ON THE LINEAR TRANSLATABLE STOCHASTIC FUNCTIONAL EQUATION.

By Takuji ONOYAMA

1. Introduction.

Our problem is how to study a special solution of the linear translatable stochastic functional equation;

$$(1.1) \qquad \Lambda f(x,\omega) \equiv \int_0^1 f(x+t,\omega)\, d\varphi(t) = g(x,\omega),$$

where

1°, Λ is a linear translatable operator,

2°, $g(x,\omega)$ is a given strictly stationary stochastic process, and

3°, $\int_0^1 \cdot\, d\varphi$ is defined as Bochner's integral according to the operator Λ.

The object of this paper is to study especially the case when zero points of generating function $G(\lambda)$ $(\Lambda e^{\lambda x} \equiv G(\lambda) e^{\lambda x})$ of Λ are only pure imaginary, because other cases are not so difficult.

Here we have to note that N.Wiener's[1] and T.Kitagawa's[2] method in the pure functional scheme are not always adoptable as they are.

2. Preliminary.

1°, We put here:

$$(2.1) \qquad G(\lambda) = \int_0^1 e^{\lambda t}\, d\varphi(t)$$

where

$G(\lambda)$ is an integral function, and let λ_0 be a zero point of order $k\ (k \geq 0)$ of $G(\lambda)$, then we can write following:

$$(2.2) \qquad \frac{(\lambda-\lambda_0)^k e^{(\lambda-\lambda_0)h}}{G(\lambda)} = \sum_{s=0}^{\infty} B^k_{s,\lambda_0}(h)(\lambda-\lambda_0)^s$$

$$(|\lambda-\lambda_0| < \rho(\lambda_0))$$

with $\rho(\lambda_0)$ which is the distance from λ_0 to the other nearest zero point of $G(\lambda)$ on the imaginary axis.

$\{B^k_{s,\lambda_0}(h)\}$ is the sequence of the generalized Bernoulli's polynomials[3].

Lemma 1[4]

$$(2.3) \qquad \Lambda B^k_{s,\lambda_0}(h) e^{\lambda_0 h} = \begin{cases} \dfrac{h^{s-k}}{(s-k)!} e^{\lambda_0 h}, & (s=k, k+1, \cdots) \\ 0 & (s=0,1,\cdots k-1) \end{cases}$$

$$(2.4) \qquad B^k_{s,\lambda_0}(h_1+h_2) = \sum_{\nu=0}^{s} \frac{h_2^{\nu}}{\nu!} B^k_{s-\nu,\lambda_0}(h_1) \qquad (s=0,1,2,\cdots)$$

2°, Regarding $f(x,\omega)$, $g(x,\omega)$ as two strictly stationary stochastic processes, we have the following definition:

$$(2.5) \qquad \text{distance } (f,g) \equiv \|f-g\| \equiv \sqrt{\int_\Omega |f(x,\omega)-g(x,\omega)|^2 dP}$$

We call this ω norm.

Lemma 2.[5] A strictly stationary stochastic process $g(x,\omega)$ and its autocorrelation coefficient $R_{|s-t|}$ are represented as follows.

$$(2.6) \qquad g(x,\omega) = \int_{-\infty}^{\infty} e^{i\lambda x} dS(\lambda,\omega)$$

$$(2.7) \qquad R_{|s-t|} = \int_{-\infty}^{\infty} e^{i\lambda(s-t)} dF(\lambda)$$

$$-1 \leq R_{|s-t|} \leq 1,$$

where $S(\lambda,\omega)$ is a differential process, and $F(\lambda)$ is a spectre function defined by $S(\lambda,\omega)$.

Lemma 3[6] If $\int_{-\infty}^{\infty} |\varphi(t)|^2 dF(\lambda) < \infty$, then

$$\int_{-\infty}^{\infty} \varphi(\lambda) e^{it\lambda} dS(\lambda,\omega) \quad \text{and} \quad R_{|t|} = \int_{-\infty}^{\infty} e^{i\lambda t} dF(\lambda)$$

exist.

Let λ_i $(i=0,1,2,\cdots)$ be zero points of $G(\lambda)$ on imaginary axis, and be non dense in any interval on imaginary axis, then the interval $(-\infty,\infty)$ can be devided into the direct sum $(I_1 \oplus I_2 \oplus \cdots)$ of enumerable subintervals I_i $(i=0,1,2,\cdots)$ by $\rho(\lambda_i)$ $(i=0,1,2,\cdots)$ in (2.2).

