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FIXED POINTS FOR CONDENSING MULTIFUNCTIONS IN METRIC SPACES WITH CONVEX STRUCTURE

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In this paper, we prove a fixed point theorem for condensing multifunctions with convex values, closed graph, and bounded range acting on a metric space endowed with a simple but powerful notion of convexity.

In Section 1, we discuss a notion of convexity for metric spaces which was introduced in [8] by W. Takahashi. We develop here some geometric and topological properties which result when a uniqueness assertion is added to Takahashi's requirements.

In Section 2, we introduce a new notion of convex structure for a metric space. This new notion is based on the Takahashi notion, but has some pleasanter geometric properties, which we investigate here. In particular, we are here able to permute the order of repeated convex combination.

In Section 3, we introduce the "measure of noncompactness" and study it in relation to a "stable" convex structure. The major fact here is that the measure of non-compactness is invariant under passage to convex hulls.

Section 4 is devoted to our major result (Theorem 4.2): A condensing multifunction with convex values, closed graph, and bounded range, which acts on a complete metric space with stable strong convex structure has a fixed point.

1. Takahashi convex structures.

1.1 DEFINITION: Let (X, d) be a metric space, and let I be the closed unit interval $[0, 1]$. A *Takahashi convex structure* (TCS) on X is a function $W: X\times X\times I\to X$ which has the property that for every $x, y\in X$ and $t\in I$ we have

$$d(z, W(x, y, t))\leqq td(z, x)+(1-t)d(z, y) \tag{1}$$

for every $z\in X$. If (X, d) is equipped with a TCS, we call X a *convex metric space*. When (X, d) is a convex metric space and $S\subset X$, we say that S is *convex* provided that $W(x, y, t)$ lies in S for each (x, y, t) in $S\times S\times I$.

Takahashi convex structures were introduced by W. Takahashi in [8], and

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have been studied by H. Machado [6] as well. The following proposition collects some results from [8] which follow immediately from the definition; see [8] for details.

1.2 PROPOSITION: *Let W be a TCS on a metric space (X, d). If $x, y \in X$ and $t \in I$, then*

(a) $W(x, y, 1)=x$ *and* $W(x, y, 0)=y$.

(b) $W(x, x, t)=x$.

(c) $d(x, W(x, y, t))=(1-t)d(x, y)$ and

$$d(y, W(x, y, t))=td(x, y).$$

(d) *Balls (either open or closed) in X are convex.*

(e) *Intersections of convex subsets of X are convex.*

1.3 DEFINITION: Let W be a TCS on a metric space (X, d). We say that W is a *strict* TCS if it has the property that whenever $w \in X$ and there is $(x, y, t) \in X \times X \times I$ for which

$$d(z, w) \leqq td(z, x)+(1-t)d(z, y), \quad \text{for every} \quad z \in X,$$

then $w=W(x, y, t)$. If W is a strict TCS on the metric space (X, d), we call X a *strictly convex metric space.*

The reader is warned that our use of the term "strictly convex" does not conform to standard usage for Banach spaces. For example, the plane equipped with the norm $\|(x_1, x_2)\|=|x_1|+|x_2|$ is strictly convex in our sense, but not in the standard sense.

1.4 LEMMA: *Let W be a strict* TCS *on the metric space (X, d). Then for every $x, y \in X$ and $t, s \in I$. we have*

$$W(W(x, y, t), y, s)=W(x, y, ts).$$

Proof: Let $z \in X$. Then

$$\begin{aligned} d(z, W(W(x, y, t), y, s)) &\leqq sd(z, W(x, y, t))+(1-s)d(z, y) \\ &\leqq s[td(z, x)+(1-t)d(z, y)]+(1-s)d(z, y) \\ &\leqq std(z, x)+(1-st)d(z, y), \end{aligned}$$

whence, by strictness, $W(W(x, y, t), y, s)=W(x, y, ts)$.

We now investigate the continuity properties of a strict TCS.

