NOTE ON LAPLACE-TRANSFORMS, (II)

ON SOME CLASS OF LAPLACE-TRANSFORMS, (I)

By Chuji TANAKA

(Communicated by Y.Komatu)

(1) THEOREM I. We consider the Laplace-transform

$$(1.1)\quad F(s) = \int_0^\infty \exp(-sx) f(x)\,dx \quad (s=\sigma+it)$$

where $f(x)$ is R-integrable in any finite interval $0 \leqq x \leqq X$, X being an arbitrary positive constant. In general, $F(s)$ has three convergence-abscisses, i.e. simple convergence-abscissa σ_s, uniform convergence-abscissa σ_u, and absolute convergence-abscissa σ_a ($\sigma_s \leqq \sigma_u \leqq \sigma_a$), whose existence is well-known ([1] p.16, p.42 - See references placed at the end -). In the present Note, we discuss the sufficient conditions for $\sigma_s = \sigma_u = \sigma_a$. We begin with some definitions:

DEFINITION I. The sequence of intervals $\{I_\nu\}$ $(\nu = 1, 2, \cdots\cdots)$ is defined as follows:

$$\begin{cases} (i) & I_\nu : \quad t_\nu - \varepsilon(t_\nu) \leqq t \leqq t_\nu + \varepsilon(t_\nu), \\ & \quad 0 < t_\nu \uparrow \infty, \\ (ii) & \lim_{\nu\to\infty} \varepsilon(t_\nu) = 0, \quad \lim_{\nu\to\infty} \frac{1}{t_\nu} \log\left(\frac{1}{\varepsilon(t_\nu)}\right) = 0. \end{cases}$$

DEFINITION II. We say that $f(t)$ belongs to the class $C\{I_\nu\}$, provided that

$$\begin{cases} (i) & \overline{\lim_{t\to\infty}} \frac{1}{t} \log|f(t)| = \lim_{\substack{t\to\infty \\ t\in\{I_\nu\}}} \frac{1}{t}\log|f(t)| \\ & = \alpha < +\infty, \\ (ii) & f(t) \text{ is continuous in } \{I_\nu\}, \\ (iii) & f_1(t) = \mathcal{J}[f(t)] \text{ is differentiable in } \{I_\nu\} \text{ and} \\ & \overline{\lim_{\substack{t\to\infty \\ t\in\{I_\nu\}}}} \frac{1}{t}\cdot \log|f_1'(t)| \leqq \alpha. \end{cases}$$

Now we can establish

THEOREM I. If $f(t)$ is R-integrable in any finite interval and belongs to $C\{I_\nu\}$, then

$$\sigma_s = \sigma_u = \sigma_a = \alpha.$$

As an immediate consequence of Theorem 1, follows

COROLLARY. If $f(t)$ is real and continuous in $0 \leqq t < +\infty$, and $\lim_{t\to\infty} \frac{1}{t}\log|f(t)| = \alpha$, then $\sigma_s = \sigma_u = \sigma_a = \alpha$.

In order to prove Theorem 1, we need next Lemma:

LEMMA.

$$\beta = \overline{\lim_{t\to\infty}} \frac{1}{t} \log|F(t;\varepsilon(t))|$$
$$\leqq \sigma_s \leqq \sigma_u \leqq \sigma_a \leqq$$
$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log|f(t)| = \alpha,$$

where

$$\begin{cases} (i) & F(t;\varepsilon(t)) = \dfrac{1}{2\varepsilon(t)} \displaystyle\int_{t-\varepsilon(t)}^{t+\varepsilon(t)} f(x)\,dx, \\ (ii) & \lim_{t\to\infty} \varepsilon(t) = 0, \\ & \lim_{t\to\infty} \frac{1}{t}\log\left[\frac{1}{\varepsilon(t)}\right] = 0 \end{cases}$$

