ON ROBIN'S CONSTANT AND A DISTORTION THEOREM.

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1. Let $\mathcal{B}$ be any domain whose Green function be denoted by $G(z,\zeta)$. The Robin constant $\gamma(\zeta)$ of $\mathcal{B}$ with respect to the pole ζ is defined by the relation

$$(1.1)\quad G(z,\zeta)+\lg|z-\zeta| = \gamma(\zeta) + \tau(z,\zeta),$$

the residual term satisfying the limit equation

$$\tau(z,\zeta) = O(|z-\zeta|) \quad \text{as} \quad z\to\zeta.$$

If the pole coincides with the point at infinity, then $|z-\zeta|$ has only to be replaced by $|z|^{-1}$. Although, in the following lines, we shall suppose, for the sake of brevity, that ζ is a finite point, the argument will easily be modified for the case $\zeta=\infty$.

Now, making use of Hadamard's variational method, Bergman[1] has recently shown that the residual term in (1.1) satisfies the inequality

$$(1.2)\quad \tau(z,\zeta)+\tau(\zeta,z)\geqq 0,$$

which, by the symmetry property of Green function, can also be expressed in the form

$$(1.3)\quad 2\left(G(z,\zeta)+\lg|z-\zeta|\right)\geqq \gamma(z)+\gamma(\zeta)$$

Moreover, he has also noticed that, in the special case of a simply-connected domain $\mathcal{B}$, this result contains, as an immediate consequence, a classical distortion theorem due to Löwner stating that

$$(1.4)\quad |f'(w)|\leqq \frac{|w|^2}{|w|^2-1}$$

is valid for any function $f(w)$ schlicht in $|w|>1$ and normalized at the point at infinity such as

$$(1.5)\quad f(w)=w+\sum_{\nu=0}^{\infty} c_\nu w^{-\nu} \qquad (|w|>1).$$

2. Now, if we restrict ourselves to simply-connected domains, the inequality (1.3) of question can conversely be deduced from Löwner's distortion formula (1.4) also by an elementary procedure, and moreover the extremal domains for the estimation (1.3) can be explicitly be determined. In the present Note, these facts will be established.

Let $z=\varphi(w)$ be a function mapping $|w|>1$ onto $\mathcal{B}$. Let ω be any point with $1<|\omega|<\infty$, and suppose temporarily $|\varphi(\bar\omega)|\neq 1$. Then

$$(2.1)\quad \Phi(w)=\varphi'(\bar\omega)\frac{|\omega|^2-1}{|\varphi(\bar\omega)|^2-1}\,\frac{\overline{\varphi(\bar\omega)}\,\varphi\left(\frac{\bar\omega w-1}{w-\omega}\right)-1}{\varphi\left(\frac{\bar\omega w-1}{w-\omega}\right)-\varphi(\bar\omega)}$$

is a function schlicht in $|w|>1$, normalized at $w=\infty$ such that $\Phi(\infty)=\infty$ and $\Phi'(\infty)=1$, and whose derivative is given by the expression

$$\Phi'(w)=\varphi'(\bar\omega)\frac{(|\omega|^2-1)^2}{(w-\omega)^2}\,\frac{\varphi'\left(\frac{\bar\omega w-1}{w-\omega}\right)}{\left(\varphi\left(\frac{\bar\omega w-1}{w-\omega}\right)-\varphi(\bar\omega)\right)^2}\qquad(|w|>1).$$

If we now put

$$\frac{\bar\omega w-1}{w-\omega}=W,\qquad \bar\omega=\Omega,$$

then $|W|>1$, $|\Omega|>1$ and the above expression becomes

$$(2.2)\quad \Phi'\left(\frac{\bar\Omega W-1}{W-\Omega}\right)=\varphi'(\Omega)\,\varphi'(W)\left(\frac{W-\Omega}{\varphi(W)-\varphi(\Omega)}\right)^2.$$

Applying here the Löwner's distortion theorem, we get

$$(2.3)\quad \left|\Phi'\left(\frac{\bar\Omega W-1}{W-\Omega}\right)\right|\leqq\frac{1}{1-\left|\frac{W-\Omega}{\bar\Omega W-1}\right|^2}=\frac{|\bar\Omega W-1|^2}{(|\Omega|^2-1)(|W|^2-1)}.$$

Hence, from (2.2), we obtain

$$(2.4)\quad |\varphi'(\Omega)\varphi'(W)|\leqq\frac{|\bar\Omega W-1|^2}{(|\Omega|^2-1)(|W|^2-1)}\left|\frac{\varphi(W)-\varphi(\Omega)}{W-\Omega}\right|^2.$$

Remembering the continuity character, we see that the last inequality holds good for any Ω and W with moduli larger than unity without the restriction $|\varphi'(\Omega)|\neq 1$. We call here, by the way, attention to the fact that if $\mathcal{B}$ is a domain containing the point at infinity and possessing the reduced modulus equal to unity and if the mapping function is normalized such as $\varphi(\infty)=\infty$ and $\varphi'(\infty)=1$, then the Löwner's distortion formula $|\varphi'(W)|\leqq |W|^2/(|W|^2-1)$ is reproduced from (2.4) as a limiting case $\Omega\to\infty$.

