ON A THEOREM OF W. GUSTIN

By Yûsaku KOMATU and Han NISHIMIYA

1. W. Gustin[1] has recently shown that any pair of functions harmonic in respective domains of a euclidean space satisfies a certain bilinear integral identity, and then applied it obtaining systematically new proofs of a few fundamental theorems in classical harmonic function theory. His principal theorem may be stated as follows:

"Let ϕ_1 be harmonic in a domain D_1 containing a point (expressed by a vector) q_1 and ϕ_2 be harmonic in D_2 containing q_2. Then the bilinear integral expression

$$\int_{\Omega} \phi_1(q_1+\rho_1 x)\,\phi_2(q_2+\rho_2 x)\,d\omega_x$$

depends only on the product $\rho_1\rho_2$, provided the closed sphere with radius ρ_1 about q_1 and the closed sphere with radius ρ_2 about q_2 are contained in D_1 and D_2, respectively. Here the integral is taken such that the unit vector x extends over the periphery Ω of the unit sphere with surface element $d\omega_x$, the dimension of the space being arbitrary."

Gustin has given two proofs of the theorem; the first being based on Poisson integral formula and the second on Green's bilinear integral identity. In this Note we shall give a brief proof which will furthermore clarify the essential nature of the theorem.

2. Now, we may suppose, without loss of generality, that q_1 and q_2 both coincide with the origin, since the harmonicity remains invariant by any translation. As well known[2], any function $\phi(\rho x)$ harmonic in a closed sphere $0 \leq \rho \leq \rho^*$ can be expanded in a uniformly convergent series of the form

$$\phi(\rho x) = \sum_{n=0}^{\infty} \rho^n Y_n(x) \qquad (0 \leq \rho \leq \rho^*),$$

$Y_n(x)$ for each n, denoting a spherical surface harmonic of order n. (As to spherical surface harmonics, cf. Remark 2 at the end of the present Note.) Hence we may put

$$\phi_1(\rho_1 x) = \sum_{n=0}^{\infty} \rho_1^n Y_n^{(1)}(x)$$

and

$$\phi_2(\rho_2 x) = \sum_{n=0}^{\infty} \rho_2^n Y_n^{(2)}(x),$$

where $Y_n^{(1)}(x)$ and $Y_n^{(2)}(x)$ are spherical surface harmonics of order n. Remembering the orthogonality character of spherical surface harmonics

$$\int_{\Omega} Y_m^{(1)}(x) Y_n^{(2)}(x)\,d\omega_x = 0 \qquad (m \neq n),$$

we deduce immediately the relation

$$\int_{\Omega} \phi_1(\rho_1 x)\phi_2(\rho_2 x)\,d\omega_x = \sum_{n=0}^{\infty} (\rho_1\rho_2)^n \int_{\Omega} Y_n^{(1)}(x) Y_n^{(2)}(x)\,d\omega_x,$$

yielding the desired result.

3. Remark 1. In Gustin's paper the dimension of basic space is assumed to be not less than two. But, if the space is one-dimensional, the bilinear integral expression may be considered to degenerate into the sum

$$\phi_1(q_1+\rho_1)\,\phi_2(q_2+\rho_2) + \phi_1(q_1-\rho_1)\,\phi_2(q_2-\rho_2).$$

On the other hand, the only harmonic functions in one-dimensional space are linear functions, i.e., of the form

$$\phi(\rho x) = a\rho x + b, \qquad \rho \geq 0,\ x = \pm 1,$$

a and b being constants. It is quite easy to see that the above expression depends on the aggregate $\rho_1\rho_2$ alone for any pair of such linear ϕ_1 and ϕ_2.