Proof. By the definition of α, for any given $\varepsilon\ (>0)$, there exists $T(\varepsilon)$ such that

$$|f(t)| < \exp((\alpha+\varepsilon)t) \quad for \quad t > T(\varepsilon).$$

Hence, denoting by $[t]$, as usual, the greatest integer contained in t, we have, for $[t] > T(\varepsilon)$,

$$\int_{[t]}^t |f(x)|\,dx < (t-[t]) \exp((\alpha+\varepsilon)t)$$
$$< \exp((\alpha+\varepsilon)t).$$

Accordingly, by K.Kurosu's formula ([2], [3]),

$$\sigma_a = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\left(\int_{[t]}^t |f(x)|\,dx\right) < (\alpha+\varepsilon).$$

Letting $\varepsilon \to 0$,

$$(1.2)\qquad \sigma_a \leqq \alpha$$

By K.Kurosu's formula, we have

$$\sigma_s = \overline{\lim_{t\to\infty}} \frac{1}{t} \log\left|\int_{[t]}^t f(x)\,dx\right|$$
$$= \overline{\lim_{t\to\infty}} \frac{1}{[t]} \log\left|\int_{[t]}^t f(x)\,dx\right|.$$

Therefore, for any given $\varepsilon\ (>0)$, there exists $T(\varepsilon)$ such that

$$(1\cdot 3)\quad \left|\int_{[t]}^{t} f(x)\,dx\right| < \exp([t](\sigma_s+\varepsilon)) \quad \text{for } [t] > T(\varepsilon)$$

By (1.3), for $t > [t] > T(\varepsilon)$,

$$|F(t;\varepsilon(t))| = \frac{1}{2\varepsilon(t)}\left|\int_{[t]}^{t+\varepsilon(t)} - \int_{[t]}^{t-\varepsilon(t)}\right|$$

$$\leqq \frac{1}{2\varepsilon(t)}\cdot 2\exp([t](\sigma_s+\varepsilon))$$

$$= \frac{1}{\varepsilon(t)}\exp([t](\sigma_s+\varepsilon)).$$

By (1.3), for $t=[t] > T(\varepsilon)+1$,

$$|F(t;\varepsilon(t))| = \frac{1}{2\varepsilon(t)}\left|\int_{[t]}^{t+\varepsilon(t)} + \int_{[t]-1}^{t-\varepsilon(t)}\right|$$

$$\leqq \frac{1}{2\varepsilon(t)}\left\{\exp([t](\sigma_s+\varepsilon)) + 2\exp(([t]-1)(\sigma_s+\varepsilon))\right\}$$

$$< \frac{3}{2\varepsilon(t)}\exp([t](\sigma_s+\varepsilon)).$$

Hence, in any case,

$$(1\cdot 4)\quad |F(t;\varepsilon(t))| < \frac{3}{2\varepsilon(t)}\exp([t](\sigma_s+\varepsilon)),$$

so that

$$\beta = \varlimsup_{t\to\infty}\frac{1}{t}\log|F(t;\varepsilon(t))|$$

$$\leqq (\sigma_s+\varepsilon)\lim_{t\to\infty}\frac{[t]}{t} + \lim_{t\to\infty}\frac{1}{t}\log\left(\frac{1}{\varepsilon(t)}\right) + \lim_{t\to\infty}\frac{1}{t}\log\left(\frac{3}{2}\right)$$

On account of (ii),

$$\beta \leqq (\sigma_s+\varepsilon).$$

Letting $\varepsilon \to 0$,

$$(1\cdot 5)\quad \beta \leqq \sigma_s.$$

By (1.2) and (1.5),

$$\beta \leqq \sigma_s \leqq \sigma_u \leqq \sigma_a \leqq \alpha,$$

which proves Lemma.

