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ON A CHARACTERISTIC PROPERTY OF THE EXPONENTIAL FUNCTION

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1. Introduction. We say that a complex number w is a linearly distributed value of the entire function $f(z)$ if there is a straight line L of the complex plane on which all the solutions of $f(z)=w$ lie. Baker [1] has shown the following characterization of the exponential function.

If $f(z)$ is a transcendental entire function for which every value is linearly distributed, then $f(z)$ is a function of the form

$$c+d\exp(az),$$

where a, c, d are constants.

The purpose of this note is to show some other characteristic properties of the exponential function. We shall prove the following theorems.

THEOREM 1. *Let $f(z)$ be a transcendental entire function. Assume that there are three distinct finite complex numbers a_j and three distinct straight lines L_j of the complex plane on which all the solutions of $f(z)=a_j$ lie ($j=1, 2, 3$). Assume further that $f(z)$ has a finite deficient value other than a_1, a_2 and a_3. Then $f(z)=P(\exp Az)$ with a quadratic polynomial $P(z)$ and a non-zero constant A.*

THEOREM 2. *Let $f(z)$ be a transcendental entire function. Assume that there are four distinct finite complex numbers b_j and four distinct straight lines L'_j of the complex plane on which all the solutions of $f(z)=b_j$ lie ($j=1, 2, 3, 4$). Then*

$$f(z)=P(\exp Az),$$

where $P(z)$ is a quadratic polynomial and A is a non-zero constant.

Without much difficulty, we can restate this theorem in the following form.

THEOREM 3. *Let $f(z)$ be a transcendental entire function which has four distinct finite linearly distributed values c_1, c_2, c_3 and c_4. If no three of the four values lie on any straight line of the complex plane, then $f(z)=P(\exp Az)$ with a quadratic polynomial $P(z)$ and a non-zero constant A.*

Combining a result of Edrei [2] and the above Theorem 2, we easily obtain the following fact.

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COROLLARY. *Let $f(z)$ be a transcendental entire function. Assume that there exists an unbounded sequence $\{w_n\}$ such that each w_n is a linearly distributed value of $f(z)$. Then*

$$f(z)=P(\exp Az),$$

where $P(z)$ is a quadratic polynomial and A is a non-zero constant.

2. Statement of known results. We shall make use of the following theorems.

THEOREM A. *Let $f(z)$ be a transcendental entire function having only real zeros. Then $f(z)$ has at most one finite deficient value. Further if $f(z)$ has a finite deficient value other than zero, then the order of $f(z)$ is not greater than one.*

This interesting result was proved by Edrei and Fuchs [3].

THEOREM B. *Let $f(z)$ be an entire function of finite genus q ($\geqq 1$). If its zeros $\{a_n\}$ satisfy*

$$\lim_{n\to\infty} \arg a_n = 0 \qquad (|\arg a_n| \leqq \pi),$$

then $f(z)$ has zero as a deficient value.

This Theorem B already appears in a weaker form in [4], and the arguments developed there are sufficient to yield this result. Further from this Theorem B, we easily have the following

THEOREM C. *Let $f(z)$ be an entire function whose genus is finite and at least two. If zero is a linearly distributed value of $f(z)$, then $f(z)$ has zero as a deficient value.*

Next fact is an improvement of Lucas' theorem [6].

THEOREM D. *Let $f(z)$ be a non-constant entire function satisfying*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

Then the smallest convex set which contains the zeros of $f(z)$ also contains the zeros of $f'(z)$.

THEOREM E. *If $f(z)$ is regular in the unit disc and fails to take there* 0 *and* 1, *then*

$$\log |f(z)| \leqq \frac{A}{1-|z|} \qquad (|z|<1),$$

where A is a positive constant depending upon $|f(0)|$ only.

This is a well known result of Bohr and Landau.

3. Consequences of Bohr and Landau's theorem. Our starting point will be the following fact.

THEOREM 4. *Let $f(z)$ be an entire function of finite lower order. If all the zero-points and the one-points of $f(z)$ lie in the strip*

$$S=\{z \mid |\mathrm{Re}\, z|\leqq h\},$$

then $f(z)$ has at most order one, mean type.

It should be remarked that the restriction "finite lower order" cannot be omitted. In order to prove this theorem, we need some preliminary facts. The first is a well known lemma of Borel.

LEMMA 1. *Let $V(r)$ be a continuous, increasing and unbounded function defined for $r\geqq r^*$, and let η (>0) be given. Then*

$$V\Big(r+\frac{r}{\log V(r)}\Big)\leqq (V(r))^{1+\eta}$$

for all $r\geqq r^*$, outside a set E whose logarithmic measure is finite.

LEMMA 2. *Let $f(z)$ be an entire function of infinite order whose zeros $\{a_n\}$ satisfy*

$$\lim_{n\to\infty} \arg a_n=0 \qquad (|\arg a_n|\leqq\pi).$$

Then the lower order of $f(z)$ is also infinite.

This lemma was proved in [5]. Using this fact, we have the following

LEMMA 3. *Let the assumptions of Theorem 4 be satisfied. Then the order of $f(z)$ is finite.*

Proof of Theorem 4. The linear transformation

$$z=\frac{1-w}{1+w}+h$$

maps the unit disc onto the half plane $\mathrm{Re}\, z>h$. Hence by the assumptions, the function defined by

$$f\Big(\frac{1-w}{1+w}+h\Big)$$

is regular and different from 0 and 1 in the unit disc. Therefore it follows from Bohr and Landau's theorem that

$$\log\Big|f\Big(\frac{1-w}{1+w}+h\Big)\Big|\leqq\frac{A}{1-|w|} \qquad (|w|<1),$$

so that

$$\log|f(z)| \leqq \frac{A(1+h+|z|)^2}{2(x-h)} \qquad (\mathrm{Re}\, z=x>h) \tag{3.1}$$

with a positive constant A. Similarly, it follows that

$$\log|f(z)| \leqq \frac{A(1+h+|z|)^2}{2(-x-h)} \qquad (\mathrm{Re}\, z=x<-h). \tag{3.2}$$

On the other hand, by Lemma 3, it is possible to find a positive number η such that

$$\lim_{r\to\infty} \frac{(T(r))^{\eta} \log T(r)}{r} = 0, \tag{3.3}$$

where $T(r)=T(r, f)$. Using the well known relation

$$\log|f(z)| \leqq \log M(r, f) \leqq \frac{R+r}{R-r} T(R, f)$$

for $0<|z|=r<R$, and the above Lemma 1, we have

$$\log^+|f(z)| \leqq (1+2\log T(r))(T(r))^{1+\eta} \qquad (|z|=r) \tag{3.4}$$

for all $r \geqq R^*$, outside a set E whose logarithmic measure is finite. Now let us set for each $r \geqq h$,

$$L^*(r)=\frac{1}{2\pi}\int_{I(r)} \log^+|f(re^{it})|\,dt,$$

where

$$I(r)=\left\{t\,\middle|\,\left|t-\frac{\pi}{2}\right| \leqq s_r\right\} \cup \left\{t\,\middle|\,\left|t-\frac{3}{2}\pi\right| \leqq s_r\right\},$$

$$s_r=\sin^{-1}\frac{h}{r} \qquad \left(0<s_r<\frac{\pi}{2}\right).$$

Then for $r \geqq h$ not in the exceptional set E, (3.4) yields

$$L^*(r) \leqq \frac{2}{\pi} s_r (1+2\log T(r))(T(r))^{1+\eta}.$$

Since rs_r is bounded for $r \geqq h$, (3.3) implies

$$L^*(r)=o(T(r))$$

outside the set E. Thus for sufficiently large r not in E, we can choose v_r so that

$$0<v_r<s_r^*=\frac{\pi}{2}-s_r \tag{3.5}$$

and

$$T(r)=\frac{1}{\pi}\int_{-v_r}^{v_r} (\log^+|f(re^{it})|+\log^+|f(-re^{it})|)\,dt. \tag{3.6}$$

Hence by means of (3.1) and (3.2), we find

$$\begin{aligned} T(r) &\leqq \frac{A}{\pi}\int_{-v_r}^{v_r} \frac{(1+h+r)^2}{r\cos t-h}dt \\ &= \frac{2A}{\pi r}\int_0^{v_r} \frac{(1+h+r)^2}{\cos t-\sin s_r}dt. \end{aligned} \tag{3.7}$$