1.5 THEOREM: *If W is a strict TCS on a metric space (X, d), then for every pair $x, y \in X$ with $x \neq y$ the function $t \mapsto W(x, y, t)$ is an embedding of I into X.*

Proof: Let $t_1, t_2 \in I$, and assume, without loss of generality, that $t_1 < t_2$. Then

$$\begin{aligned} d(W(x, y, t_1), W(x, y, t_2)) &= d(W(x, y, t_2(t_1/t_2)), W(x, y, t_2)) \\ &= d(W(W(x, y, t_2), y, t_1/t_2), W(x, y, t_2)) \\ &= [1-(t_1/t_2)]d(W(x, y, t_2), y) \\ &= (t_2 - t_1)d(x, y), \end{aligned}$$

which establishes the theorem.

1.6 REMARK: The above argument shows that the map $W(x, y, t) \mapsto td(x, y)$ gives an isometry of the subspace $\{W(x, y, t) : t \in I\}$ of X onto the closed interval $[0, d(x, y)]$. In particular, $\{W(x, y, t) : t \in I\}$ is homeomorphic with I if $x \neq y$, and is a singleton if $x = y$. Thus the strictly convex metric spaces are seen to be SC_2-metric spaces as defined by J. Dydak [1].

It does not appear that even a strict TCS is necessarily continuous as a function from $X \times X \times I$ to X. However, we have the following:

1.7 THEOREM: *Let W be a TCS on a metric space (X, d). Then W is continuous at each point (x, x, t) of $X \times X \times I$.*

Proof. Let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times I$ which converges to (x, x, t). In view of Proposition 1.2(b), it suffices to show that $\{W(x_n, y_n, t_n)\}_{n=1}^{\infty}$ converges to x. But this is immediate, since the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converge to x, and (1) yields, for each n, $d(x, W(x_n, y_n, t_n)) \leqq t_n d(x, x_n) + (1-t_n)d(x, y_n)$.

The difficulty in obtaining continuity of W as a map from the product lies in the fact that there seems to be no way to guarantee that the sequence $\{W(x_n, y_n, t_n)\}_{n=1}^{\infty}$ will converge when $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ converges to (x, y, t) with $x \neq y$. When X is compact, we can eliminate this difficulty. I am indebted to Prof. C.J. Himmelberg for the proof of the following theorem.

1.8 THEOREM: *(C.J. Himmelberg. unpublished notes) Let W be a strict TCS on a compact metric space (X, d). Then W is continuous as a function from* $X \times X \times I$ to X.

Proof: Let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times I$ which converges to (x, y, t), and let w be a limit point of the sequence $\{W(x_n, y_n, t_n)\}_{n=1}^{\infty}$. Select a subsequence $\{W(x_{n_k}, y_{n_k}, t_{n_k})\}_{k=1}^{\infty}$ which converges to w. Then for any $z \in X$, we have $d(z, W(x_{n_k}, y_{n_k}, t_{n_k})) \leqq t_{n_k} d(z, x_{n_k}) + (1-t_{n_k})d(z, y_{n_k})$ for $k = 1, 2, \cdots$. By continuity of the metric, we conclude that $d(z, w) \leqq td(z, x) + (1-t)d(z, y)$. Strictness now guarantees that $w = W(x, y, t)$. It follows that $W(x, y, t)$ is the only limit point of the sequence $\{W(x_n, y_n, t_n)\}_{n=1}^{\infty}$. Since X is compact, $\{W(x_n, y_n, t_n)\}_{n=1}^{\infty}$ must converge to $W(x, y, t)$, and we are done.

2. Strong convex structures.

Takahashi convex structures provide a reasonably rich notion of convexity for abstract metric spaces. However, they suffer from two serious flaws. The first, which we have already noted, lies in their rather weak continuity properties. The second lies in a lack of identities which would allow permutation of the order of repeated convex combination, *i.e.*, given $t_1, t_2 \in I$, we would like to know that we can find $s_1, s_2 \in I$ in such a way that $W(W(x, y, t_1), z, t_2) = W(W(x, z, s_1), y, s_2)$.

The first difficulty seems to be inherent to the Takahashi notion, and must be circumvented by explicit assumption of compactness on the one hand or continuity itself on the other. Surprisingly, the second difficulty can be rectified within the spirit of Takahashi's definition by simply "going up a dimension". This is the purpose of our next definition.

