z_m', t) = 0$ and $\overline{\lim_{m\to\infty}}\, g(z_m, t) > 0$. Hence we can select a subsequence $g(z_{m_n}, t)$ whose limit is not zero. Then $\lim_{n\to\infty} g(t, z_{m_n})$ does not vanish identically while it vanishes continuously on Γ, and hence it represents a non-trivial harmonic function non-negative on Ω.

Such a classification of the ideal boundary element with respect to Ω is evidently locally invariant and does not depend on a special choice of a point t.

2. Harmonic dimension of a C-end.

If the ideal boundary element is regular, then we call that Ω is of zero harmonic dimension. If not so, then we define the harmonic dimension of Ω by the number of non-proportional limit functions $\lim_{n\to\infty} g(t, z_{m_n})$ not vanishing identically. Let P_Ω be a family generated, with non-negative coefficients, by such limit functions which are non-trivial positive harmonic on Ω and vanish identically on Γ.

Now we should notice a distinction between the definition by Kjellberg [1], [2] and ours. Kjellberg has introduced a similar notion "harmonically simple or multiple". However, by his definition, an ordinary finite-ply connected planar region bounded by analytic curves where a boundary point is regarded as an ideal element is harmonically simple. For instance, $\mathrm{Re}\, z$ is only one generator of positive harmonic function on the right half-plane if the point at infinity is regarded as an ideal element; but in our definition there is no limiting function, that is, the harmonic dimension is equal to zero.

Next we shall prove that the notion of harmonic dimension of a C-end is a local property. Let $\Omega^{(1)}$, $\Omega^{(2)}$ be two C-ends satisfying the following conditions:

(i) $\Omega^{(1)} \supset \Omega^{(2)}$,

(ii) $\Omega^{(1)} - \Omega_m^{(1)} \subseteq \Omega^{(2)} - \Omega_m^{(2)}$ for all $m > N$,

(iii) every boundary component of $\Omega^{(1)} - \Omega^{(2)}$ contains a subarc belonging to the boundary $\Gamma^{(1)}$ of $\Omega^{(1)}$. Here we assume that $\Gamma^{(1)}$ and $\Gamma^{(2)}$, the boundary of $\Omega^{(2)}$, do not touch each other.

We then denote by $\Omega^{(1)} \succ \Omega^{(2)}$.

Let $g(z, t_m^{(i)})$ be the Green function of $\Omega^{(1)}$ with the pole at $t_m^{(i)}$, and let $\lim_{m\to\infty} g(z, t_m^{(i)})$ exist and be a non-trivial positive harmonic function $G_i(z)$ on $\Omega^{(1)}$. Then $g(z, t_m^{(i)})$ is bounded on $\Gamma^{(2)}$. Let $b_{g(z,t_m^{(i)})}(z)$ be a bounded harmonic function on $\Omega^{(2)}$ such that $= g(z, t_m^{(i)})$ on $\Gamma^{(2)}$. Then $b_{g(z,t_m^{(i)})}(z)$ converges, as $m \to \infty$, uniformly to a bounded harmonic function $b_{G_i(z)}(z)$ such that $= G_i(z)$ on $\Gamma^{(2)}$ on which $G_i(z)$ is uniformly bounded. On the other hand $h(z, t_m^{(i)}) = g(z, t_m^{(i)}) - b_{g(z,t_m^{(i)})}(z)$ is the Green function of $\Omega^{(2)}$ with the pole at $t_m^{(i)}$. Hence there holds

$$\begin{aligned} H_i(z) &\equiv \lim_{m\to\infty} h(z, t_m^{(i)}) \\ &= \lim_{m\to\infty} \left(g(z, t_m^{(i)}) - b_{g(z,t_m^{(i)})}(z) \right) \\ &= G_i(z) - b_{G_i(z)}(z) \equiv T(G_i(z)), \end{aligned}$$

and $H_i(z)$ is a non-trivial positive harmonic function on $\Omega^{(2)}$. The T-operator defined above is evidently a positively linear mapping from $P_{\Omega^{(1)}}$ to $P_{\Omega^{(2)}}$.

Next T-operator is one-to-one. In fact, if $T(G_1) = T(G_2)$ holds, then

$$G_1(z) - G_2(z) = b_{G_1}(z) - b_{G_2}(z).$$

Since the right-hand member is bounded on $\Omega^{(2)}$ and the left-hand member is bounded on $\Omega^{(1)} - \Omega^{(2)}$, the left-hand member is bounded and harmonic on the whole $\Omega^{(1)}$. The latter is identically zero on $\Gamma^{(1)}$ and hence also on $\Omega^{(1)}$. Thus $G_1(z) \equiv G_2(z)$.

Next we shall prove that T-operator is an onto-mapping from $P_{\Omega^{(1)}}$ to $P_{\Omega^{(2)}}$.