From (3.5), an elementary calculation gives

$$\begin{aligned} \cos t-\sin s_r &= \cos t-\cos s_r^* \\ &\geqq \frac{2}{\pi}(s_r^*-t)\sin\frac{s_r^*}{2}>0 \end{aligned} \tag{3.8}$$

for $0\leqq t\leqq v_r$. Therefore it follows from (3.7) and (3.8) that

$$\frac{T(r)}{r}\sin\frac{s_r^*}{2}\leqq\frac{A(1+h+r)^2}{r^2}\log\left(\frac{s_r^*}{s_r^*-v_r}\right).$$

Then for sufficiently large r not in E, we find

$$\frac{T(r)}{r}\leqq B\log\left(\frac{s_r^*}{s_r^*-v_r}\right) \tag{3.9}$$

with a positive constant B. Further from (3.3), (3.4), (3.5) and (3.6),

$$\begin{aligned} \frac{\pi}{4}T(r) &\leqq \left(\frac{\pi}{2}-v_r\right)(1+2\log T(r))(T(r))^{1+\eta} \\ &= (s_r^*-v_r)(1+2\log T(r))(T(r))^{1+\eta}+o(T(r)) \end{aligned} \tag{3.10}$$

for values of r not in E. Combining (3.9) and (3.10), we thus have

$$1+o(1)\leqq(2+4\log T(r))(T(r))^{\eta}\exp\left(-\frac{T(r)}{Br}\right) \tag{3.11}$$

for sufficiently large r not in E. Hence by this inequality (3.11), we easily find

$$\limsup_{\substack{r\to\infty \\ r\notin E}}\frac{\log T(r)}{\log r}\leqq 1.$$

Since the logarithmic measure of the set E is finite, we conclude that

$$\limsup_{r\to\infty}\frac{\log T(r)}{\log r}\leqq 1,$$

so that the order of $f(z)$ is at most one.

It remains to prove that

$$\limsup_{r\to\infty}\frac{T(r)}{r}<+\infty. \tag{3.12}$$

Since the genus of $f(z)$ is at most one, we easily obtain

$$f(z)f(-z)=C\,z^{2m}\prod_n\left(1-\frac{z^2}{a_n^2}\right) \qquad (C\neq 0),$$

where $\{a_n\}$ are the zeros of $f(z)$. Evidently, for all $r\geqq 0$,

$$\left|1-\frac{r^2}{a_n^2}\right|^2=1+\frac{r^4}{|a_n|^4}-2\frac{r^2}{|a_n|^2}\cos 2\alpha_n,$$

where $\alpha_n=\arg a_n$. By the assumptions, $|\operatorname{Re} a_n|\leqq h$. Hence if $|a_n|\geqq 2h$, then $\cos 2\alpha_n\leqq 0$. Hereby

$$\left|1-\frac{r^2}{a_n^2}\right|\geqq 1, \qquad \left|1-\frac{r^2}{a_n^2}\right|\geqq\frac{r^2}{|a_n|^2}$$

for all $|a_n|\geqq 2h$ and all $r\geqq 0$. Therefore we obtain

$$\begin{aligned}\log|f(r)f(-r)| &\geqq O(\log r)+\sum_{2h\leqq|a_n|}\log\left|1-\frac{r^2}{a_n^2}\right| \\ &\geqq O(\log r)+\sum_{|a_n|\leqq r}2\log\frac{r}{|a_n|} \\ &=O(\log r)+2N(r,0,f).\end{aligned}$$

Thus it follows from (3.1) and (3.2) that

$$A\frac{(1+h+r)^2}{r-h}\geqq O(\log r)+2N(r,0,f),$$

so that

$$\limsup_{r\to\infty}\frac{N(r,0,f)}{r}<+\infty. \tag{3.13}$$

Similarly, applying the above argument to $1-f(z)$, we also find

$$\limsup_{r\to\infty}\frac{N(r,1,f)}{r}<+\infty. \tag{3.14}$$

Then by (3.13) and (3.14), the second main theorem yields the desired result (3.12). The proof of Theorem 4 is now complete.

Minor modifications in the above proof lead to the following facts which we state without proof.

THEOREM 5. *Let $f(z)$ be a transcendental entire function having only real zeros. Assume that there exist two distinct finite values a, b ($a\neq 0$, $b\neq 0$) such that all the solutions of $f(z)=a$ and $f(z)=b$ are contained in* $\operatorname{Im} z\geqq 0$ *and* $\operatorname{Im} z\leqq 0$, *respectively. Then the order of $f(z)$ is at most one.*

THEOREM 6. *Let $f(z)$ be a transcendental entire function which has three distinct finite linearly distributed values. Then the order of $f(z)$ is finite.*

4. A property of entire functions of genus at most one. First of all, we remark the following

LEMMA 4. *Let $g(z)$ be an entire function of finite genus. Then the genus of the entire function defined by*

$$G(z)=g(az+b)\,,$$

where a ($\neq 0$), b are constants, coincides with that of $g(z)$.

This lemma is an immediate consequence of the well known result [7: p. 230].

Now, let $f(z)$ be an entire function of genus less than two. Assume that all the zeros $\{a_n\}$ of $f(z)$ lie on the imaginary axis. Then

$$f(z)=z^m \exp(Az+B)\prod_{a_n\neq 0} E\left(\frac{z}{a_n}, q\right), \tag{4.1}$$

where q is either zero or one, and $E(z, q)$ is the Weierstrass primary factor of genus q. Since $\bar{a}_n=-a_n$, we find

$$\begin{gathered}\overline{f(\bar{z})}=(-1)^m f(-z)\exp(2Cz-2iD)\,,\\ \operatorname{Re} A=C\,,\qquad \operatorname{Im} B=D\,.\end{gathered} \tag{4.2}$$

Further from the expression (4.1), we also obtain

$$\operatorname{Re}\frac{f'(z)}{f(z)}=\operatorname{Re} A+\sum_n \frac{\operatorname{Re} z}{|z-a_n|^2}\,. \tag{4.3}$$

To generalize these facts, we introduce a notation. Every straight line of the complex plane can be expressed as

$$\{z\,|\operatorname{Re}(e^{is}z)=r\} \tag{4.4}$$

with suitable real numbers s and r. In what follows, we denote the line (4.4) by $L(s, r)$.

LEMMA 5. *Let $f(z)$ be an entire function of genus at most one. If all the zero-points $\{a_n^*\}$ of $f(z)$ lie on the line $L(s, r)$, then*

$$\overline{f(e^{-is}\bar{z}+e^{-is}r)}=f(-e^{-is}z+e^{-is}r)\exp(2Cz+iC') \tag{4.5}$$

and

$$\operatorname{Re}\frac{e^{-is}f'(z)}{f(z)}=C+\sum_n \frac{\operatorname{Re}(e^{is}z)-r}{|z-a_n^*|^2} \tag{4.6}$$

with suitable real constants C and C'.

Proof. Consider the function defined by

$$F(z)=f(e^{-is}z+e^{-is}r).$$

Then by Lemma 4, the genus of $F(z)$ is at most one. Further all the zeros of $F(z)$ are $\{e^{is}a_n^*-r\}$. So by the assumptions, $\mathrm{Re}\,(e^{is}a_n^*-r)=0$. Therefore by (4.2) and (4.3), we find

$$\overline{F(\bar{z})}=F(-z)\exp(2Cz+iC') \tag{4.7}$$

and

$$\mathrm{Re}\,\frac{F'(z)}{F(z)}=C+\sum_n \frac{\mathrm{Re}\,z}{|z-e^{is}a_n^*+r|^2} \tag{4.8}$$

with suitable real constants C and C'. From (4.7) and (4.8), we at once obtain the desired (4.5) and (4.6).

In the above Lemma 5, we further assume that $f(z)$ has at least one zero-point. If the real constant C which appears in (4.5) and (4.6), is equal to zero, then

$$\mathrm{Re}\,\frac{e^{-is}f'(z)}{f(z)}\neq 0$$

for all $\mathrm{Re}\,(e^{is}z)>r$ and for all $\mathrm{Re}\,(e^{is}z)<r$. In particular, $f'(z)$ fails to take the value zero there. Hence all the zeros of $f'(z)$, if exist, must lie on the line $L(s,r)$.