<u>Proof of Theorem 1.</u> On account of definition 2, putting $f(t) = f_2(t) + i f_1(t)$, we have

$$F(t_\nu;\varepsilon(t_\nu)) = \frac{1}{2\varepsilon(t_\nu)}\int_{t_\nu-\varepsilon(t_\nu)}^{t_\nu+\varepsilon(t_\nu)} f(x)\,dx$$

$$= \frac{1}{2\varepsilon(t_\nu)}\int_{t_\nu-\varepsilon(t_\nu)}^{t_\nu+\varepsilon(t_\nu)}\{f_2(x)+i f_1(x)\}\,dx$$

$$= f_2(t_1) + i f_1(t_2) \quad (t_1, t_2 \in I_\nu)$$

$$= f(t_1) + i(t_2-t_1) f_1'(t_3) \quad (t_3 \in I_\nu)$$

Hence,

$$(1\cdot 6)\quad \log|F(t_\nu;\varepsilon(t_\nu))| = \log|f(t_1)| + \log\left|\left\{1 + i(t_2-t_1) f_1'(t_3)\cdot\frac{1}{f(t_1)}\right\}\right|.$$

By definition 2, we can put

$$|f(t)| = \exp(t(\alpha+\Delta(t))).\quad t\in\{I_\nu\}$$

$$\lim_{\substack{t\to\infty\\ t\in\{I_\nu\}}} \Delta(t) = 0$$

Accordingly,

$$(1\cdot 7)\quad |f(t_1)| \geqq \exp\{(t_\nu-\varepsilon(t_\nu))(\alpha+\Delta(t_1))\} \quad \text{if } \alpha \geqq 0,$$

$$\geqq \exp\{(t_\nu+\varepsilon(t_\nu))(\alpha+\Delta(t_1))\} \quad \text{if } \alpha < 0.$$

Taking account of definition 2, for any given $\varepsilon\ (>0)$, there exists $T(\varepsilon)$ such that

$$|f_1'(t)| < \exp((\alpha+\varepsilon)t)$$

for $t > T(\varepsilon)$, $t \in \{I_\nu\}$.

Hence

$$(1\cdot 8)\quad |f_1'(t_3)| \leqq \exp\{(\alpha+\varepsilon)(t_\nu+\varepsilon(t_\nu))\} \quad \text{if } \alpha \geqq 0,$$

$$\leqq \exp\{(\alpha+\varepsilon)(t_\nu-\varepsilon(t_\nu))\} \quad \text{if } \alpha < 0,$$

provided that $t_\nu - \varepsilon(t_\nu) > T(\varepsilon)$.

By (1.7), (1.8) and $|t_2-t_1| < 2$,

$$\left|1 + i(t_2-t_1) f_1'(t_3)\cdot\frac{1}{f(t_1)}\right| \leqq 1$$

$$+2\exp\{(\alpha+\varepsilon)(t_\nu \pm \varepsilon(t_\nu)) - (\alpha+\Delta(t_1))(t_\nu \mp \varepsilon(t_\nu))\}$$

$$= 1 + 2\exp\{(\varepsilon-\Delta(t_1))t_\nu \pm (2\alpha+\varepsilon+\Delta(t_1))\varepsilon(t_\nu)\}$$

$$< \exp\{2\varepsilon t_\nu\}.$$

Hence,

$$(1\cdot 9)\quad \log\left|1 + i(t_2-t_1) f_1'(t_3)\cdot\frac{1}{f(t_1)}\right| < 2\varepsilon t_\nu$$

By virtue of (1.6), (1.9) and definition 2,

$$\varlimsup_{t\to\infty}\frac{1}{t}\log|F(t,\varepsilon(t))|$$

$$\geqq \varlimsup_{\nu\to\infty}\frac{1}{t_\nu}\log|F(t_\nu,\varepsilon(t_\nu))|$$

$$\geq \lim_{\nu\to\infty} \frac{t_1}{t_\nu}\cdot \frac{1}{t_1}\log|f(t_1)| - 2\varepsilon$$

$$\geq \alpha - 2\varepsilon$$

Letting $\varepsilon \longrightarrow 0$,

$$\varliminf_{t\to\infty} \frac{1}{t}\log|F(t;\varepsilon(t))| \geq \alpha$$

On account of Lemma, we have $\sigma_s = \sigma_u = \sigma_a = \alpha$, which is to be proved.