2.1 DEFINITION: Let (X, d) be a metric space, and let $P = \{(t_1, t_2, t_3) \in I \times I \times I : t_1 + t_2 + t_3 = 1\}$. A *strong convex structure* (SCS) on X is a continuous function $K: X \times X \times X \times P \to X$ with the property that for each $(x_1, x_2, x_3, t_1, t_2, t_3) \in X \times X \times X \times P$, $K(x_1, x_2, x_3, t_1, t_2, t_3)$ is the unique point of X which satisfies

$$d(y, K(x_1, x_2, x_3, t_1, t_2, t_3)) \leqq \sum_{k=1}^{3} t_k d(y, x_k) \tag{2}$$

for every $y \in X$. A metric space with a strong convex structure will be called *strongly convex.*

2.2 REMARK. The uniqueness assumption in Def. 2.1 guarantees that if p is a permutation of $\{1, 2, 3\}$, then, for $(x_1, x_2, x_3, t_1, t_2, t_3) \in X \times X \times X \times P$, we have

$$K(x_1, x_2, x_3, t_1, t_2, t_3) = K(x_{p(1)}, x_{p(2)}, x_{p(3)}, t_{p(1)}, t_{p(2)}, t_{p(3)}) .$$

It is from this trivial observation and the following lemma that we obtain a well-behaved TCS on X.

2.3 LEMMA: *Let (X, d) be a strongly convex metric space, K its SCS. Define W_K: $X \times X \times I \to X$ by $W_K(x_1, x_2, t) = K(x_1, x_2, x_1, t, 1-t, 0)$. Then W_K is a continuous strict TCS on X. Moreover, for any $x_1, x_2, x_3 \in X$ and $t, s \in I$, we have*

$$W_K(W_K(x_1, x_2, s), x_3, t) = K(x_1, x_2, x_3, st, t(1-s), 1-t) . \tag{3}$$

Proof. That W_K is a continuous strict TCS for X is immediate from the properties of K and the definition of a strict TCS; the details are left to the reader.

To prove (3), let $y \in X$ be arbitrary. We then have

$$\begin{aligned} &d(y, W_K(W_K(x_1, x_2, s), x_3, t)) \\ &\quad \leqq td(y, W_K(x_1, x_2, s)) + (1-t)d(y, x_3) \end{aligned}$$

$$\leqq st\, d(y, x_1)+(1-s)t\, d(y, x_2)+(1-t)d(y, x_3)\,.$$

Thus, $W_K(W_K(x_1, x_2, s), x_3, t)$ satisfies (2) with $t_1=st$, $t_2=(1-s)t$, and $t_3=1-t$. (3) follows by uniqueness.

We remark that the value of the third argument for K in the definition of W_K is irrelevant, as is easily seen. We have chosen x_1 for definiteness.

In the remainder of the paper (X, d) will denote a strongly convex metric space; K, its SCS; and $W=W_K$, the induced strict TCS provided by Lemma 2.3.

2.4 PROPOSITION. *If* $x_1, x_2, x_3 \in X$ *and* $t, s \in I$, *we have*

(a) $W(x_1, x_2, s)=W(x_2, x_1, 1-s)$

(b) $W(W(x_1, x_2, s), x_3, t)$

$$=W(W(x_1, x_3, st[1-t(1-s)]^{-1}), x_2, 1-t(1-s))\,,$$

where the right side of (*b*) *should be interpreted as meaning* x_2 *when* $1-t(1-s)=0$.

Proof. (a) is immediate from the strictness of W and Proposition 1.2 (in fact, (a) holds for any strict TCS).

(b) follows easily from Remark 2.2 and Lemma 2.3. The details are left to the reader.

2.5 DEFINITION: A subset S of a strongly convex metric space is said to be *convex* if it is convex for the induced Takahashi convex structure. If $H \subset X$, then $C(H)$ denotes the intersection of all convex subsets M of X for which $H \subset M$. $C(H)$ is called the *convex hull of* H.

2.6 REMARK: $C(H)$ is convex by Proposition 1.2(e), so that the terminology "convex hull" is reasonable. Moreover, W is continuous; and this guarantees that the closure of a convex set is convex. It is, however, not true that the convex hull of a closed set is necessarily closed. (Consider the set $\{(x, y) \in R \times R : y=(1+x^2)^{-1}\}$.)

Our next lemma depends heavily on Prop. 2.4, and is the fundamental property of convex hulls in strongly convex metric spaces.