Next let us assume that $C>0$. Then (4.6) implies

$$\mathrm{Re}\,\frac{e^{-is}f'(z)}{f(z)}\geqq C>0$$

for $\mathrm{Re}\,(e^{is}z)\geqq r$. Thus we find

$$\begin{aligned}\log|f(e^{-is}b)|-\log|f(e^{-is}a)|&=\int_a^b \mathrm{Re}\left(\frac{e^{-is}f'(e^{-is}t)}{f(e^{-is}t)}\right)dt\\ &\geqq C(b-a)\end{aligned}$$

for $r<a\leqq b$. From this inequality, $f(e^{-is}t)$ tends to infinity as t does to infinity along the positive real axis.

Similarly, if $C<0$, then $f(e^{-is}t)$ tends to infinity as t does to infinity along the negative real axis.

LEMMA 6. *Let $f(z)$ be a transcendental entire function. Assume that there are three distinct finite complex numbers a_j and three distinct straight lines L_j of the complex plane on which all the solutions of $f(z)=a_j$ lie ($j=1,2,3$). Assume further that no two of the three lines L_j are parallel to each other. Then at least one of the three values a_j is a radially distributed value of $f(z)$.*

Proof. By the assumptions, $f(z)$ has three linearly distributed finite values. So by virtue of Theorem 6, the order of $f(z)$ must be finite. For a moment, we assume that the genus of $f(z)-a_1$ is greater than one. Clearly it is possible to find a linear transformation $L(z)=cz+d$ so that the function defined by

$$F(z)=f(L(z))-a_2$$

has only real zeros. Of course, the value a_1-a_2 ($\neq 0$) is a linearly distributed value of $F(z)$ and the genus of $F(z)-a_1+a_2$ coincides with that of $f(z)-a_1$. So the order of $F(z)$ must be at least two. Further by Theorem C, $F(z)$ has the value a_1-a_2 as a deficient value. However by means of Theorem A, the order of $F(z)$ is less than or equal to one. This is absurd. Therefore the genera of $f(z)-a_j$ ($j=1, 2, 3$) are at most one.

Let us set

$$L_j=L(s_j, r_j)$$

with real numbers s_j and r_j ($j=1, 2, 3$). Then by Lemma 5, denoting the a_j-points of $f(z)$ by $\{a_{nj}^*\}$, we have

$$\mathrm{Re}\,\frac{\bar{u}_j f'(z)}{f(z)-a_j}=C_j+\sum_n \frac{\mathrm{Re}\,(u_j z)-r_j}{|z-a_{nj}^*|^2},$$

$$u_j=\exp(is_j)$$

with suitable real constants C_j ($j=1, 2, 3$). Firstly, assume that $C_1=C_2=C_3=0$. Then by what mentioned above, all the zeros of $f'(z)$ must lie on the three distinct lines L_j simultaneously. Hence $f'(z)$ has at most one zero-point. Since the genus of $f(z)-a_1$ is at most one, $T(r, f)=o(r^2)$. So $T(r, f')=o(r^2)$. Therefore $f'(z)$ can be expressed as

$$f'(z)=(z-z_*)^n \exp(Az+B),$$

where n is a non-negative integer and A, B are constants. Hereby

$$f(z)=P(z)\exp(Az)+D \tag{4.9}$$

with a suitable polynomial $P(z)$ and a suitable constant D. From this representation (4.9), using asymptotic properties of the exponential function, we thus obtain the desired result.

Secondly, assume that $C_1>0$. Then $f(\bar{u}_1 t)$ tends to infinity when t tends to infinity along the positive real axis. On taking into account that $L(s+\pi, r)=L(s, -r)$, we may assume that

$$s_1<s_2<s_3<s_1+\pi.$$

There occur two cases. Either (I) $s_3\leqq s_1+\pi/2$, or (II) $s_1+\pi/2<s_3<s_1+\pi$.

In the case (I), since $s_2-s_1<s_3-s_1\leqq\pi/2$, $t\bar{u}_1$ must be contained in the angular domain

$$A=\{z\,|\,\mathrm{Re}\,(u_1 z)>r_1 \text{ and } \mathrm{Re}\,(u_2 z)>r_2\}$$

for sufficiently large real positive t. Observe that $f(z)$ fails to take the two values a_1 and a_2 in this domain A. Then using Lindelöf-Iversen-Gross' theorem [8], we find for arbitrarily fixed t_1 and t_2 ($-s_1-\pi/2<t_1<t_2<-s_2+\pi/2$),

$$\lim_{r\to+\infty} |f(re^{it})| = +\infty$$

uniformly for $t_1 \leqq t \leqq t_2$. On the other hand, from $s_3 - s_1 < \pi$, the intersection of the line L_3 and the domain A is a half straight line. Therefore $f(z)$ tends to infinity when z tends to infinity along the unbounded part of L_3 which is contained in A. Consequently, the value a_3 is a radially distributed value of $f(z)$.

Next in the case (II), for sufficiently large real positive t, $t\bar{u}_1$ must lie in the angular domain

$$B = \{z \mid \mathrm{Re}\,(u_1 z) > r_1 \text{ and } \mathrm{Re}\,(u_3 z) < r_3\}.$$

Hence using Lindelöf-Iversen-Gross' theorem again, we have for arbitrarily fixed t_1 and t_2 $(-s_3 + \pi/2 < t_1 < t_2 < -s_1 + \pi/2)$,

$$\lim_{r\to+\infty} |f(re^{it})| = +\infty$$

uniformly for $t_1 \leqq t \leqq t_2$. Further in this case, the intersection of the line L_2 and the domain B is a half straight line. By these facts, the value a_2 must be a radially distributed value of $f(z)$.

All other cases, say $C_2 < 0$, can be treated by the same fashion as above. The proof of Lemma 6 is hereby complete.

5. Entire functions with two linearly distributed values. Let $G(z)$ be a transcendental entire function of finite lower order such that all the zero-points $\{a_n\}$ of $G(z)$ lie on the line $\mathrm{Re}\, z = 0$, and all the one-points $\{b_n\}$ of $G(z)$ lie on the line $\mathrm{Re}\, z = 1$. Then by Theorem 4, $G(z)$ has at most order one, mean type. So the genera of $G(z)$ and $G(z)-1$ are at most one. Hence by means of Lemma 5, we find

$$\mathrm{Re}\,\frac{G'(z)}{G(z)} = A + \sum_n \frac{\mathrm{Re}\, z}{|z-a_n|^2},$$

(5.1)

$$\mathrm{Re}\,\frac{G'(z)}{G(z)-1} = B + \sum_n \frac{\mathrm{Re}\, z - 1}{|z-b_n|^2}$$

and

$$\overline{G(\bar{z})} = G(-z) \exp(2Az + iA'),$$

(5.2)

$$\overline{G(\bar{z}+1)-1} = (G(-z+1)-1) \exp(2Bz + iB')$$

with suitable real constants A, A', B and B'. The quantities A and B satisfy the following

LEMMA 7. *If $AB < 0$, then $A < 0$ and $B > 0$. Further if $AB = 0$, then either $A < 0$ or $B > 0$.*

Proof. Suppose that $A > 0$ and $B < 0$. Then by (5.1),

$$\operatorname{Re}\frac{G'(z)}{G(z)} \geqq A > 0$$

for $\operatorname{Re} z > 0$, and

$$\operatorname{Re}\frac{G'(z)}{G(z)-1} \leqq B < 0$$

for $\operatorname{Re} z < 1$. In particular, $G'(z)$ has no zeros for $\operatorname{Re} z > 0$ and $\operatorname{Re} z < 1$. Therefore $G'(z)$ fails to take the value zero. Since the order of $G'(z)$ is at most one,

$$G'(z) = \exp(Cz + D)$$

with constants C and D. So we find

$$G(z) = E\exp(Cz) + F \tag{5.3}$$

with suitable constants C, E and F. By this representation (5.3), using an elementary calculation, we easily arrive at a contradiction. Accordingly. $A < 0$ and $B > 0$.