Remark 2. In an N-dimensional euclidean space, the rectangular cartesian and polar coordinates, $(\xi_1, \ldots, \xi_N)$ and $(\rho, \vartheta_1, \ldots, \vartheta_{N-1})$, are connected in the following manner[3]:

$$\xi_j = \rho\left(\prod_{k=1}^{j-1} \sin\vartheta_k\right)\cos\vartheta_j \qquad (1 \leq j \leq N-1),$$

$$\xi_N = \rho \prod_{k=1}^{N-1} \sin\vartheta_k;$$

$$\rho \geq 0, \quad 0 \leq \vartheta_j \leq \pi \quad (1 \leq j \leq N-2), \quad 0 \leq \vartheta_{N-1} \leq 2\pi;$$

the empty product being understood, in the usual way, to denote unity. The square of line element is given by

$$ds^2 = \sum_{j=1}^{N} d\xi_j^2 = d\rho^2 + \rho^2 \sum_{j=1}^{N-1}\left(\prod_{k=1}^{j-1} \sin^2\vartheta_k\right) d\vartheta_j^2.$$

On the other hand, by introducing general orthogonal curvilinear coordinates ($\sigma_1, \cdots, \sigma_N$) with $ds^2 = \sum_{j=1}^{N} g_j d\sigma_j^2$, the Laplacian operator

$$\Delta \equiv \sum_{j=1}^{N} \frac{\partial^2}{\partial \xi_j^2}$$

is transformed into [4]

$$\Delta = \frac{1}{\sqrt{g}} \sum_{j=1}^{N} \frac{\partial}{\partial \sigma_j} \left(\frac{\sqrt{g}}{g_j} \frac{\partial}{\partial \sigma_j} \right), \quad g \equiv \prod_{j=1}^{N} g_j ,$$

which reduces, in our case of polar coordinates, to

$$\Delta = \frac{1}{\rho^{N-1}} \frac{\partial}{\partial \rho} \left(\rho^{N-1} \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \Delta^* \equiv \frac{\partial^2}{\partial \rho^2} + \frac{N-1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta^*$$

with

$$\Delta^* = \sum_{j=1}^{N-1} \left(\prod_{k=1}^{j-1} \operatorname{cosec}^2 \vartheta_k \right) \cdot \left(\frac{\partial^2}{\partial \vartheta_j^2} + (N-j+1) \cot \vartheta_j \frac{\partial}{\partial \vartheta_j} \right).$$

Hence, for any solid harmonics of the form $\rho^n Y_n(\vartheta_1, \cdots, \vartheta_{N-1})$, we have

$$0 = \Delta(\rho^n Y_n) = \rho^{n-2} (n(n+N-2) Y_n + \Delta^* Y_n),$$

i.e.,

$$\Delta^* Y_n + n(n+N-2) Y_n = 0 .$$

The last relation is the self-adjoint partial differential equation for spherical surface harmonics Y_n of order n, which belong to the eigen-value $n(n+N-2)$ [5]. In case of N variables, a general homogeneous function (polynomial) of order n (with respect to cartesian coordinates) possesses $\binom{n+N-1}{N-1}$ coefficients. Hence, the maximal number of linearly independent Y_n is, in general, equal to

$$\binom{n+N-1}{N-1} - \binom{n+N-3}{N-1} = 2\binom{n+N-3}{N-2} + \binom{n+N-3}{N-3}.$$

(*) Received September 1, 1950.

(1) William Gustin, A bilinear integral identity for harmonic functions. Amer. Journ. of Math. 70(1948), 212-220.

(2) Cf., e.g., R.Courant u. D.Hilbert, Methoden der mathematischen Physik, I. Berlin (1931), p.443, where the completeness of the system is shown for three-dimensional case.

(3) A.Dinghas, Geometrische Anwendungen der Kugelfunktionen. Göttinger Nachr. Neue Folge 1, No.18 (1940), 213-235.

(4) See, for instance, H.Cartan, Leçons sur le géométrie des espaces de Riemann. Paris (1928), pp.48-49; for particular case of $N=3$ see also Courant-Hilbert, loc. cit., pp.194-195.

(5) Courant-Hilbert, loc. cit., pp. 270 and 441.

Tokyo Institute of Technology.