2.7 LEMMA: *Let* $H \subset X$ *be convex, and let* $x_0 \in X \backslash H$. *Define a subset* $H(x_0)$ *of* X *by* $H(x_0)=\{W(x_0, y, t) : y \in H$ *and* $t \in I\}$. *Then* $C(H \cup \{x_0\})=H(x_0)$.

Proof: It is clear that $H \subset H(x_0) \subset C(H \cup \{x_0\})$, so it suffices to prove that $H(x_0)$ is convex. To this end, let $y_1, y_2 \in H(x_0)$, and let $t \in I$. We show that $W(y_1, y_2, t) \in H(x_0)$. By definition of $H(x_0)$, we can find $x_1, x_2 \in H$ and $r_1, r_2 \in I$ so that $y_i = W(x_0, x_i, r_i)$, $i=1, 2$. Applying Prop. 2.4 several times, and using Prop. 1.2(b), we obtain a succession $s_1, s_2, \cdots, s_6$ of elements of I such that

$$\begin{aligned} W(y_1, y_2, t) &= W(W(x_0, x_1, r_1), W(x_0, x_2, r_2), t) \\ &= W(W(x_0, W(x_0, x_2, r_2), s_1), x_1, s_2) \\ &= W(W(W(x_0, x_2, r_2), x_0, 1-s_1), x, s_2) \\ &= W(W(W(x_0, x_0, s_3), x_2, s_4), x_1, s_2) \\ &= W(W(x_0, x_2, s_4), x_1, s_2) \\ &= W(W(x_2, x_0, 1-s_4), x_1, s_2) \\ &= W(W(x_2, x_1, s_5), x_0, s_6) \\ &= W(x_0, W(x_2, x_1, s_5), 1-s_6)\,. \end{aligned}$$

But H is convex, and $x_2, x_1 \in H$, whence $W(x_2, x_1, s_5) \in H$, so that $W(x_0, W(x_2, x_1, s_5), 1-s_6) \in H(x_0)$. The argument is complete.

3. Convex hulls and compactness; the measure of non-compactness.

3.1 THEOREM: *If $S \subset X$ is a finite set, then $C(S)$ is compact.*

Proof: We proceed by induction on $|S|$, the number of elements in S. Since the assertion is trivial for $|S|=1$, we assume that for some integer, n, $C(S)$ is compact whenever $|S| \leqq n$. Suppose that $S \subset X$ is a set with $|S| = n+1$. Let x_0 be any element of S. The induction hypothesis guarantees that $C(S \setminus \{x_0\})$ is compact, so if $x_0 \in C(S \setminus \{x_0\})$, we are through. If $x_0 \notin C(S \setminus \{x_0\})$, then Lemma 2.7 yields that $C(S) = C(C(S \setminus \{x_0\}) \cup \{x_0\}) = \{W(x_0, y, t) : y \in C(S \setminus \{x_0\}), t \in I\}$. Thus, $C(S) = W(\{x_0\} \times C(S \setminus \{x_0\}) \times I)$, and, as W is continuous and $\{x_0\} \times C(S \setminus \{x_0\}) \times I$ is compact, the proof is complete.

In what follows, when $S \subset X$ and r is a positive real number, we shall use the notation S_r for the set $\{x \in X : d(x, S) < r\}$.

3.2 DEFINITION: Let X be a strongly convex metric space, and let $S \subset X$ be convex. We say that S is *stable* if S_r is convex for every $r > 0$. We say that the SCS on X is stable if the set $\{W(x, y, t) : t \in I\}$ is stable for every pair $x, y \in X$. (Note that $\{W(x, y, t) : t \in I\} = C(\{x, y\})$ is convex by Lemma 2.7.)

3.3 THEOREM: *Let X be a strongly convex metric space. The SCS on X is stable iff every convex subset of X is stable.*

Proof: If every convex subset of X is stable, it is clear that the SCS is stable.

Conversely, suppose that the SCS is stable and let $S \subset X$ be convex. If

$r>0$, let $x_0, y_0 \in S_r$. We can then find $x_1, y_1 \in S$ with $d(x_0, x_1)<r$ and $d(y_0, y_1)<r$. Since S is convex, if we let $J=\{W(x_1, y_1, t): t\in I\}$, we must have $J\subset S$. But J is stable, so that J_r is convex. Moreover, $x_0, y_0\in J_r$. Thus $\{W(x_0, y_0, t): t\in I\}\subset J_r$. Clearly, $J_r\subset S_r$, and this implies that $\{W(x_0, y_0, t): t\in I\}\subset S_r$, *i.e.*, S_r is convex.