Next we prove the latter statement. Suppose that $A \geqq 0$ and $B \leqq 0$. Then using (5.1) again, we conclude that $G'(z)$ has no zeros. Thus we also arrive at a contradiction. Hereby $A < 0$ or $B > 0$. This completes the proof of Lemma 7.

LEMMA 8. *If $AB \leqq 0$, then $G(z)$ has no finite deficient values.*

Proof. Firstly, consider the case $AB < 0$. Then by Lemma 7, $A < 0$ and $B > 0$. Hence we find

$$\lim_{x \to \pm\infty} |G(x)| = +\infty. \tag{5.4}$$

Since $G(z)$ fails to take the values zero and one in the open half planes $\operatorname{Re} z > 1$ and $\operatorname{Re} z < 0$, Lindelöf-Iversen-Gross' theorem and (5.4) imply

$$\lim_{r \to +\infty} |G(re^{it})| = +\infty \tag{5.5}$$

uniformly for $|t| \leqq t^*$ and for $|t - \pi| \leqq t^*$, where t^* is an arbitrarily fixed number in $(0, \pi/2)$. Now let w be an arbitrary finite complex number. Then by (5.5), it is possible to find a positive number R so that

$$|G(re^{it}) - w| \geqq 1$$

for $r \geqq R$, $|t| \leqq t^*$ and for $r \geqq R$, $|t - \pi| \leqq t^*$. Therefore on putting

$$M(r, w, G) = \sup_{|z|=r} \frac{1}{|G(z) - w|},$$

we obtain

$$m(r, w, G) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|G(re^{it}) - w|}\, dt$$

$$= \frac{1}{2\pi}\left\{\int_{t^*}^{\pi - t^*} + \int_{\pi + t^*}^{2\pi - t^*}\right\}\left(\log^+ \frac{1}{|G(re^{it}) - w|}\right) dt$$

$$\leqq\left(1-\frac{2}{\pi}t^*\right)\log^+ M(r, w, G)$$

for each $r\geqq R$. According to Petrenko's result [9], we thus have

$$\liminf_{r\to\infty}\frac{m(r, w, G)}{T(r, G)}\leqq\left(1-\frac{2}{\pi}t^*\right)\pi$$

for an arbitrarily fixed number t^* $(0<t^*<\pi/2)$, so that $\delta(w, G)=0$ for every finite complex number w. Hereby we have the desired result.

Secondly, if $AB=0$, then either $A<0$ or $B>0$. Hence we find either

$$\lim_{x\to-\infty}|G(x)|=+\infty\,,$$

or

$$\lim_{x\to+\infty}|G(x)|=+\infty\,.$$

On the other hand, by means of (5.2), $A=0$ implies

$$|G(\bar{z})|=|G(-z)|,$$

and $B=0$ implies

$$|G(\bar{z}+1)-1|=|G(-z+1)-1|.$$

By these facts, we at once obtain (5.4). Therefore we also have the desired result by the same process as above. Lemma 8 is thus proved.

Next we consider the case $AB>0$. The following lemma plays an important role in what follows.

LEMMA 9. *Assume that A and B are both positive. If the curve $\{G(1+iy)\mid -\infty<y<+\infty\}$ meets the open half line $\{1<t<+\infty\}$, then $G(z)$ approaches the value one as z tends to infinity along the negative real axis.*

Proof. Let y^* and X be real numbers such that $G(1+iy^*)=X$, $X>1$. By (5.1),

$$\mathrm{Re}\,\frac{G'(1+iy^*)}{X}\geqq A$$

and

$$\mathrm{Re}\,\frac{G'(1+iy^*)}{X-1}=B\,,$$

so that

$$\mathrm{Re}\,G'(1+iy^*)=B(X-1)\geqq XA\,.$$

Since A and B are both positive,

(5.6) $$B-A>0\,.$$

Here we assume that $G(z^*)>1$ for some z^* with $0\leqq\mathrm{Re}\,z^*\leqq1$. Then using (5.1) again, we have

$$AG(z^*) \leqq \mathrm{Re}\, G'(z^*) \leqq B(G(z^*)-1)\,,$$

so that

$$1<S=\frac{B}{B-A} \leqq G(z^*)\,.$$

Consequently, if $1<G(z)<S$ for some z, then either $\mathrm{Re}\, z<0$ or $\mathrm{Re}\, z>1$. Therefore we have the following two possibilities.

1) For every r $(1<r<S)$, $G(z) \neq r$ in the closed half plane $\mathrm{Re}\, z \leqq 1$.

2) There exists a real number r^* $(1<r^*<S)$ such that $G(z_*)=r^*$ for some z_* with $\mathrm{Re}\, z_*<0$.

Firstly, consider the case 2). By $G^{-1}(w)$, we denote the inverse function of $G(z)$. Of course, $G^{-1}(w)$ is an infinitely many valued analytic function with algebraic character. By $E(w, r^*)$, we also denote the element of $G^{-1}(w)$ with center r^* and satisfying $E(r^*, r^*)=z_*$. Now let us continue analytically $E(w, r^*)$ along the segment $I=\{1 \leqq t \leqq r^*\}$ toward the point $t=1$. Then we have an analytic continuation $G^{-1}(I_h)$ with algebraic character along the segment I up to some point $t=h$ $(1 \leqq h<r^*)$, with the possible exception of this end point. Hereby from this continuation $G^{-1}(I_h)$, we can define the simple path $\gamma=\{z(t) \mid 0 \leqq t<r^*-h\}$ such that $z(0)=z_*$ and

$$G(z(t))=r^*-t \qquad (0 \leqq t<r^*-h)\,. \tag{5.7}$$

Clearly, by (5.7), this path γ must be contained in the open half plane $\mathrm{Re}\, z<0$ and the continuation $G^{-1}(I_h)$ does not continue to the point $t=1$. So we may assume that this continuation $G^{-1}(I_h)$ defines a transcendental singularity at the point $t=h$. Therefore by Iversen's theorem [7], the path γ is an asymptotic path of $G(z)$ and as z tends to infinity along γ, $G(z)$ tends to the real value h. By this fact, using Lindelöf-Iversen-Gross' theorem, we obtain

$$\lim_{r \to +\infty} G(re^{it})=h \tag{5.8}$$

uniformly for $|t-\pi| \leqq t^*<\pi/2$. On the other hand, by means of (5.2), we have

$$\begin{aligned} &A_1 G(z-1) \exp(2(B-A)z) \\ &=A_2(G(z+1)-1)+\exp(2Bz) \end{aligned} \tag{5.9}$$

with non-zero constants A_1 and A_2. Combining (5.6), (5.8) and (5.9), we at once conclude that

$$A_2(h-1)=0\,,$$

so that $h=1$. Consequently, $G(z)$ tends to the value one when z does to infinity along the negative real axis.

Secondly, consider the case 1). By (5.1),

$$\mathrm{Re}\, \frac{G'(z)}{G(z)} \geqq A>0$$

for $\operatorname{Re} z>0$. Hence $G'(z)$ omits the value zero there. Especially, $G'(1+iy^*)\neq 0$. Further we can claim that $S<X$. For if $S=X$, then

$$\operatorname{Re}\frac{G'(1+iy^*)}{X}=\operatorname{Re}\frac{G'(1+iy^*)}{G(1+iy^*)}=A.$$

Thus from (5.1), $G(z)$ has no zero. So we obtain

$$G(z)=D\exp(Az),$$

where D is a suitable constant with $|D|=e^{-A}$. This clearly contradicts the assumptions.