For that purpose we construct the inverse operator. Let $H_i(z)$ denote a generator of $P_{\Omega^{(2)}}$, that is, a limit function obtained from a sequence of the Green functions $h(z, t_m^{(i)})$ as $m \to \infty$ on $\Omega^{(2)}$. From $h(z, t_m^{(i)})$ we construct two functions such that

$$\underline{h} = \begin{cases} h & \text{on} \quad \Omega^{(2)} \\ 0 & \text{on} \quad \Omega^{(1)} - \Omega^{(2)} \end{cases}$$

and

$$\overline{h} = \begin{cases} h + a & \text{on} \quad \Omega^{(2)} \\ a\omega_j & \text{on} \quad \Omega^{(1)} - \Omega^{(2)}, \end{cases}$$

where ω_j is the harmonic measure on the j th component of $\Omega^{(1)} - \Omega^{(2)}$ which is $= 1$ on $\Gamma^{(2)} - \Gamma^{(1)}$ and $= 0$ on $\Gamma^{(1)} - \Gamma^{(2)}$, and a is a finite number such that $\overline{h}$ is superharmonic. Existence of ω_j is guaranteed by $\Omega^{(1)} \succ \Omega^{(2)}$ and that of such a finite a is easily proved by means of the compactness of $\overline{\Omega^{(1)} - \Omega^{(2)}}$. Here $\Gamma^{(1)}$ and $\Gamma^{(2)}$ do not touch each other and $\Gamma^{(2)}$ consists of a finite number of smooth Jordan curves which are not mutually tangent. Then there exists a function g harmonic on $\Omega^{(1)}$ except at $t_m^{(i)}$ where it has an expansion

$$\log \frac{1}{|\lambda|} \quad + \quad \text{(a function harmonic around } t_m^{(i)} \text{)}$$

by a local parameter λ at $t_m^{(i)}$ and satisfying the inequalities $\underline{h} \leq g \leq \overline{h}$ on $\Omega^{(1)}$. We denote this by $g = g(z, t_m^{(i)})$. Then $g(z, t_m^{(i)})$ is the Green function of $\Omega^{(1)}$. Now we remark that a can be chosen independently of each m, since $h(z, t_m^{(i)})$ is positive and uniformly bounded in a neighborhood of $\Gamma^{(2)} - \Gamma^{(1)}$ with respect to m. Thus $g(z, t_m^{(i)})$ is uniformly bounded on $\Omega^{(1)} - \Omega^{(2)}$. A suitable subsequence of $\{g(z, t_m^{(i)})\}$, $g(z, t_{m_n}^{(i)})$ tends uniformly in the wider sense to a non-negative harmonic function on $\Omega^{(1)}$ as $n \to \infty$. The limit function $G_i(z)$ does not vanish identically by virtue of $g \geq h$ on $\Omega^{(2)}$, while it vanishes identically on $\Gamma^{(1)}$. Therefore $G_i(z)$ belongs to the class $P_{\Omega^{(1)}}$. Now we put $G_i(z) - H_i(z) = b_{G_i}(z)$ on $\Omega^{(2)}$. Then $b_{G_i}(z)$ is bounded uniformly on $\Gamma^{(2)}$ and $\leq a$ on $\Omega^{(2)}$. We now define the inverse operator T^{-1} by the relation $T^{-1}(H_i(z)) = G_i(z)$. Evidently T^{-1}-operator can be extended in the positively linear manner. And moreover it is evident that $T \circ T^{-1} = 1$ and $T^{-1} \circ T = 1$.

Thus we have the desired result, that is, the harmonic dimension is a local invariant.

4. Example.

We shall give here an example of C-end of $CH(\Omega) \geqq 1$. Let Ω_1 denote the unit circular disc with an infinite number of slits lying on the positive real axis. Here we assume that the slits are as the whole very small in length and are rarely distributed such that $\lim_{r \to 0} g(-r, z, \Omega_1) > 0$ for a fixed point z in Ω_1, where $g(z, -r, \Omega_1)$ is the Green function of Ω_1 with pole at a point $-r$ (< 0) on the negative real axis. Let Ω_2 be a replica of Ω_1 and Ω_3 be an H-end which is constructed from Ω_1 and Ω_2 by making the standard identification along the corresponding slits. Let Ω_4 be constructed from Ω_3 by omitting off the left semi-circular disc $\{\operatorname{Re} z \leq 0,\ |z| < 1\}$ on Ω_2. Then Ω_4 is a C-end.

Let $g(z, -r, \Omega_4)$ be the Green function of Ω_4 with the pole at $-r$ on Ω_1. Evidently $g(z, -r, \Omega_4) \geq g(z, -r, \Omega_1)$. By the construction of Ω_1, we see

$$\lim_{r_n \to 0} g(z, -r_n, \Omega_4) \geq \lim_{r_n \to 0} g(z, -r_n, \Omega_1) > 0$$

on Ω_1. Therefore $\lim_{r_n \to 0} g(z, -r_n, \Omega_4)$ is a non-degenerate limit function which is to be constructed.

