The object of this note is to give a proof of the following theorem and certain of its applications.

Theorem. If \( S_n^p = o(n^{\beta + \rho}) \), \( \beta > 0 \), \( p > 1 \), where \( S_n^p \) denotes the \( n \)-th Cesaro sum of order \( p \) for the series \( \sum a_n \), then the series \( \sum n^\beta a_n \) is either summable \((C.\, p)\) or not summable by any Cesaro means. Conversely if the series \( \sum n^\beta a_n \) is summable \((C.\, p)\), \( \beta > 0 \), \( p > 1 \), then \( S_n^p = o(n^{\beta + \rho}) \).

Hardy and Littlewood [2] have proved this theorem for a non-negative integer \( p \). Prof. Bosanquet kindly remarked to me that the first part of the theorem is contained in a paper by A. Zygmund [4]. But it seems to me that this theorem is not popular (see Hyslop [3]). On the other hand Bosanquet [1] has succeeded in completing the convergence and summability factor theorem. Following his method we can prove the theorem. The method of proof is different from that of Zygmund. The converse part is also well known, but we give a new proof in the same idea.

Before going to the proof, we need some lemmas.

Lemma 1. If \( \alpha + p > 1 \), \( \beta > 0 \), and \( S_n^p = o(n^{\alpha + p}) \), then \( S_n^p = o(n^{\alpha + p + \beta}) \).

Proof. We have
\[
S_n^p = \frac{1}{n^p} \sum_{n=1}^{\infty} S_n^p \leq \frac{1}{n^p} o\left( n^{\alpha + p} \right) = o\left( n^{\alpha + p + \beta} \right).
\]

Lemma 2. (Bosanquet [1]). If
\[
J = \sum_{\rho} A^{\beta + p - \rho} A^{\alpha - p} A_{\alpha - \rho}^{-\alpha - \beta}
\]

then, for \( 0 < \mu < \nu \), \( 0 < \beta < \mu \),

\[
|J| \leq K A_{\alpha - \rho}^{-\alpha - \beta} A_{\alpha - \rho}^{-\alpha - \beta},
\]

where \( K \) is independent of \( \mu \), \( \nu \), and \( n \).

Proof. We first observe that \( A_{\alpha - \rho}^{-\alpha - \beta} = \mu > 0 \) and
\[
A_{\alpha - \rho}^{-\alpha - \beta} = \frac{1}{\mu} \left( \frac{\mu}{\mu - 1} \right)^{\alpha + \beta} < O(\mu^2) \quad \text{for} \quad 0 < \mu < \lambda.
\]

For \( 0 < \beta < \mu \). Moreover \( A_{\alpha - \rho}^{-\alpha - \beta} = 1 > 0 \) and
\[
0 < A_{\alpha - \rho}^{-\alpha - \beta} = \frac{1}{\mu - 1} A_{\alpha - \rho}^{-\alpha - \beta} < A_{\alpha - \rho}^{-\alpha - \beta} (\mu = 2) \quad \text{for} \quad 0 < \beta < \mu.
\]

It follows by (2) and (3) that
\[
J = \sum_{\rho} A^{\beta + p - \rho} A^{\alpha - p} A_{\alpha - \rho}^{-\alpha - \beta}
\]

Moreover
\[
J = \sum_{\rho} (A_{\alpha - \rho}^{-\alpha - \beta} - A_{\alpha - \rho}^{-\alpha - \beta} - \ldots - A_{\alpha - \rho}^{-\alpha - \beta})^n A_{\alpha - \rho}^{-\alpha - \beta}
\]

Conversely if the series \( \sum n^\beta a_n \) is summable \((C.\, p)\), \( \beta > 0 \), \( p > 1 \), then
\[
S_n^p = o(n^{\beta + \rho})
\]

On the other hand by (2) and (3), if
\[
0 < \mu \leq n \leq \lambda
\]

Thus (1) holds, with \( K_{\mu} = \max (I, \lambda) \).

Proof of the first part of Theorem. The thesis is equivalent to the problem that if \( \sum S_n^p = o(n^{\beta + \rho}) \), \( \beta > 0 \), \( p > 1 \), then the series \( \sum n^\beta a_n \) is summable \((C.\, p)\), provided that it is summable \((C.\, p + 1)\). We divide the proof into the case \( \mu < p < \mu + 1 \) \((\mu = 0, 1, 2, \ldots)\).

Case 1. Suppose that \( \beta > 0 \), \( \mu > 0 \).