Now by $E(w, X)$, denote the regular element of $G^{-1}(w)$ with center X and satisfying $E(X, X)=1+iy^*$. In this case, we continue analytically $E(w, X)$ along the segment $J=\{1\leqq t\leqq X\}$ toward the point $t=1$. Then we have an analytic continuation $G^{-1}(J_u)$ with algebraic character along the segment J up to some point $t=u$ $(1\leqq u<X)$, with the possible exception of this end point. From this continuation $G^{-1}(J_u)$, let us define the simple path $\beta=\{Z(t)\,|\,0\leqq t<X-u\}$ such that $Z(0)=1+iy^*$ and

$$G(Z(t))=X-t \qquad (0\leqq t<X-u). \tag{5.10}$$

For a moment, we assume that the path β intersects the line $\operatorname{Re} z=1$ at some point $Z(t_*)$ $(0\leqq t_*<X-u)$. Then the function $Z(t)$ is differentiable at t_* and from (5.10), we find

$$G'(Z(t_*))Z'(t_*)=-1.$$

Since

$$\operatorname{Re} G'(Z(t_*))=B(X-t_*-1)>0,$$

we thus conclude that

$$\operatorname{Re} Z'(t_*)<0.$$

Consequently, if $\operatorname{Re} Z(t_*)=1$ for some t_*, then $\operatorname{Re} Z'(t_*)<0$. By this fact, the path β must be contained in the open half plane $\operatorname{Re} z<1$ save for the initial point $Z(0)$ and further the continuation $G^{-1}(J_u)$ does not continue along the segment J to the point S. Hereby we may assume that this continuation $G^{-1}(J_u)$ defines a transcendental singularity at the point $t=u$ $(S\leqq u<X)$. Accordingly, the path β must be an asymptotic path of $G(z)$ and as z tends to infinity along this path β, $G(z)$ approaches the value u. Since $G(z)$ fails to take infinitely many values in the open half plane $\operatorname{Re} z<1$, we thus find

$$\lim_{x\to-\infty} G(x)=u\geqq S>1. \tag{5.11}$$

However in view of (5.6), (5.9) and (5.11), we arrive at a contradiction. Therefore the case 1) never occurs. Lemma 9 is thus proved.

In addition to the above lemma, for the case $AB>0$, we further obtain the following

LEMMA 10. *Assume that A and B are both positive. Then*

$$\lim_{r\to\infty}\frac{N(r,1,G)}{r}=\frac{2}{\pi}B,$$

provided that the curve $\{G(1+iy)\mid -\infty<y<+\infty\}$ does not cut off the open half line $\{1<t<+\infty\}$.

Proof. Using (5.1), we find

$$\mathrm{Re}\,\frac{G'(z)}{G(z)}\geqq A \tag{5.12}$$

for $\mathrm{Re}\,z>0$ and

$$\mathrm{Re}\,\frac{G'(z)}{G(z)-1}\geqq B$$

for $\mathrm{Re}\,z\geqq 1$, provided $G(z)\neq 1$. Hence for every real number t with $0\leqq t\leqq 1$,

$$\mathrm{Re}\,\frac{G'(z)(G(z)-t)}{G(z)(G(z)-1)}\geqq tA+(1-t)B>0$$

for $\mathrm{Re}\,z\geqq 1$, provided $G(z)\neq 1$. Especially, the curve $\{G(1+iy)\mid -\infty<y<+\infty\}$ never intersects the segment $0\leqq t<1$. So by the assumptions, if $G(1+iy_*)\geqq 0$ for some real number y_*, then $G(1+iy_*)=1$. Further by (5.12),

$$\begin{aligned}&\arg G(1+iy_2)-\arg G(1+iy_1)\\&=\int_{y_1}^{y_2}\mathrm{Re}\,\frac{G'(1+it)}{G(1+it)}dt\geqq A(y_2-y_1)\end{aligned}$$

for $y_1<y_2$. Thus as the real parameter y traverses the real axis increasingly, this curve winds around the origin infinitely many times in the anti-clockwise direction. Therefore $G(z)$ must have infinitely many one-points. From (5.12), all the one-points of $G(z)$ are simple.

Now let us denote the set of all the one-points of $G(z)$ by $\{1+ic_n\}$. We may assume, by what mentioned above, that

$$\cdots<c_{-2}<c_{-1}<c_0<c_1<c_2<\cdots$$

and

$$\lim_{n\to+\infty}c_n=+\infty,\qquad \lim_{n\to+\infty}c_{-n}=-\infty.$$

In view of (5.12),

$$\mathrm{Re}\,\frac{G'(1+ic_n)}{G(1+ic_n)}=\mathrm{Re}\,G'(1+ic_n)\geqq A>0,$$

so that

$$-\frac{\pi}{2}<\alpha_n=\arg G'(1+ic_n)<\frac{\pi}{2} \tag{5.13}$$

for every integer n. Since $G(1+it)-1$ does not take any non-negative real number for each open interval (c_n, c_{n+1}), we can define the function $I_n(t)$ such that

$$0<I_n(t)=\arg(G(1+it)-1)<2\pi \tag{5.14}$$

for $c_n<t<c_{n+1}$. By means of (5.1), we thus find

$$I_n(t)-I_n(s)=\int_s^t \operatorname{Re}\frac{G'(1+iy)}{G(1+iy)-1}\,dy=B(t-s)\,,$$

so that

$$\left(I_n(t)-\frac{3}{2}\pi\right)-\left(I_n(s)-\frac{1}{2}\pi\right)=B(t-s)-\pi \tag{5.15}$$

for $c_n<s<t<c_{n+1}$. On the other hand, from (5.14),

$$-\frac{3}{2}\pi<I_n(t)-\frac{3}{2}\pi=\arg\frac{G(1+it)-G(1+ic_{n+1})}{it-ic_{n+1}}<\frac{\pi}{2}$$

for every real number t with $c_n<t<c_{n+1}$. Hence we have

$$-\frac{3}{2}\pi\leqq\lim_{t\to c_{n+1}-0}\left(I_n(t)-\frac{3}{2}\pi\right)=\arg G'(1+ic_{n+1})\leqq\frac{\pi}{2}\,.$$

Taking into account of (5.13), we conclude that

$$\lim_{t\to c_{n+1}-0}\left(I_n(t)-\frac{3}{2}\pi\right)=\alpha_{n+1}\,. \tag{5.16}$$

Similarly, it follows from (5.13) and (5.14) that

$$\lim_{s\to c_n+0}\left(I_n(s)-\frac{1}{2}\pi\right)=\alpha_n\,. \tag{5.17}$$

Accordingly, by (5.15), (5.16) and (5.17), we obtain

$$\alpha_{n+1}-\alpha_n=B(c_{n+1}-c_n)-\pi$$

for each integer n. Therefore

$$\alpha_m-\alpha_n=B(c_m-c_n)-(m-n)\pi\,,$$

so that from (5.13),

$$(m-n-1)\pi\leqq B(c_m-c_n)\leqq(m-n+1)\pi \tag{5.18}$$

for arbitrary integers m, n with $n\leqq m$. From these inequalities (5.18), using the usual method, we have the desired result.

In the above Lemmas 9 and 10, we assume that A and B are both positive. For the case where A and B are both negative, we can prove similar results. Indeed, when A and B are both negative, we consider the function defined by

$$F(z)=-G(-z+1)+1\,.$$

Clearly all the zero-points and all the one-points of $F(z)$ lie on the lines $\operatorname{Re} z=0$

and $\operatorname{Re} z=1$, respectively. Further from (5.1),

$$\operatorname{Re}\frac{F'(z)}{F(z)}=-B+\sum_n\frac{\operatorname{Re} z}{|z-(1-b_n)|^2}$$

and

$$\operatorname{Re}\frac{F'(z)}{F(z)-1}=-A+\sum_n\frac{\operatorname{Re} z-1}{|z-(1-a_n)|^2}.$$

Therefore we can apply Lemmas 9 and 10 to this function $F(z)$. Turning back to $G(z)$, we at once obtain the following.