From (2.6), we can write

$$(2.8)\quad g(x,\omega)=\int_{-\infty}^{\infty}e^{i\lambda x}\,dS(\lambda,\omega)=\sum_{n=0}^{\infty}\int_{I_n}e^{i\lambda x}\,dS(\lambda,\omega)\equiv\sum_{n=0}^{\infty}g_n(x,\omega),$$

where $g_n(x,\omega)=\int_{I_n}e^{i\lambda x}\,dS(x,\omega).$

3. Main Theorem.

Theorem I. The stochastic functional equations which are obtained from (1.1) and (2.8)

$$(3.1)\quad \Lambda f_n(x,\omega)=g_n(x,\omega)\quad (n=0,1,2,\cdots)$$

have the special solutions

$$(3.2)\quad f_n(x,\omega)=\int_{I_n}\sum_{s=0}^{k-1}B^{k_n}_{s,i\lambda_n}(0)\,(D_x-i\lambda_n)^s \times\Big\{e^{i\lambda x}-\sum_{\ell=0}^{k-1}\frac{(i\lambda x-i\lambda_n x)^{\ell}}{\ell!}e^{i\lambda x}\Big\}\frac{1}{(i\lambda-i\lambda_n)^k}\,dS(\lambda,\omega)$$
$$+\int_{I_n}\sum_{s=k}^{\infty}B^{k_n}_{s,i\lambda_n}(0)(i\lambda-i\lambda_n)^{s-k}e^{i\lambda x}\,dS(\lambda,\omega)\quad (n=0,1,2,\cdots\cdots)$$

where zero points of $G(\lambda)$ are pure imaginary, and $(D_x-i\lambda_n)$ $(\equiv(\frac{d}{dx}-i\lambda_n))$ is a differential operator.

Here we have to prove a lemma before a proof of Theorem I.

Lemma 4. In any finite interval of x

$$\Lambda\int_{I_n}h(x,\lambda)\,dS(\lambda,\omega)=\int_{I_n}\Lambda h(x,\lambda)\,dS(\lambda,\omega),$$

where $h(x,\lambda)$ is a uniformly bounded function on $\lambda\in I_n$ and any finite interval of x.

Proof

$$\Big\|\int_{I_n}h(x,\lambda)\,dS(\lambda,\omega)\Big\|^2=\int_{\Omega}\Big(\int_{I_n}h(x,\lambda)\,dS(\lambda,\omega)\Big)^2dP$$
$$=\int_{\Omega}\Big\{\int_{I_n}h(x,\lambda)\,dS(\lambda,\omega)\cdot\int_{I_n}\overline{h(x,\mu)}\,dS(\mu,\omega)\Big\}dP$$
$$=\int_{I_n}h(x,\lambda)\overline{h(x,\mu)}\cdot\int_{\Omega}dS(\lambda,\omega)\,dS(\mu,\omega)\,dP$$
$$=\int_{I_n}h(x,\lambda)\overline{h(x,\lambda)}\,dF(\lambda)\quad(\text{by Lemma 2.2})$$
$$\leqq \underset{\lambda\in I_n}{\text{l.u.b}}\,|f(x,\lambda)|^2\{F(\lambda_n+\rho(\lambda_n))-F(\lambda_n-\rho(\lambda_n))\}$$

in any finite interval of x.

$< K$ (= absolute constant).

Therefore, by the property of Bochner Integral

$$\Lambda\int_{I_n}h(x,\lambda)\,dS(\lambda,\omega)=\int_0^1\int_{I_n}h(x+t,\lambda)\,dS(\lambda,\omega)\,d\varphi(t)$$
$$=\int_{I_n}\int_0^1h(x+t,\lambda)\,d\varphi(t)\,dS(\lambda,\omega)=\int_{I_n}\Lambda h(x,\lambda)\,dS(\lambda,\omega).$$

Proof of Theorem 1.