3.4 THEOREM: *If X has stable SCS, then the convex hull of any precompact subset of X is precompact.*

Proof: Let $S\subset X$ be precompact, and let $r>0$ be given. Choose a finite set $F\subset X$ in such a way that $S\subset F_t$, where $t=r/2$. Since F is finite, $C(F)$ is compact, and there is a finite set $T\subset X$ such that $C(F)\subset T_t$. But then $S\subset F_t\subset [C(F)]_t$, which is convex. Hence, $C(S)\subset[C(F)]_t\subset[T_t]_t\subset T_r$, and $C(S)$ is precompact.

The reader should note that the convex hull of a compact set need not be compact, even under the stability assumption. For let X be the subspace of l^2 which consists of all sequences which are zero from some point on. If $\{e_n\}_{n=1}^{\infty}$ denotes the usual basis for l^2, then $S=\{0\}\cup\{n^{-1}e_n\}_{n=1}^{\infty}\subset X$ is a compact set whose convex hull is not compact.

3.5 DEFINITION: Let (X, d) be a metric space, and let $S\subset X$ be a bounded subset of X. Then $m(S)$, the *measure of non-compactness of* S, is the real number defined by

$$m(S)=\inf\{r>0: S\subset Q_r \text{ for some precompact } Q\subset X\}.$$

The above notion of measure of non-compactness has been called the Hausdorff measure of non-compactness by some authors, and the ball measure of non-compactness by others. For references, as well as a treatment of this and other measures of non-compactness, see Sadovskii [7]. For a somewhat novel approach to the idea of measuring the way in which a set fails to be compact, see [5]. A non-numerical measure of non-compactness was used in [4] to obtain the analogue for locally convex topological vector spaces of Theorem 4.2, below.

We digress momentarily to give an alternate description of m.

3.6 PROPOSITION: *Let (X, d) be a metric space, and let m be the measure of non-compactness on X. If $S\subset X$ is bounded, then*

$$m(S)=\inf\{r>0: S\subset F_r \text{ for some finite } F\subset X\}. \tag{4}$$

Proof: Let $a(S)$ denote the quantity on the right side of (4). Clearly $m(S)\leqq a(S)$, so let $r>m(S)$ be given. We choose a precompact set $Q\subset X$ so that $S\subset Q_r$, and let $\varepsilon>0$ be arbitrary. We then choose a finite set $F\subset X$ in such a way that $Q\subset F_\varepsilon$. But then $S\subset Q_r\subset[F_\varepsilon]_r\subset F_{r+\varepsilon}$. Since $\varepsilon>0$ was arbitrary, we see that $a(S)\leqq r$. But $r>m(S)$ was arbitrary, so it follows that $a(S)\leqq m(S)$, and the proposition is proved.

It is clear that a (bounded) subset S of a metric space X is precompact iff $m(S)=0$. The following theorem is thus an extension of Theorem 3.4.

3.7 THEOREM: *Let X be a strongly convex metric space with stable SCS. If $S\subset X$ is bounded, then $m(S)=m(C(S))$.*

Proof: Let $r>m(S)$. There is a precompact set $Q\subset X$ such that $S\subset Q_r$. By Theorem 3.3, $C(Q)$ is stable, so $[C(Q)]_r$ is convex. Thus $S\subset Q_r\subset[C(Q)]_r$, and it follows that $C(S)\subset[C(Q)]_r$. But, by Theorem 3.4, $C(Q)$ is precompact, so that $m(C(S))\leqq m(S)$. The reverse inequality is clear, and we conclude that $m(S)=m(C(S))$.

Observe that we could have based the proof of Theorem 3.7 on Prop. 3.6 and Theorem 3.1, and then deduced Theorem 3.4 as a corollary.

4. Condensing multifunctions; a fixed point theorem.

Recall that a multifunction $F: X\to Y$ is a function which assigns to each $x\in X$ a subset $F(x)$ of Y. If $F: X\to Y$ is a multifunction, and $S\subset X$, then by $F(S)$ we mean the set $\{y\in Y: y\in F(x) \text{ for some } x\in S\}$. The graph of F is the set $\{(x, y): y\in F(x)\}\subset X\times Y$. A fixed point of a multifunction $F: X\to X$ is a point which satisfies $x\in F(x)$.