LEMMA 10′. *If A and B are both negative, then either $G(z)$ approaches the value zero as z tends to infinity along the positive real axis or*

$$\lim_{r\to\infty}\frac{N(r, 0, G)}{r}=\frac{2}{\pi}|A|.$$

6. The case $A=B$. In this section, using the same notations as in the previous section 5, we consider the case $A=B$. Lemma 7 asserts that $A=B\neq 0$. In what follows, we assume that $A=B>0$. For this case, we make use of the functional equations (5.2). By means of (5.2), we find

$$\begin{aligned}&G(z-1)=S_*(G(z+1)-1)+\exp(2Az-2A-iA'),\\ (6.1)\qquad&\\ &S_*=\exp(-2A-iA'+iB').\end{aligned}$$

So by the induction,

$$\begin{aligned}&G(z-1-2n)-1=S_*^{n+1}(G(z+1)-1)-\sum_{k=0}^{n}S_*^k\\ (6.2)\qquad&\\ &\qquad+\exp(2Az-4nA-2A-iA')\sum_{k=0}^{n}\left(\frac{1}{\bar{S}_*}\right)\end{aligned}$$

for each non-negative integer n. Since $A>0$,

$$|S_*|=\exp(-2A)<1.$$

Further

$$\begin{aligned}\exp(-4nA)\left|\sum_{k=0}^{n}\left(\frac{1}{\bar{S}_*}\right)^k\right|&\leqq|S_*|^{2n}\sum_{k=0}^{n}|S_*|^{-k}\\ &=|S_*|^n\sum_{k=0}^{n}|S_*|^k.\end{aligned}$$

Therefore from (6.2), we easily have

$$\lim_{x\to-\infty}G(x)=\frac{S_*}{S_*-1}. \tag{6.3}$$

Since $G(z)$ omits the values zero and one in the open half plane $\operatorname{Re} z<0$, Lindelöf-Iversen-Gross' theorem and (6.3) yield

(6.4) $$\lim_{r\to+\infty} G(re^{it})=\frac{S_*}{S_*-1}$$

uniformly for $|t-\pi|\leqq t^*<\pi/2$.

From now on let us consider the functions defined by

$$H(z)=\frac{S_*}{S_*-1}+\frac{\bar{S}_*}{\bar{S}_*-1}\exp(2Az-iA')$$

and

$$F(z)=G(z)-H(z).$$

By an elementary calculation,

$$\overline{H(\bar{z})}=H(-z)\exp(2Az+iA')$$

and

$$H(z-1)=S_*(H(z+1)-1)+\exp(2Az-2A-iA').$$

So the entire function $F(z)$ must satisfy

(6.5) $$\overline{F(\bar{z})}=F(-z)\exp(2Az+iA')$$

and

(6.6) $$F(z-1)=S_*F(z+1).$$

Further by (6.4), the function $F(z)$ also satisfies

(6.7) $$\lim_{r\to+\infty} F(re^{it})=0$$

for each real number t with $|t-\pi|<\pi/2$.

We now assume that $F(z)$ has no zero-points. Since the order of $F(z)$ is at most one, $F(z)$ can be written as

(6.8) $$F(z)=K\exp(Dz)$$

with suitable constants D and K. By means of (6.5), we thus find

$$\bar{K}\exp(\bar{D}z)=K\exp(-Dz+2Az+iA'),$$

so that

$$D+\bar{D}=2A.$$

On the other hand, from (6.7) and (6.8), the constant D must be real. Consequently,

$$F(z)=K\exp(Az).$$

Therefore

$$G(z)=H(z)+K\exp(Az)$$
$$=P(\exp Az),$$

where

$$P(z)=\frac{S_*}{S_*-1}+Kz+\frac{\bar{S}_*}{\bar{S}_*-1}\exp(-iA')z^2.$$

Next we consider the case where $F(z)$ has at least one zero-point. In this case, by virtue of (6.6), $F(z)$ must be identically equal to zero. On the contrary, we assume that $F(z)$ is not constant zero. Then by (6.6), if $F(z)$ has a zero-point of order n at z^*, all the points of the form z^*+2m $(m=0, \pm 1, \pm 2, \cdots)$ are also zeros of $F(z)$ of order exactly n. Further if $z_1, z_2, \cdots, z_k$ are zero-points of $F(z)$ such that each $(z_i-z_j)/2$ is not integer $(1\leqq i<j\leqq k)$, then

$$\limsup_{r\to\infty}\frac{T(r,F)}{r}\geqq\limsup_{r\to\infty}\frac{N(r,0,F)}{r}\geqq\lim_{r\to\infty}\frac{N^*(r)}{r}=k,$$

where $N^*(r)$ is the counting function for the set

$$\{z_j+2m\,|\,m=0, \pm 1, \pm 2, \cdots, j=1, 2, \cdots, k\}.$$

Here we observe that $F(z)$ has at most order one, mean type. Then by these facts, the function $F(z)$ can be expressed as with suitable a finite number of points $z_1^*, z_2^*, \cdots, z_n^*$,

$$F(z)=E_1(z)E_2(z)\cdots E_n(z)\exp(Cz+D),$$

where C, D are constants and

$$E_j(z)+1=\exp(i\pi(z-z_j^*)) \qquad (j=1, 2, \cdots, n).$$

However using (6.7) and asymptotic properties of $E_j(z)$, we easily arrive at a contradiction. Therefore if $F(z)$ has zero-points, then $F(z)$ is identically equal to zero. Consequently,

$$G(z)=\frac{S_*}{S_*-1}+\frac{\bar{S}_*}{\bar{S}_*-1}\exp(2Az-iA').$$

Thus we obtain the following lemma.

LEMMA 11. *Let $G(z)$ be a transcendental entire function of finite lower order such that all the zero-points and all the one-points of $G(z)$ lie on the lines* $\mathrm{Re}\,z=0$ *and* $\mathrm{Re}\,z=1$, *respectively. If $G(z)$ satisfies*

$$\overline{G(\bar{z})}=G(-z)\exp(2Az+iA'),$$

$$\overline{G(\bar{z}+1)-1}=(G(-z+1)-1)\exp(2Az+iB')$$

with some real constants A $(\neq 0)$, A' and B', then

$$G(z)=P(\exp Az),$$

where $P(z)$ is a quadratic polynomial.

For the case $A<0$, as in the proof of Lemma 10′, considering the function defined by

$$G^*(z)=-G(-z+1)+1,$$

we at once obtain the desired result.

7. Proof of Theorem 1. Now we are in a position to piece together the foregoing lemmas and prove our Theorem 1.

Let $f(z)$ be an entire function satisfying the assumptions of Theorem 1. By Theorem 6, the order of $f(z)$ must be finite. Further Theorem D asserts that the characteristic function of $f(z)$ must satisfy

$$\liminf_{r\to\infty}\frac{T(r,f)}{r}\neq 0\,. \tag{7.1}$$

For otherwise, all the zero-points of $f'(z)$ must lie on the three distinct lines L_j simultaneously. Hence $f'(z)$ has at most one zero-point, so that $f'(z)$ reduces to a polynomial.

To prove our theorem, it is sufficient to consider the following three cases.

1) No two of the three lines L_j are parallel to each other.

2) L_1 and L_2 are parallel, but L_1 and L_3 are not.

3) The three lines L_j are parallel to one another.

Firstly, we consider the case 1). By Lemma 6, at least one of the three values, say a_1, is a radially distributed value of $f(z)$. Let us choose two complex numbers c and d such that the function defined by

$$f^*(z)=f(cz+d)-a_2$$

has only real zeros. Then the value a_1-a_2 $(\neq 0)$ is a radially distributed value of $f^*(z)$ and the genus of $f^*(z)-a_1+a_2$ is at least one by (7.1). Hence from Theorem B, a_1-a_2 is a deficient value of $f^*(z)$. So the order of $f^*(z)$ is at most one by Theorem A. Thus from (7.1), the original function $f(z)$ is of order one and regular growth. By the assumptions, $f(z)$ has a finite deficient value, say a^*, other than a_1, a_2 and a_3. Since $f(z)$ is of regular growth, the value a^*-a_2 must be a deficient value of $f^*(z)$. Clearly, a_1-a_2 is different from a^*-a_2. So the auxiliary function $f^*(z)$ has at least two finite deficient values. This is absurd by Theorem A. Consequently, the case 1) never occurs.

Next we consider the case 2). In this case, using Theorem 4 and (7.1), we at once conclude that $f(z)$ is of order one and regular growth. So we may assume, without loss of generality, that $a_1=0$, $a_2=1$, $L_1:\operatorname{Re} z=0$ and $L_2:\operatorname{Re} z=1$. From Lemma 5, we find

$$\begin{aligned}\operatorname{Re}\frac{f'(z)}{f(z)}&=A+\sum_n\frac{\operatorname{Re} z}{|z-a_n^*|^2}\,,\\ \operatorname{Re}\frac{f'(z)}{f(z)-1}&=B+\sum_n\frac{\operatorname{Re} z-1}{|z-b_n^*|^2}\,,\end{aligned} \tag{7.2}$$

where $\{a_n^*\}$ and $\{b_n^*\}$ denote the zero-points and the one-points of $f(z)$, respectively. By virtue of Lemma 8, A and B are both positive or both negative. Thus by (7.2) and by Lindelöf-Iversen-Gross' theorem, either

$$\lim_{r\to+\infty} |f(re^{it})| = +\infty$$

uniformly for $|t| \leqq t^* < \pi/2$, or

$$\lim_{r\to+\infty} |f(re^{it})| = +\infty$$

uniformly for $|t-\pi| \leqq t^* < \pi/2$. Therefore by the assumptions, the value a_3 must be a radially distributed value of $f(z)$. Hence using the same argument as in the above case 1), we arrive at a contradiction. Consequently, this case 2) does not occur either.