Since $e^{i\lambda_n x}\int_0^x g_n(x,\omega)e^{-i\lambda_n t}dt$ exists with ω^2-norm, we can put

$$(3.3)\quad g_n^{(-1)}(x,\omega)\equiv e^{i\lambda_n x}\int_0^x g_n(t,\omega)e^{-i\lambda_n t}dt$$
$$h_n(x,\omega)\equiv e^{i\lambda_n x}\int_0^x\frac{(x-t)^{k-1}}{(k-1)!}g_n(t,\omega)e^{-i\lambda_0 t}dt,$$

Then we have easily

$$(3.4)\quad (D_x-i\lambda_n)^k h_n(x,\omega)=g_n(x,\omega)$$

$$(3.5)\quad g_n^{(-1)}(x,\omega)\equiv e^{i\lambda_n x}\int_0^x g_n(t,\omega)e^{-i\lambda_n t}dt$$
$$=e^{i\lambda_n x}\int_0^x\Big\{\int_{I_n}e^{it(\lambda-\lambda_n)}dS(\lambda,\omega)\Big\}dt$$
$$=\int_{I_n}\frac{e^{i\lambda x}-e^{i\lambda_n x}}{(i\lambda-i\lambda_n)}\,dS(\lambda,\omega)$$

with ω^2-norm.

In the same way

$$(3.6)\quad h_n(x,\omega)=\int_{I_n}\frac{\Big\{e^{i\lambda x}-\sum_{\ell=0}^{k-1}\frac{\{i(\lambda-\lambda_n)\}^{\ell}}{\ell!}e^{i\lambda_n x}\Big\}}{(i\lambda-i\lambda_n)^k}\,dS(\lambda,\omega)$$

with ω^2-norm.

Let s be any positive integer, then $\int_{I_n}(i\lambda)^s e^{i\lambda x}\,dS(x,\omega)$ always exists, and is considered as $(k+s)$'s differential of $h_n(x,\omega)$ (or s' differential of $g_n(x,\omega)$).

And so

$$\|(D_x-i\lambda_n)^{k+s}h_n(x,\omega)\|^2=\Big\|\int_{I_n}(i\lambda-i\lambda_n)^{k+s}e^{i\lambda x}\,dS(\lambda,\omega)\Big\|^2$$
$$\leqq\int_{I_n}|i\lambda-i\lambda_n|^{2(s+k)}\,dF(\lambda)$$
$$\leqq\rho^{2(s+k)}(\lambda_n)\int_{I_n}dF(\lambda)$$
$$=\rho^{2(s+k)}(\lambda_n)\{F(\lambda_n+\rho(\lambda_n))-F(\lambda_n-\rho(\lambda_n))\}$$

(not relative to x)

We put

$$f_{n,N} \equiv \sum_{S=0}^{N} B_{S,i\lambda_n}^{k}(0)(D_x - i\lambda_n)^S h_n(x,\omega)$$

(3.7) $$f_n(x,\omega) \equiv \sum_{S=0}^{\infty} B_{S,i\lambda_n}^{k}(0)(D_x - i\lambda_n)^S h_n(x,\omega),$$

then $f_{n,N}$ converges to $f_n(x,\omega)$ uniformly in any finite interval of x with ω^2-norm, because $s \geq M$

$$\|f_{n,N} - f_{n,M}\|^2 \leq \left\{ \sum_{S=M}^{N} |B^k(0)| \rho^{2S}(\lambda_n) F(\lambda_n + \rho(\lambda_n)) - F(\lambda_n - \rho(\lambda_n)) \right\}$$

this convergency is easily seen from (2.2).

We have, therefore, from (3.6), (3.8)

$$\sum_{S=0}^{\infty} B_{S,i\lambda_n}^{k_n}(0)(D_x - i\lambda_n)^S h_n(x,\omega)$$

$$= \int_{I_n} \sum_{S=0}^{\infty} B_{S,i\lambda_n}^{k}(0)(D_x - i\lambda_n)^S \frac{e^{i\lambda x} - \sum_{\ell=0}^{k-1} \frac{i(\lambda - \lambda_n)}{\ell!} e^{i\lambda_n x}}{(i\lambda - i\lambda_n)^k} dS(\lambda,\omega)$$

and

$$f_n(x,\omega) = \int_{I_n} \sum_{S=0}^{k-1} B_{S,i\lambda_n}^{k_n}(0)(D_x - i\lambda_n)^S \left\{ e^{i\lambda x} - \sum_{\ell=0}^{k-1} \frac{(i\lambda x - i\lambda_n x)}{\ell!} e^{i\lambda x} \right\} \frac{1}{(i\lambda - i\lambda_n)^k} dS(\lambda,\omega)$$

$$+ \int_{I_n} \sum_{S=k}^{\infty} B_{S,i\lambda_n}^{k_n}(0)(i\lambda - i\lambda_n)^{S-k} e^{it\lambda} dS(\lambda,\omega)$$

$$\equiv f_{n,1}(x,\omega) + f_{n,2}(x,\omega).$$

Next we prove

$$\Lambda f_n(x,\omega) = \Lambda f_{n,1} + \Lambda f_{n,2} \; (\equiv F_1 + F_2)$$

$$= g_n,$$

Where

Λ is the integral operator of (1.1) in the sence of Bochner integral.