5. A characterization of a C-end of zero harmonic dimension.

Let $u(z)$ be an arbitrary harmonic function bounded on $\Omega - \Omega_m$ which reduces to a constant k on $\Gamma_m' (= \Gamma \cap \overline{\Omega - \Omega_m})$. If Ω is of zero harmonic dimension, then we assert that $u(z)$ has a limit at the ideal boundary, and vice versa. This fact may be proved as follows.

Sufficiency. Let $g(z, t)$ be the Green function of Ω with the pole at t which lies in a fixed compact part of Ω . $g(z,t)$ is a harmonic function bounded on $\Omega - \Omega_m$ for sufficiently large m and is equal to zero on Γ_m' . Thus $g(z,t)$ must have a limit at the ideal boundary which is equal to zero. Therefore the ideal boundary is regular with respect to $\Omega - \Omega_m$.

Necessity. Let $u(z)$ satisfy the assumption, then $u(z) - k$ is bounded harmonic on Ω and is constant zero on Γ_m' . Then we see easily that

$$u(q) - k = \frac{1}{2\pi}\int_{\gamma_0} (u(p)-k) \frac{\partial}{\partial\nu} g(p,q)\, ds$$

holds, γ_0 being the remaining boundary $\Gamma - \Gamma_m'$ of Γ . For a suitable sequence $\{q_n^{(\mu)}\}$ we have

$$\lim_{n\to\infty} g(p, q_n^{(\mu)}) = G_\mu(p)$$

by definition. However if $G_\mu(p) \equiv 0$ holds for any non-compact sequence $\{q_n^{(\mu)}\}$, then

$$\lim_{n\to\infty} \left(u(q_n^{(\mu)}) - k\right) = \frac{1}{2\pi}\int_{\gamma_0} (u(p)-k)\frac{\partial}{\partial\nu} G_\mu(p)\, ds \equiv 0 .$$

Therefore, under our assumption, there holds

$$\lim_{n\to\infty} u(q_n^{(\mu)}) = k .$$

As an application we shall establish a sufficient condition for $CH(\Omega) = 0$. A simply connected domain with four-vertices, i. e. a "curvilinear quadrilateral", A_n is said to have the modulus a_n if A_n is mapped one-to-one and conformally onto a rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq 2\pi$ in such a manner the vertices correspond to the vertices in an assigned order. Let Ω be a C-end and $\{A_n\}$ be a sequence of curvilinear quadrilaterals lying on Ω such that A_{n+1} separates A_n from the ideal boundary. Let f_n^{-1} be a mapping function such that $\Gamma \wedge \bar{A}_n$ correspond to two sides ($x=0$, $0 \leqq y \leqq 2\pi$) and ($x = a_n$, $0 \leqq y \leqq 2\pi$).

If there is a sequence $\{A_n\}$ of curvilinear quadrilaterals on Ω such that

$$\sum_{n=1}^{\infty} a_n = \infty ,$$

then Ω has zero harmonic dimension.

Proof can be done under a similar manner as for theorem 13.1 in Heins' paper. Here use is made of a perfect condition stated above.

6. Unsolved problems.

We shall now explain some unsolved problems. Let Ω be an H-end and Ω_1 be a C-end which satisfy the following conditions:

(i) $\Omega \supset \Omega_1$,

(ii) every components of $\Omega - \Omega_1$ contains at least a subarc of Γ as a part of its boundary, where Γ is the relative boundary of Ω . This positional postulate will be denoted by $\Omega \Supset \Omega_1$.

Let Ω be a subsurface of an admitted surface satisfying the following conditions:

(i) Ω has an infinite number of analytic Jordan curves as its boundary,

(ii) the same as the condition (ii) for C-end.

This end Ω is abbreviated by an extended C-end. We first suppose that Ω and Ω_1 are an H-end and a C-end (not in the extended sense), respectively, and that $\Omega \Supset \Omega_1$ holds.

Problem 1. Does $H(\Omega) \geqq CH(\Omega_1) + 1$ always hold or not ?

If we admit an extended C-end in place of a C-end and assume merely that $\Omega \supset \Omega_1$ in place of $\Omega \Supset \Omega_1$, then the relation in Problem 1 does not hold in general. In fact, let Ω_1 denote an extended C-end constructed in § 4. Evidently $\Omega = \{|z| < 1\}$ is an H-end such that $H(\Omega) = 1$ and Ω_1 is also an extended C-end such that $CH(\Omega_1) = 1$ which however does not satisfy $\Omega \Supset \Omega_1$.

Let next Ω be a C-end and Ω_1 the same as in Problem 1 and further the similar positional postulate as in Problem 1 be assumed to hold.

Problem 2. Does $CH(\Omega) \geqq CH(\Omega_1)$ hold ?

References.

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Department of Mathematics,
Tokyo Institute of Technology.

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