Denoting now by $w=\psi(z)$ the inverse function of $z=\varphi(w)$, the Green function of $\mathcal{B}$ is given by

$$(2.5)\quad G(z,\zeta)=\lg\left|\frac{\overline{\psi(\zeta)}\,\psi(z)-1}{\psi(z)-\psi(\zeta)}\right|.$$

Hence, the Robin constant is, by definition (1.1), expressed in the form

$$(2.6)\quad \gamma(\zeta)=\lim_{z\to\zeta}\left(G(z,\zeta)+\lg|z-\zeta|\right)=\lg\frac{|\psi(\zeta)|^2-1}{|\psi'(\zeta)|}.$$

On the other hand, we get from (2.4) an equivalent inequality

$$(2.7)\quad |\psi'(\zeta)\psi'(z)| \geq \frac{(|\psi(\zeta)|^2-1)(|\psi(z)|^2-1)}{|\overline{\psi(\zeta)}\psi(z)-1|^2}\left|\frac{\psi(z)-\psi(\zeta)}{z-\zeta}\right|^2.$$

Combining these relations, we conclude the inequality

$$\gamma(z)+\gamma(\zeta) = \lg\frac{(|\psi(z)|^2-1)(|\psi(\zeta)|^2-1)}{|\psi'(z)||\psi'(\zeta)|}$$

$$\leq \lg\left(|z-\zeta|^2\left|\frac{\overline{\psi(\zeta)}\psi(z)-1}{\psi(z)-\psi(\zeta)}\right|^2\right) = 2\left(G(z,\zeta)+\lg|z-\zeta|\right),$$

yielding the desired result (1.3).

3. We shall next investigate the extremal domains for the estimation (1.3). This has been derived from the inequality (2.7), which is quite trivial for $z=\zeta$. Suppose now that $z\neq\zeta$.

The extremal functions for Löwner's distortion theorem (1.5) are those which map $|w|>1$ onto circular slit domains about the origin. More precisely, the only functions for which the equality sign in (1.5) holds at a given point w_0 $(1<|w_0|<\infty)$ are rational functions of the form

$$(3.1)\quad f_0(w) = f_0(w_0) + \frac{\overline{w}_0 w(w-w_0)}{\overline{w}_0 w-1},$$

its image domain being the whole plane cut along a circular arc about the point $f_0(w_0)$ with radius $|w_0|$. Hence, the extremal functions for (2.3) are given by

$$(3.2)\quad \Phi_0(w) = \Phi_0(w_0) + \frac{\overline{w}_0 w(w-w_0)}{\overline{w}_0 w-1},$$

for which we have, in fact,

$$(3.3)\quad \Phi_0'(w_0) = \frac{|w_0|^2}{|w_0|^2-1}.$$

Therefore, the relation (2.1) shows that the extremal functions for (2.4) must be of the form

$$(3.4)\quad \varphi_0(W) = \frac{1-\alpha\beta\Phi_0(w)}{\beta-\alpha\Phi_0(w)},\qquad w = \frac{\overline{\Omega}W-1}{W-\Omega},$$

α and β being constants given by

$$(3.5)\quad \alpha = \frac{1}{\varphi_0'(\overline{\omega})}\frac{|\varphi_0(\overline{\omega})|^2-1}{|\omega|^2-1},\qquad \beta = \varphi_0(\overline{\omega})$$

$$(\overline{\omega}=\Omega),$$

and the point corresponding to $w=w_0$ is just

$$(3.6)\quad W = W_0 \equiv \frac{w_0\Omega-1}{w_0-\overline{\Omega}}.$$

The function $\varphi_0(W)$ in (3.4) maps $|W|>1$ also onto a circular slit domain.

We shall now show that, for any given w_0 $(1<|w_0|<\infty)$ and Ω $(|\Omega|>1)$, the function $\varphi_0(W)$ of (3.4), with arbitrary constants α and β, is the one for which the equality sign in (2.4) is really attained at the point (3.6).

Since the point (3.6) corresponds to $w=w_0$ and the relation (3.3) holds at this point, we get, from (3.4),

$$\varphi_0(W_0) = \frac{\alpha(1-|\beta|^2)}{(\beta-\alpha\Phi_0(w_0))^2}\Phi_0'(w_0)\left[\frac{dw}{dW}\right]^{w=w_0}$$

$$= \frac{\alpha(1-|\beta|^2)}{(\beta-\alpha\Phi_0(w_0))^2}\frac{|w_0|^2}{|w_0|^2-1}\frac{(w_0-\overline{\Omega})^2}{1-|\Omega|^2}.$$