$$F_1 = \Lambda_x \int_{I_n} \sum_{S=0}^{k-1} B_{S,i\lambda_n}^{k_n}(0)(D_x - i\lambda_n)^S V \, dS(\lambda,\omega)$$

$$F_2 = \Lambda_x \int_{I_n} \sum_{S=k}^{\infty} B_{S,i\lambda_n}^{k_n}(0)(D_x - i\lambda_n)^{S-k} e^{it\lambda} dS(\lambda,\omega),$$

$$V \equiv \left\{ e^{i\lambda x} - \sum_{\ell=0}^{k-1} \left\{ \frac{i(\lambda x - \lambda_n x)^\ell}{\ell!} e^{i\lambda_n x} \right\} \frac{1}{(i\lambda - i\lambda_n)^k} \right.$$

$$(D_x - i\lambda_n)^S V = (i\lambda - i\lambda_n)^S e^{i\lambda_n x} - \sum_{\ell=0}^{k-1} (D_x - i\lambda_n)^S \left\{ \frac{(i\lambda x - i\lambda_n x)^\ell}{\ell!} e^{i\lambda_n x} \right\} \frac{1}{(i\lambda - i\lambda_n)^k}$$

$$= \left[(i\lambda - i\lambda_n)^S e^{i\lambda x} - \sum_{\ell=S}^{k-1} \frac{(i\lambda x - i\lambda_n x)^{\ell-S} (i\lambda - i\lambda_n)}{(\ell - S)!} e^{i\lambda_n x} \right] \cdot \frac{1}{(i\lambda - i\lambda_n)^k}.$$

By uniform boundedness of f_n. of (3.7) in finite interval of x, and Lemma 4

$$F_1 = \int_{I_n} \Lambda \left(\sum_{S=0}^{k_n - 1} B_{S,i\lambda_n}^{k_n}(0)(D_x - i\lambda_n)^S V \right) dS(\lambda,\omega)$$

$$= \int_{I_n} \Lambda \sum_{S=0}^{k_n-1} B_{S,i\lambda_n}^{k_n}(0) \left\{ (i\lambda - i\lambda_n)^S e^{i\lambda x} - \sum_{\ell=S}^{k-1} \frac{(i\lambda x - i\lambda_n x)^{(\ell-S)} (i\lambda - i\lambda_n)^S}{(\ell - S)!} e^{i\lambda_n x} \right\} \cdot \frac{1}{(i\lambda - i\lambda_n)^{k_n}}$$

$$= \int_{I_n} dS(\lambda,\omega) \left\{ \sum_{S=0}^{k_n-1} B_{S,i\lambda_n}^{k_n}(0) \left\{ (i\lambda - i\lambda_n)^S G(i\lambda) e^{i\lambda x} - P(\lambda,x) \right\} \cdot \frac{1}{(i\lambda - i\lambda_n)^{k_n}} \right\}$$

Here

$$P(\lambda,x) = \Lambda \left\{ \sum_{S=0}^{k-1} B_{S,i\lambda_n}^{k_n}(0) \sum_{\ell=0}^{k_n-1} \frac{(i\lambda x - i\lambda_n x)^{\ell-S} (i\lambda - i\lambda_n)^S}{(\ell - S)!} \cdot e^{i\lambda_n x} \right\}$$

$$= \Lambda \sum_{S=0}^{k_n-1} \sum_{\ell=0}^{k_n-1} B_{S,i\lambda_n}^{k_n}(0) \frac{(i\lambda x - i\lambda_n x)^{\ell-S} (i\lambda - i\lambda_n)^S}{(\ell - S)!} e^{i\lambda_n x}$$

$$= \sum_{\ell=0}^{k_n-1} \Lambda B_{\ell,i\lambda_n}^{k_n}(x) (i\lambda - i\lambda_n)^\ell e^{i\lambda_n x}$$

$$= 0$$

By Lemma 2.