(2) THEOREM II. In this section, we shall give sufficient conditions for $f(t)$ to belong to $C\{I_\nu\}$. The theorem states as follows:

THEOREM II. $f(t)$ belongs to $C\{I_\nu\}$, provided that

(a) $f(z)$ $(z = re^{i\theta})$ is regular in P : $r > r_0$, $|\theta| \leq \vartheta < \frac{\pi}{2}$.

(b) $f(z)$ is of exponential type in P for sufficiently large r .

As its consequence, by Theorem I, we have

COROLLARY. Under the same conditions as in Theorem 2, if $f(t)$ is R-integrable in $0 \leq t \leq r_0$, then $\sigma_s = \sigma_u = \sigma_a = \varlimsup_{t\to\infty} \frac{1}{t}\log|f(t)|$.

To establish Theorem 2, we need next Lemma.

LEMMA. Under the same conditions as in Theorem 2, we have

$$\varlimsup_{t\to\infty} \frac{1}{t}\log|f'(t)|$$
$$\leq \varlimsup_{t\to\infty} \frac{1}{t}\log|f(t)| = \alpha.$$

Proof. Let us put

$$\varphi(\theta) = \varlimsup_{r\to\infty} \frac{1}{r}\log|f(re^{i\theta})| , \quad |\theta| \leq \vartheta < \frac{\pi}{2},$$

which is finite on account of (b). By well-known Phragmén-Lindelöf's theorem, for any given $\varepsilon\ (>0)$, there exists $\delta(\varepsilon)$ such that

$$\varphi(\theta) < \varphi(0) + \varepsilon = \alpha + \varepsilon \quad \text{for } |\theta| < \delta(\varepsilon)$$

Hence we have uniformly

$$(2.1) \quad |f(re^{i\theta})| < \exp((\alpha+\varepsilon)r)$$

$$\text{for } |\theta| < \delta(\varepsilon), \quad r > R(\varepsilon).$$

By Cauchy's theorem,

$$f'(t) = \frac{1}{2\pi i}\oint_{|\zeta - t| = \rho} f(\zeta)\frac{d\zeta}{\zeta - t} ,$$

where t : real number, ρ : fixed positive constant. Therefore, by (2.1),

$$|f'(t)| \leq \frac{1}{2\pi}\exp((\alpha+\varepsilon)(t+\rho))\cdot\int_0^{2\pi}\rho\frac{d\theta}{\rho^2}$$

$$= \exp((\alpha+\varepsilon)(t+\rho))\cdot\frac{1}{\rho},$$

so that

$$\varlimsup_{t\to\infty} \frac{1}{t}\log|f'(t)| \leq \alpha+\varepsilon$$

Letting $\varepsilon \to 0$,

$$\varlimsup_{t\to\infty} \frac{1}{t}\log|f'(t)| \leq \alpha,$$

which completes the proof.

Proof of Theorem 2. By (a) and Lemma, we have evidently

$$\begin{cases} \text{(i)} & f(t) \text{ : continuous for } t > r_0 , \\ \text{(ii)} & \varlimsup_{t\to\infty} \frac{1}{t}\log|\mathcal{J}f'(t)| \leq \varlimsup_{t\to\infty} \frac{1}{t}\log|f'(t)| \\ & \leq \alpha \end{cases}$$