At the point $W=\Omega$ corresponding to $w=\infty$, we obtain

$$\varphi_0'(\Omega) = \frac{1-|\beta|^2}{\alpha}\frac{1}{1-|\Omega|^2}$$

On the other hand, we have successively

$$\overline{\Omega}W_0-1 = \frac{w_0(|\Omega|^2-1)}{w_0-\overline{\Omega}}$$

$$|W_0|^2-1 = \frac{(|w_0|^2-1)(|\Omega|^2-1)}{|w_0-\overline{\Omega}|^2},$$

$$\varphi_0(W_0)-\varphi_0(\Omega) = \frac{1-\alpha\beta\Phi_0(w_0)}{\beta-\alpha\Phi_0(w_0)}-\beta = \frac{1-|\beta|^2}{\beta-\alpha\Phi_0(w_0)},$$

$$W_0-\Omega = \frac{|\Omega|^2-1}{w_0-\overline{\Omega}}.$$

Substituting these values in each side of (2.4), we see that the equality sign is realized for $\varphi_0(W)$ at $W=W_0$.

The inequality (1.3) is equivalent to the relation (2.7) which is itself equivalent to the distortion inequality (2.4). Hence, we obtain the following result:

In our estimation (1.3), the equality sign holds for $z\neq\zeta$ —— the excluded case $z=\zeta$ is trivial —— if and only if $\mathcal{B}$ is a circular slit domain which is obtained as the image of $|W|>1$ by a function of the form

$$z = \varphi_0(W) \equiv \frac{1-\alpha\beta\Phi_0(w)}{\beta-\alpha\Phi_0(w)};$$

$$w \equiv \frac{\overline{\Omega}W-1}{W-\Omega}\qquad(|\Omega|>1),$$

$$\Phi_0(w) \equiv \Phi_0(w_0) + \frac{\overline{w}_0 w(w-w_0)}{\overline{w}_0 w-1}.$$

Moreover, for any given point $\zeta\in\mathcal{B}$ we have $\zeta=\varphi_0(\Omega)=\beta$ and hence the equality sign is realized only for the point $z_0\in\mathcal{B}$ given by

$$z_0 = \varphi_0(W_0) = \frac{1-\alpha\beta\Phi_0(w_0)}{\beta-\alpha\Phi_0(w_0)}$$

$$= \frac{1-\alpha\zeta\Phi_0(w_0)}{\zeta-\alpha\Phi_0(w_0)}.$$

4. The inequality (1.3) of the question is also written in the form

$$(4.1)\quad 2\left(G(z,\zeta)+\lg|z-\zeta|\right)-\left(\gamma(z)+\gamma(\zeta)\right) \geq 0.$$

We notice here that the expression standing in the left hand side of this

inequality remains invariant under any linear transformations

$$(4.2)\qquad z=\frac{az^*+b}{cz^*+d},\qquad \zeta=\frac{a\zeta^*+b}{c\zeta^*+d}$$

$$(ad-bc\neq 0).$$

In fact, denoting the quantities which refer to z^* **-plane by asterisking the corresponding quantities which refer to** z **-plane, we have successively**

$$\psi^*(z^*)=\psi(z)\equiv\psi\left(\frac{az^*+b}{cz^*+d}\right),$$

$$\psi^{*\prime}(z^*)=\psi'(z)\frac{dz}{dz^*}=\psi'(z)\frac{ad-bc}{(cz^*+d)^2},$$

$$G^*(z^*,\zeta^*)=\lg\left|\frac{\overline{\psi^*(\zeta^*)}\,\psi^*(z^*)-1}{\psi^*(z^*)-\psi^*(\zeta^*)}\right|$$

$$=\lg\left|\frac{\overline{\psi(\zeta)}\,\psi(z)-1}{\psi(z)-\psi(\zeta)}\right|=G(z,\zeta),$$

$$\gamma^*(\zeta^*)=\lg\frac{|\psi^*(\zeta^*)|^2-1}{|\psi^{*\prime}(\zeta^*)|}$$

$$=\lg\frac{|\psi(\zeta)|^2-1}{|\psi'(\zeta)|}-\lg\left|\frac{ad-bc}{(c\zeta^*+d)^2}\right|$$

$$=\gamma(\zeta)-\lg\left|\frac{ad-bc}{(c\zeta^*+d)^2}\right|,$$

$$z-\zeta=\frac{ad-bc}{(cz^*+d)(c\zeta^*+d)}(z^*-\zeta^*).$$

These relations yield indeed the invariance of the referring expression that is

$$2\left(G^*(z^*,\zeta^*)+\lg|z^*-\zeta^*|\right)-\left(\gamma^*(z^*)+\gamma^*(\zeta^*)\right)$$

$$=2\left(G(z,\zeta)+\lg|z-\zeta|\right)-\left(\gamma(z)+\gamma(\zeta)\right).$$

(*) Received May 29, 1950.

(1) S.Bergman: Complex orthogonal functions and conformal mapping (mimeographed note). (1949), pp. 117-120; revised edition (1950), VIII. pp. 9-12.

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