And so

$$\Lambda f_n(x,\omega) = F_1 + F_2$$

$$= \int_{I_n} e^{i\lambda x} dS(\lambda,\omega)\Big\{\sum_{s=0}^{k_n-1} B^{k}_{s,i\lambda_n}(0)(i\lambda - i\lambda_n)^s G(i\lambda)\Big\} \frac{1}{(i\lambda - i\lambda_n)^{k_n}}$$

$$+ \int_{I_n} e^{i\lambda x} dS(\lambda,\omega)\Big\{\sum_{s=k}^{\infty} B^{k_n}_{s,i\lambda_n}(0)(i\lambda - i\lambda_n)^{s-k} G(i\lambda)\Big\}$$

$$= \int_{I_n} e^{i\lambda x} dS(\lambda,\omega)\Big\{\sum_{s=0}^{\infty} B^{k_n}_{s,i\lambda_n}(0)(i\lambda - i\lambda_n)^{s-k} G(i\lambda)\Big\}$$

$$= g_n(x,\omega)$$

by (2.2).

Theorem is proved.

We can not get so easily some condition for convergence of $\sum_{n=0}^{\infty} f_n(x,\omega)$,

But it is easily seen that, if K such that $|\varphi_n(\lambda,x)| \leq K$ (= absolute constant $< \infty$)

$$f_n(x,\omega) \equiv \int_{I_n} \varphi_n(\lambda,x)\, dS(\lambda,\omega)$$

exists uniformly in any finite interval of x and no relative to n and λ ,

then $\sum_{n=0}^{\infty} f_n(x,\omega)$ converges uniformly in any finite interval of x with respect to ω^2 -norm.

4.

When no zero point of $G(\lambda)$ is pure imaginary, we have the following result.

$G(\lambda)$ is written as the sum of three parts, $H(\lambda)$, $F(\lambda)$ and $S(\lambda)$, step, absolutely continuous, and singular respectively. Then

Theorem II. If $\underline{Bd}\, |G(i\xi)| > 0$ (1°)

$$\int_E |dS(i\xi)| < \underline{Bd}\, |H(i\xi)| \quad (2°)$$

ξ : real
E : a certain interval on the real axis

Then, the functional equation (1.1) has the special solution

$$f(x,\omega) = \int_E g(x-t,\omega)\, d\alpha(t)$$

where $\alpha(t)$ is defined as

$$\frac{1}{G(i\xi)} = \int_E e^{-i\xi t} d\alpha(t)$$

Proof

By the condition (1°) and (2°) there exist E and $\alpha(t)$ such that (7)

$$\frac{1}{G(i\xi)} = \int_E e^{-i\xi t} d\alpha(t)$$

$$\Lambda f(x,\omega) = \int_0^1\int_E g(x-t+s,\omega)\, d\alpha(t)\, d\varphi(s)$$

$$= \int_0^1\int_E\int_{-\infty}^{\infty} e^{i\lambda(x-t+s)} dS(\lambda,\omega)\, d\alpha(t)\, d\varphi(s)$$

$$= \int_{-\infty}^{\infty} e^{i\lambda x} dS(\lambda,\omega)\int_0^1 e^{i\lambda s} d\varphi(s)\int_E e^{-i\lambda t} d\alpha(t)$$

(by Lemma 4)

$$= g(x,\omega)\, G(i\lambda)\cdot\frac{1}{G(i\lambda)}$$

$$= g(x,\omega).$$

When no zero points of $G(\lambda)$ is pure imaginary, the condition I° is satisfied.

In the Wold's Theorem (8), I° and II° are satisfied. In this sence, Theorem II gives the generalization of Wold's Theorem.

(*) Received July 4th, 1949.

(1) Wiener, N., The solution of a difference equation by trigonometrical integrals. Jour.Math.Phys. M.I.T. 4(1926).
(2) Kitagawa, T., The special solution of the non-homogeneous linear translatable functional equation. The TOHOKU Math.Jour. Vol.43, Part II.
(3) Kitagawa, T., On the theory of linear translatable functional equation and Cauchy's series. Japanese Jour.Math. Vol.XIII, 1937.
(4) See (3).
(5) Maruyama, G., The memoirs of the faculty of Science, Kyusyu University Vol.2, No.1.
(6) See (5).
(7) Wiener, N. and Pitt, R.H., On absolutely convergent Fourier-Stietjes transform. Duke Math.Jour.1938.
(8) Wold, H., A study in the analysis of stationary time series, Uppsala 1938.

Tokyo Institute of Technology.