A NOTE ON NORMED RING.

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1. I. Gelfand has shown in his first paper on Normierte Ringe (Recueil Mathematique, T.9 (51), 1941) that if R satisfies four conditions (α), (β), (γ), and (δ) given below, then R is algebraically isomorphic and topologically homeomorphic to R' with the same three conditions (α), (β), (γ), and (δ') which is strictly stronger than (δ).

According to his proof, he assumed commutativity of R or, at least, the existence of right unit element of R. In this note, we shall show that his assertion is still valid in the case without assumption such as commutativity of R.

It is to be mentioned, however, that our condition (γ) has a right unit element, while Gelfand's (γ) has a left unit element.

2. Let R be a set of elements x, y, z, . . . which satisfies the following four conditions (α), (β), (γ) and (δ).

(α) R is a Banach space with complex numbers as its coefficient field.

(β) R is a ring:

$$x(\lambda y + \mu z) = \lambda x y + \mu x z$$

(λ, μ are complex numbers),

$$x(yz) = (xy)z$$

(γ) R has a right unit element e :

$$xe = x$$

moreover $\|e\| \neq 0$

(δ) Operation of Multiplication is continuous, i.e.,

$$x_n \to x \text{ implies } y x_n \to y x,$$

$$\text{and } x_n y \to x y.$$

Let Q be a Banach space of all linear operators on R into R itself. And let R' be the totality of A_x in Q such that

$$A_x y = x y,$$

i.e., $$R' = (A_x ; x \in R)$$

Then, for the mapping $\varphi : x \longleftrightarrow A_x$ between R and R', we can easily show that

(1) $x \neq x'$ implies $A_x \neq A_{x'}$,

which evidently asserts a one-to-one mapping of φ between R and R'.

(2) φ is algebraic isomorphism.

(3) φ is continuous from R' onto R.

(4) R' is closed in Q; thus R' is complete.

Therefore by the known theorem of Banach,

(5) φ is continuous from R onto R'.

We can then conclude that

R <u>and</u> R' <u>are isomorphic and homeomorphic, and moreover</u> R' <u>satisfies the stronger conditions</u>

(γ') $\|e\| = 1.$

(δ') $\|xy\| \leq \|x\| \cdot \|y\|$

Proof (1)

If $x \neq x'$, then

$$A_x e = xe = x \neq x' = x'e = A_{x'} e$$

Hence $A_x \neq A_{x'}$

In the case of Gelfand, (1) is not satisfied, and we shall give its counter example at the end of this note (4. (b)).

(2) Obvious.

(3) By the inequality $\|A_x\| \geq \frac{1}{\|e\|}\|x\|$.

(4) If $A_{x_n} \to A \in Q$, then $\{x_n\}$ is a Cauchy sequence, for

$$\|x_n - x_m\| \leq \|e\| \|A_{x_n} - A_{x_m}\| \to 0$$

$$(n, m \to \infty)$$

R being complete, there exists an element $x \in R$, such that

$$x_n \to x \quad (n \to \infty)$$

For any element $y \in R$

$$x_n y \to x y \quad (\text{by } (\delta))$$

and $A_{x_n} y \to A y$ (by assumption),

i.e., $x_n y \to A y$

Hence it must be

$$A_x = xy$$

Thus

$$A = A_x \in R'$$

This asserts the closedness of R' in Q.

(5) By (1), (3) and (4).

3. Let

$$M = (x;\ ex = x)$$

$$N = (x;\ ex = 0)$$

Then M is a right ideal with e as a unit element, and N is an ideal. To be explicit

$$M^2 = M,\ NM = N,\ RN = (0),$$

and $R = M + N$ (direct sum).

According as the direct decomposition of R, R' can be expressed in a matrix form such that

$$A_z = \begin{pmatrix} B_x & 0 \\ C_y & 0 \end{pmatrix}$$

where $z = x + y,\ x \in M,\ y \in N,$

and for any $w \in M$, $B_x w = xw,\ C_y w = yw.$

4. Examples.

(a) Let

$$R = ((\lambda, \mu);\ \lambda, \mu \text{ any complex numbers})$$

and $\|(\lambda, \mu)\| = |\lambda| + |\mu|$

Then R is a Banach space with respect to this norm.

By the operation of multiplication $(\lambda, \mu)\cdot(\lambda', \mu') = (\lambda\lambda', \mu\lambda')$, R forms a normed ring.

In this case (1,0) is a right unit element, but not a left unit element, for

$$(1,0)(\lambda', \mu') = (\lambda', 0)$$

And

$$A_{(\lambda,\mu)} = \begin{pmatrix} \lambda, & 0 \\ \mu, & 0 \end{pmatrix}$$

(b) In place of the multiplication in (a), a little change of multiplication such that

$$(\lambda, \mu)(\lambda', \mu') = (\lambda\lambda', \lambda\mu')$$

yields a counter example of Gelfand s case.

(*) Received July 12, 1949.

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