Now let us discuss the case 3). By the same reason as in the case 2), we may assume that $L_1: \operatorname{Re} z = x_1$, $L_2: \operatorname{Re} z = x_2$ and $L_3: \operatorname{Re} z = x_3$ with $x_1 < x_2 < x_3$. From Lemma 5,

$$\overline{f(\bar{z}+x_j)-a_j} = (f(-z+x_j)-a_j) \exp(2C_j z + iC_j') \tag{7.3}$$

with suitable real constants C_j and C_j' $(j=1, 2, 3)$. For each pair j and k with $1 \leqq j < k \leqq 3$, consider the function defined by

$$G_{jk}(z) = \frac{f((x_k-x_j)z+x_j)-a_j}{a_k-a_j}. \tag{7.4}$$

Then all the zero-points and all the one-points of $G_{jk}(z)$ lie on the lines $\operatorname{Re} z=0$ and $\operatorname{Re} z=1$, respectively. Since the original function $f(z)$ is of regular growth, $G_{jk}(z)$ has also a finite deficient value other than the three values

$$\frac{a_m-a_j}{a_k-a_j} \qquad (m=1, 2, 3).$$

Further using (7.3) and (7.4), we easily obtain

$$\overline{G_{jk}(\bar{z})} = G_{jk}(-z) \exp(2(x_k-x_j)C_j z + iC_{jk}') \tag{7.5}$$

and

$$\overline{G_{jk}(\bar{z}+1)-1} = (G_{jk}(-z+1)-1) \exp(2(x_k-x_j)C_k z + iC_{kj}') \tag{7.6}$$

with suitable real constants C_{jk}' and C_{kj}'.

From now on, applying the foregoing lemmas to these functions $G_{jk}(z)$, we prove that either $C_1=C_2$ or $C_2=C_3$. Since $G_{jk}(z)$ has a finite deficient value, Lemma 8 asserts that the three real constants C_1, C_2 and C_3 are positive or negative simultaneously. For a moment we assume that C_1, C_2 and C_3 are all positive. Here recall that the auxiliary function $G_{jk}(z)$ omits the two values zero and one in the open half plane $\operatorname{Re} z < 0$. Then if $G_{jk}(z)$ approaches the value one when z tends to infinity along the negative real axis, by the same fashion as in the proof of Lemma 8, $G_{jk}(z)$ has no finite deficient values with the possible exception of the value one. So by virtue of Lemma 10, we obtain

$$\lim_{r\to\infty} \frac{N(r, 1, G_{jk})}{r} = \frac{2}{\pi}(x_k-x_j)C_k. \tag{7.7}$$

Returning to the original function $f(z)$, we thus find

$$\lim_{r\to\infty}\frac{N(r, a_k, f)}{r}=\frac{2}{\pi}C_k \qquad (k=2, 3). \tag{7.8}$$

In fact, by the definition of $G_{jk}(z)$,

$$n(t, 1, G_{jk})\leqq n((x_k-x_j)t+|x_j|, a_k, f) \tag{7.9}$$

and

$$n(t, a_k, f)\leqq n\Big(\frac{t+|x_j|}{x_k-x_j}, 1, G_{jk}\Big) \tag{7.10}$$

for each positive number t, where $n(t, a, F)$ denotes the number of a-points of the function $F(z)$ in $|z|<t$. Hence (7.9) implies

$$\begin{aligned} N(r, 1, G_{jk})-N(s, 1, G_{jk}) &\leqq \int_{(x_k-x_j)s+|x_j|}^{(x_k-x_j)r+|x_j|}\frac{n(t, a_k, f)}{t}\,\frac{t}{t-|x_j|}dt \\ &\leqq \frac{(x_k-x_j)s+|x_j|}{(x_k-x_j)s}\;N((x_k-x_j)r+|x_j|, a_k, f) \end{aligned}$$

for every positive numbers r and s with $s\leqq r$, so that

$$\liminf_{r\to\infty}\frac{N(r, 1, G_{jk})}{r}\leqq\Big(\frac{(x_k-x_j)s+|x_j|}{s}\Big)\liminf_{r\to\infty}\frac{N(r, a_k, f)}{r}$$

for every positive number s. Since s can be chosen as large as we please,

$$\liminf_{r\to\infty}\frac{N(r, 1, G_{jk})}{r}\leqq(x_k-x_j)\liminf_{r\to\infty}\frac{N(r, a_k, f)}{r}.$$

Similarly, (7.10) implies

$$(x_k-x_j)\limsup_{r\to\infty}\frac{N(r, a_k, f)}{r}\leqq\limsup_{r\to\infty}\frac{N(r, 1, G_{jk})}{r}.$$

Therefore by means of (7.7), we have the desired (7.8).

On the other hand, by Theorem A and by the fact that $f(z)$ is of regular growth, $f(z)$ has exactly one finite deficient value. Hence $\delta(a_j, f)=0$ $(j=1, 2, 3)$. Thus from (7.8), we have

$$\liminf_{r\to\infty}\frac{T(r, f)}{r}=\frac{2}{\pi}C_2=\frac{2}{\pi}C_3,$$

so that $C_2=C_3$.

In the case where C_1, C_2 and C_3 are all negative, in virtue of Lemma 10′, we conclude that $C_1=C_2$ by the same way as above.

Now return to $G_{12}(z)$ or $G_{23}(z)$, again. Then from Lemma 11, either

$$G_{12}(z)=P(\exp{(x_2-x_1)C_2z})$$

or

$$G_{23}(z)=P(\exp{(x_3-x_2)C_2z})$$

with a quadratic polynomial $P(z)$. Turning back to the original function $f(z)$ once more, we have the desired result. This completes the proof of Theorem 1.

8. Proof of Theorem 2. Let $f(z)$ be a transcendental entire function which satisfies the assumptions of Theorem 2. We may assume that the four straight lines L'_j run parallel with one another. For otherwise, by the same way as in the proof of Lemma 6, at least one of the four values b_j must be a radially distributed value of $f(z)$. So $f(z)$ has at least one finite deficient value, since the characteristic function of $f(z)$ satisfies (7.1). Hence by virtue of Theorem 1, we have nothing to prove. Further we may assume that $L'_j : \operatorname{Re} z=x_j$ $(j=1, 2, 3, 4)$ with $x_1<x_2<x_3<x_4$, and that

$$\delta(b_j, f)=0 \qquad (j=1, 2, 3, 4). \tag{8.1}$$

Evidently, $f(z)$ is a function of order one and regular growth. From Lemma 5, with suitable real constants A_j and B_j,

$$\overline{f(\bar{z}+x_j)-b_j}=(f(-z+x_j)-b_j) \exp(2A_j z+iB_j) \tag{8.2}$$

and

$$\operatorname{Re} \frac{f'(z)}{f(z)-b_j}=A_j+\sum_n \frac{\operatorname{Re} z-x_j}{|z-c_{nj}|^2}, \tag{8.3}$$

where $\{c_{nj}\}$ are the b_j-points of $f(z)$ $(j=1, 2, 3, 4)$.

Hereafter, our proof is divided into the consideration of several cases.

The first step. As before, let us set

$$G^*_{jk}(z)=\frac{f((x_k-x_j)z+x_j)-b_j}{b_k-b_j} \tag{8.4}$$

for each pair j and k with $1\leqq j<k\leqq 4$. Clearly, all the zero-points and all the one-points of $G^*_{jk}(z)$ lie on the lines $\operatorname{Re} z=0$ and $\operatorname{Re} z=1$, respectively. Further (8.2) and (8.4) imply

$$\overline{G^*_{jk}(\bar{z})}=G^*_{jk}(-z) \exp(2(x_k-x_j)A_j z+iB'_{jk}) \tag{8.5}$$

and

$$\overline{G^*_{jk}(\bar{z}+1)-1}=(G^*_{jk}(-z+1)-1 (\exp(2(x_k-x_j)A_k z+iB'_{kj}) \tag{8.6}$$

with suitable real constants B'_{jk} and B'_{kj}.