4.1 DEFINITION: Let (X, d) be a metric space, m the measure of non-compactnes on X, and $F: X\to X$ a multifunction. We say that F is *condensing* if for every bounded subset $S\subset X$, the relation $m(S)>0$ implies that $m(F(S))<m(S)$.

4.2 THEOREM: *Let X be a complete strongly convex metric space whose SCS is stable, and let $F: X\to X$ be a condensing multifunction with convex values, closed graph, and bounded range. Then F has a fixed point.*

Proof: The argument parallels, in large measure, that of Himmelberg, Porter, and Van Vleck in [4]. By hypothesis, $F(X)$ is bounded. Let S be the closure of the convex hull of $F(X)$. Then $F(S)\subset S$, and S is bounded, complete, and convex.

Let $x\in S$, and let $A=\{x\}\cup F(x)\cup F^2(x)\cup F^3(x)\cup\cdots$. Then $A\subset S$ is bounded, and $A=\{x\}\cup F(A)$. Hence, $m(A)=m(\{x\}\cup F(A))=m(F(A))$. Since F is condensing, A must be precompact. Thus, $cl\, A$ is compact. By Lemma 1 of [4], there is a non-empty compact subset K of S such that $F(K)\supset K$.

Now put $S_0=S$, and if α is an ordinal, put

$$S_\alpha=\begin{cases} C(F(S_{\alpha-1})) & \text{when } \alpha \text{ is not a limit ordinal,} \\ \bigcap_{\beta<\alpha} S_\beta & \text{when } \alpha \text{ is a limit ordinal.} \end{cases}$$

Then for every α,

(i) S_α is a bounded convex subset of S;
(ii) $F(S_\alpha)\subset S_\alpha$;
(iii) $S_\alpha\supset K\neq\phi$.

It follows that there is an ordinal, γ, such that $S_{\gamma+1}=C(F(S_\gamma))=S_\gamma$, and $m(F(S_\gamma))=m(C(F(S_\gamma)))=m(S_{\gamma+1})=m(S_\gamma)$. Since F is condensing, S_γ is precompact and $cl\, S_\gamma$ is a convex compact subset of S.

Now define a multifunction $G: cl\, S_\gamma\to cl\, S_\gamma$ by $G(x)=F(x)\cap cl\, S_\gamma$ for every $x\in cl\, S_\gamma$. G has convex values and closed graph. Moreover, $cl\, S_\gamma$ is compact and acyclic (in fact, contractible). By Theorem 4 of [3], $cl\, S_\gamma$ is an ANR. Since $G(x)$ is convex for each $x\in cl\, S_\gamma$, G has acyclic images. It therefore follows from Theorem 1 of [2] that G has a fixed point, which is also a fixed point for F.

REFERENCES

[1] J. DYDAK, On Convex Metric Spaces, Bull. Acad. Polonaise Sci., Serie sci., math., astr., phys., **20** (1972), 667–672.

[2] S. EILENBERG AND D. MONTGOMERY, Fixed Point Theorems for Multi-valued Transformations, Amer. J. Math., **68** (1946), 214–222.

[3] C.J. HIMMELBERG, Some Theorems on Equiconnected and Locally Equiconnected Spaces, Trans. Amer. Math. Soc., **115** (1965), 43–53.

[4] C.J. HIMMELBERG, J.R. PORTER AND F.S. VAN VLECK, Fixed Point Theorems for Condensing Multifunctions, Proc. Amer. Math. Soc., **23** (1969), 635–641.

[5] G.S. JONES, A Functional Approach to Fixed-Point Analysis of Noncompact Operators, Math. Systems Theory, **6** (1972), 375–382.

[6] H.V. MACHADO, A Characterization of Convex Subsets of Normed Spaces, Kodai Math. Sem. Rep., **25** (1973), 307–320.

[7] B.N. SADOVSKII, Limit-Compact and Condensing Operators, Russian Math. Surveys, **27** (1972), 85–155.

[8] W. TAKAHASHI, A Convexity in Metric Space and Nonexpansive Mappings, I, Kodai Math. Sem Rep., **22** (1970), 142–149.

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