For the convenience we replace \( \rho \) by \( -p \), then \( 0 < p < 1 \) and
\[
S_n^p = o(n^{\beta + \rho})
\]

Since \( \sum n^\beta a_n \) is \((C.\, p + 1)\) summable, the necessary and sufficient condition that it is summable \((C.\, p)\) is
\[
T_n^p = \sum_{\rho} A_{n, n}^{\mu + \beta} A_{n, n}^{\nu - \beta} = o(n^{\beta + \rho}).
\]

Now
\[
T_n^p = \sum_{\rho} A_{n, n}^{\mu + \beta} A_{n, n}^{\nu - \beta} = o(n^{\beta + \rho}).
\]
Case 2. Suppose that $\beta > 0$.

It is sufficient to prove

$$\mathcal{L}_m = \mathcal{L}_m^{\beta} 0 = O(n^{-1}) \quad (\text{80})$$

If we denotes

$$\mathcal{L}_m^{\beta}(\alpha\nu) = \frac{n}{m^{\beta}} A_{m-\nu}^{\beta} \alpha\nu,$$

then

$$\mathcal{L}_m^{\beta} = \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu) \quad (\text{81})$$

We have by partial summation

$$\mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu) = \sum_{\nu=0}^{\infty} \frac{\nu^{\beta}}{\nu^{\beta} + \rho} \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu),$$

i.e.

$$\mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu) = \sum_{\nu=0}^{\infty} \frac{\nu^{\beta}}{\nu^{\beta} + \rho} \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu),$$

and hence

$$\mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu) = \sum_{\nu=0}^{\infty} \frac{\nu^{\beta}}{\nu^{\beta} + \rho} \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu).$$

Then

$$\mathcal{L}_m^{\beta} = \sum_{\nu=0}^{\infty} \frac{\nu^{\beta}}{\nu^{\beta} + \rho} \mathcal{L}_m^{\beta} + \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu),$$

Since $-1 < p - 1 < 0$ and $-2 < -p - 1 < -1$, $K$ is analogous to $I$ in Case 1, and we get

$$\mathcal{L}_m^{\beta} = 0 (n^{-p}).$$

Next

$$\mathcal{L}_m^{\beta} = \sum_{\nu=0}^{\infty} \frac{\nu^{\beta}}{\nu^{\beta} + \rho} \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu).$$

Estimating similarly as $I$ in Case 1, we have

$$\mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu) = \sum_{\nu=0}^{\infty} \frac{\nu^{\beta}}{\nu^{\beta} + \rho} \mathcal{L}_m^{\beta} (\nu^{\beta} \alpha\nu).$$

Hence

$$\mathcal{L}_m^{\beta} = \mathcal{L}_m^{\beta} (0(\nu^{\beta})) = 0 (n^{-p}).$$
Collecting the estimations (11), (12), (13) and (14), we get (10).

Proof of the second part of Theorem. If

\[ \sum_{n=0}^{\infty} n^p a_n = J(C, p), \]

then

\[ a_n - J + \sum_{n=1}^{\infty} n^p a_n = O(C, p). \]

But

\[ \sum_{n=0}^{\infty} n^p a_n = \sum_{n=0}^{\infty} (a_n - J - \cdots), \]

and

\[ \sum_{n=0}^{\infty} n^p a_n = \sum_{n=0}^{\infty} a_n - J - \cdots. \]

Hence we can suppose without loss of generality

\[ \sum_{n=0}^{\infty} n^p a_n = O(C, p). \]

Now

\[ \sum_{\nu=0}^{\infty} \nu^p a_{\nu} = \sum_{\nu=0}^{\infty} \nu^p a_{\nu}, \]

where

\[ \sum_{\nu=0}^{\infty} \nu^p a_{\nu} = \sum_{\nu=0}^{\infty} \nu^p a_{\nu} \cdot \nu^p = O(C, p). \]

Thus we can now follow the line of proof of the first part to reach the result.

We will now give some applications to the theory of Fourier series.

Let \( f(x) \) be a function absolutely integrable in \((0, 2\pi)\) and of period \( 2\pi \) and let its Fourier series be

\[ f(x) \sim \frac{1}{2\pi} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]

Then we have

Theorem. If

\[ \int_{0}^{t} |f'(u)| du = o(t), \]

then

\[ f(x) = \frac{1}{2} \left( f(x+t) + f(x-t) - 2f(x) \right) \]

(5)

then the series \( \sum_{n=0}^{\infty} a_n(x)/n^p \) \((0 < s < 1)\) is summable \((C, s)\).