Accordingly, it is sufficient to prove the existence of sequence of intervals $\{I_\nu\}$ such that

$$\lim_{\substack{t\to\infty \\ t\in\{I_\nu\}}} \frac{1}{t}\log|f(t)| = \alpha$$

There exists evidently sequence $\{x_\nu\}$ such that

$$\begin{cases} \text{(i)} & \alpha = \varlimsup_{t\to\infty} \frac{1}{t}\log|f(t)| = \lim_{\nu\to\infty} \frac{1}{x_\nu}\log|f(x_\nu)| \\ \text{(ii)} & f(x_\nu) \neq 0 \qquad (\nu = 1, 2, \cdots\cdots) \end{cases}$$

We now consider sequence of intervals $\{\bar{I}_\nu\}$: $x_\nu - \theta_\nu \leq t \leq x_\nu + \theta_\nu$, where θ_ν $(0 < \theta_\nu \leq 1)$ will be determined later. By the meanvalue theorem,

$$(2.2) \quad |f(t)| = |f(x_\nu)| + (t - x_\nu)\left\{\frac{d}{dt}|f(t)|\right\}_{t = x_\nu'}$$

$$(t, x_\nu' \in \bar{I}_\nu)$$

By the inequality $\left|\frac{d}{dt}|f(t)|\right| \leq |f'(t)|$,

$$(2.3) \quad |t - x_\nu|\cdot\left|\frac{d}{dt}|f(t)|\right|_{t = x_\nu'}$$
$$\leq \theta_\nu \max_{x_\nu - 1 \leq t \leq x_\nu + 1} |f'(t)| = \theta_\nu |f'(x_\nu'')|$$
$$(|x_\nu - x_\nu''| \leq 1)$$

On account of (2.2) and (2.3),

(2.4) $$|f(x_\nu)|\left\{1-\theta_\nu\left|\frac{f'(x_\nu'')}{f(x_\nu)}\right|\right\} \leqq |f(t)| \leqq |f(x_\nu)|\left\{1+\theta_\nu\left|\frac{f'(x_\nu'')}{f(x_\nu)}\right|\right\}$$
$$(t \in \bar{I}_\nu)$$

By Lemma,

$$\beta = \varlimsup_{\nu\to\infty} \frac{1}{x_\nu''} \log|f'(x_\nu'')| \leqq \lim_{\nu\to\infty} \frac{1}{x_\nu} \log|f(x_\nu)| = \alpha$$

Taking suitable subsequence of $\{x_\nu''\}$, if necessary, we can assume that

(2.5) $$\begin{cases} \text{(i)} & |f'(x_\nu'')| = \exp(\beta(\nu)x_\nu''), \quad \lim_{\nu\to\infty}\beta(\nu) = \beta \\ \text{(ii)} & |f(x_\nu)| = \exp(\alpha(\nu)x_\nu), \quad \lim_{\nu\to\infty}\alpha(\nu) = \alpha \end{cases}$$

We distinguish two cases:

Case: $\beta<\alpha$ In this case, by (2.5),

(2.6) $$\left|\frac{f'(x_\nu'')}{f(x_\nu)}\right| = \exp\{\beta(\nu)x_\nu''-\alpha(\nu)x_\nu\} < \exp\{x_\nu(\beta(\nu)-\alpha(\nu))+|\beta(\nu)|\}.$$

Now we determine $\{\theta_\nu\}$ as follows:

$$\begin{cases} \text{(i)} & \lim_{\nu\to\infty}\theta_\nu = 0, \\ \text{(ii)} & \lim_{\nu\to\infty}\frac{1}{x_\nu}\log\left(\frac{1}{\theta_\nu}\right) = 0 \end{cases}$$

(for instance, $\theta_\nu = \frac{1}{x_\nu}$).