In this step, we prove that $A_1\neq 0$ and $A_4\neq 0$. If $A_1=0$, then applying Lemma 7 to the above functions $G^*_{1k}(z)$, we find $A_k>0$ $(k=2, 3, 4)$. Hence

$$\lim_{x\to+\infty} |G^*_{1k}(x)|=+\infty. \tag{8.7}$$

By means of (8.5),

$$|G^*_{1k}(\bar{z})|=|G^*_{1k}(-z)|,$$

so that (8.7) implies

(8.8) $$\lim_{x\to-\infty} |G_{1k}^*(x)|=+\infty\,.$$

Turning back to the original function $f(z)$, we thus have

(8.9) $$\lim_{x\to-\infty} |f(x)|=+\infty\,.$$

Therefore it follows from (8.4) and (8.9) that

(8.10) $$\lim_{x\to-\infty} |G_{jk}^*(x)|=+\infty$$

for each function $G_{jk}^*(z)$. Since $A_j>0$ ($j=2, 3, 4$), using Lemmas 9 and 10, w. hence obtain

(8.11) $$\lim_{r\to\infty} \frac{N(r, 1, G_{jk}^*)}{r}=\frac{2}{\pi}(x_k-x_j)A_k$$

for each pair j and k with $2\leqq j<k\leqq 4$. Return to $f(z)$ again. Then from (8.11),

(8.12) $$\lim_{r\to\infty} \frac{N(r, b_k, f)}{r}=\frac{2}{\pi}A_k \qquad (k=3, 4)\,.$$

Therefore by making use of (8.1) and (8.12), we deduce

$$\liminf_{r\to\infty} \frac{T(r, f)}{r}=\frac{2}{\pi}A_3=\frac{2}{\pi}A_4\,,$$

so that $A_3=A_4$. Hereby Lemma 11 asserts that

$$G_{34}^*(z)=P(\exp(x_4-x_3)A_3z)$$

with a quadratic polynomial $P(z)$. However, since $A_3>0$, we easily have

$$\lim_{x\to-\infty} G_{34}^*(x)=P(0)\,,$$

which contradicts (8.10). Consequently, $A_1\neq 0$.

By the similar way as above, we can prove that $A_4\neq 0$.

The second step. In this step, we prove that $A_2\neq 0$ and $A_3\neq 0$. If $A_2=0$, then $A_1<0$, $A_3>0$ and $A_4>0$ by Lemma 7. Hence as in the above step, we find

(8.13) $$\lim_{x\to\pm\infty} |f(x)|=+\infty\,.$$

Especially,

$$\lim_{x\to-\infty} |G_{34}^*(x)|=+\infty\,.$$

Therefore applying Lemmas 9 and 10 to this auxiliary function $G_{34}^*(z)$, we obtain

$$\lim_{r\to\infty} \frac{N(r, 1, G_{34}^*)}{r}=\frac{2}{\pi}(x_4-x_3)A_4\,,$$

so that

$$\lim_{r\to\infty}\frac{N(r,b_4,f)}{r}=\frac{2}{\pi}A_4\,. \tag{8.14}$$

It thus follows from (8.1) and (8.14) that

$$\liminf_{r\to\infty}\frac{T(r,f)}{r}=\frac{2}{\pi}A_4\,. \tag{8.15}$$

On the other hand, since $f(z)$ fails to take the four values b_j for $\operatorname{Re} z<x_1$ and for $\operatorname{Re} z>x_4$, Lindelöf-Iversen-Gross' theorem and (8.13) imply

$$\lim_{r\to+\infty}|f(re^{it})|=+\infty$$

uniformly for $|t|\leqq t^*$ and for $|t-\pi|\leqq t^*$, where t^* is an arbitrarily fixed number in $(0,\pi/2)$. Hence it is possible to find a positive number R so that

$$|f(-re^{-it}+2x_4)-b_4|\geqq 1$$

for $r\geqq R$, $|t|\leqq t^*$ and that

$$|f(-re^{-it}+2x_1)-b_1|\geqq 1$$

for $r\geqq R$, $|t-\pi|\leqq t^*$. Thus by means of (8.2),

$$\begin{aligned}|f(re^{it})-b_4|&=|f(-re^{-it}+2x_4)-b_4|\exp(2A_4r\cos t-2A_4x_4)\\&\geqq\exp(2A_4r\cos t-2A_4x_4)\end{aligned}$$

for $r\geqq R$, $|t|\leqq t^*$ and

$$\begin{aligned}|f(re^{it})-b_1|&=|f(-re^{-it}+2x_1)-b_1|\exp(2A_1r\cos t-2A_1x_1)\\&\geqq\exp(2A_1r\cos t-2A_1x_1)\end{aligned}$$

for $r\geqq R$, $|t-\pi|\leqq t^*$. Therefore using these inequalities, we obtain

$$\begin{aligned}T(r,f)&\geqq\frac{1}{2\pi}\int_{-t^*}^{t^*}\log^+|f(re^{it})|\,dt+\frac{1}{2\pi}\int_{\pi-t^*}^{\pi+t^*}\log^+|f(re^{it})|\,dt\\&\geqq\frac{1}{\pi}(A_4+|A_1|)r\int_{-t^*}^{t^*}\cos t\,dt+O(1)\end{aligned}$$

for $r\geqq R$, so that

$$\liminf_{r\to\infty}\frac{T(r,f)}{r}\geqq\frac{2}{\pi}(A_4+|A_1|)\sin t^*\,. \tag{8.16}$$

Since (8.16) holds for every number t^* with $0<t^*<\pi/2$,

$$\liminf_{r\to\infty}\frac{T(r,f)}{r}\geqq\frac{2}{\pi}(A_4+|A_1|)\,.$$

This is clearly absurd by (8.15). Consequently, $A_2\neq 0$.

Similarly, we can prove that $A_3\neq 0$.

The third step. By the above two steps and by Lemma 7, the following five cases may occur.

1) A_1, A_2, A_3 and A_4 are all positive.
2) A_1 is negative and the other three are positive.
3) A_1, A_2 are negative and A_3, A_4 are positive.
4) A_4 is positive and the other three are negative.
5) The four are all negative.

However, we can easily prove the impossibility of the cases 2) and 4) by the same arguments which are used in the first step. Further we can also prove the impossibility of the case 3) using the arguments developed in the second step.

The final step. Let us consider the case 1). Since the four real numbers A_j are all positive, we can apply Lemmas 9 and 10 to the auxiliary functions $G_{jk}^*(z)$. If some $G_{1k}^*(z)$, say $G_{12}^*(z)$, tends to the value one when z does to infinity along the negative real axis, then by (8.4),

$$\lim_{x\to-\infty} f(x)=b_2 .$$

Therefore $G_{13}^*(z)$ and $G_{14}^*(z)$ never approach the value one as z tends to infinity along the negative real axis. So Lemma 10 yields

$$\lim_{r\to\infty} \frac{N(r, 1, G_{1k}^*)}{r} = \frac{2}{\pi}(x_k - x_1)A_k \qquad (k=3, 4) .$$

Turning back to the original function $f(z)$, we thus find

$$\lim_{r\to\infty} \frac{N(r, b_k, f)}{r} = \frac{2}{\pi} A_k \qquad (k=3, 4) . \tag{8.17}$$

Hence (8.1) and (8.17) imply

$$\liminf_{r\to\infty} \frac{T(r, f)}{r} = \frac{2}{\pi} A_3 = \frac{2}{\pi} A_4 ,$$

so that $A_3=A_4$. By this consideration, we can claim that at least two of the three numbers A_2, A_3 and A_4 coincide with each other. Hereby Theorem 2 follows at once from Lemma 11.

For the case 5), by making use of Lemma 10′, we can also conclude that at least two of the three numbers A_1, A_2 and A_3 must be equal to each other. So by virtue of Lemma 11, Theorem 2 follows immediately. The proof of Theorem 2 is now complete.

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