By (2.6), for sufficiently large ν,

$$\theta_\nu\left|\frac{f'(x_\nu'')}{f(x_\nu)}\right| < \exp\left\{-(\alpha-\beta)\cdot\frac{x_\nu}{2}\right\} < \frac{1}{2}.$$

Hence, by (2.4),

$$\frac{1}{2}|f(x_\nu)| < |f(t)| < \frac{3}{2}|f(x_\nu)|.$$
$$(t\in \bar{I}_\nu)$$

Accordingly,

$$\lim_{\substack{t\to\infty\\ t\in\{\bar{I}_\nu\}}} \frac{1}{t}\log|f(t)| = \lim_{\nu\to\infty}\frac{1}{x_\nu}\log|f(x_\nu)| = \alpha.$$

Then, the sequence of intervals $\{\bar{I}_\nu\}$ is desired one.

Case: $\beta=\alpha$. In this case we determine θ_ν such that

(2.7) $$\begin{cases} \text{(i)} & \theta_\nu = \exp\{x_\nu\cdot\gamma(\nu)\}, \\ \text{(ii)} & \gamma(\nu) = -2\,|\beta(\nu)-\alpha(\nu)| - \frac{|\beta(\nu)|}{x_\nu} - \frac{1}{2x_\nu}. \end{cases}$$

Then, evidently

(2.8) $$\begin{cases} \text{(i)} & \lim_{\nu\to\infty}\gamma(\nu) = 0, \\ \text{(ii)} & 0<\theta_\nu<\exp(-\frac{1}{2})<1 \end{cases}$$

By (2.6) and (2.7),

$$\theta_\nu\left|\frac{f'(x_\nu'')}{f(x_\nu)}\right| < \exp\left\{x_\nu\left(\gamma(\nu)+\beta(\nu)-\alpha(\nu)+\frac{|\beta(\nu)|}{x_\nu}\right)\right\} < \exp(-\tfrac{1}{2}).$$

Hence, by (2.4),

$$|f(x_\nu)|\{1-\exp(-\tfrac{1}{2})\} < |f(t)| < |f(x_\nu)|\{1+\exp(-\tfrac{1}{2})\}$$
$$(t\in\bar{I}_\nu),$$

so that

(2.9) $$\lim_{\substack{t\to\infty\\ t\in\{\bar{I}_\nu\}}}\frac{1}{t}\log|f(t)| = \lim_{\nu\to\infty}\frac{1}{x_\nu}\log|f(x_\nu)| = \alpha.$$

If $\varlimsup_{\nu\to\infty}\theta_\nu>0$, we can determine a sequence of intervals $\{I_\nu\}$ such that

$$\begin{cases} \text{(i)} & I_\nu:\ x_\nu-\varepsilon_\nu \leqq t \leqq x_\nu+\varepsilon_\nu, \text{ and } I_\nu\in\bar{I}_\nu, \\ \text{(ii)} & \lim_{\nu\to\infty}\varepsilon_\nu = 0, \quad \lim_{\nu\to\infty}\frac{1}{x_\nu}\log\left(\frac{1}{\varepsilon_\nu}\right)=0, \\ \text{(iii)} & \lim_{\substack{t\to\infty\\ t\in\{I_\nu\}}}\frac{1}{t}\log|f(t)| = \alpha. \end{cases}$$

(for instance, $\varepsilon_\nu = \frac{1}{x_\nu}$). In this case, $\{I_\nu\}$ is desired one. If $\varlimsup_{\nu\to\infty}\theta_\nu=0$, taking suitable subsequence of $\{\theta_\nu\}$, if necessary, we can assume that $\lim_{\nu\to\infty}\theta_\nu=0$. Then, by (2.8) and (2.9), $\{\bar{I}_\nu\}$ is desired one. Thus, in any case, there exists $\{I_\nu\}$ having desired properties, which completes our proof.

(*) Received July 28, 1951.

[1] G.Doetsh: Theorie und Anwendung der Laplace-Transformation. 1937.
[2] K.Kurosu: On the convergence-abscissa of a certain-definite integral. Tôhoku Math. Jour. Bd. 16, 1919.
[3] C.Tanaka: Note on Laplace-transforms. (I) On the determination of three convergence-abscisses. In the press.

Mathematical Institute, Waseda University, Tokyo.