TWO REMARKS ON H.WEYL'S THEOREMS.

By Yukiyosi KAWADA

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This note consists of two independent parts. In §1 we concern with the paper: H.Weyl, Almost periodic invariant vector sets in a metric vector space, Amer. J. Math., 71 (1949) (W. I). In this paper the whole theory was established upon 4 axioms. First the Parseval equation was proved by assuming Axioms I,II,III and then the approximation theorem was proved by assuming a further Axiom IV. We shall prove here first the approximation theorem without Axiom IV and then we shall obtain the Parseval equation. In §2 we shall give a simple proof of Lemma 2 of the paper: H.Weyl, The method of orthogonal projection in potential theory, Duke Math. J., 7 (1940) (W.II).

§1. We start from the following axioms:

Axiom I (W.I, p.178) Let Σ be a complex vector space and $\sigma=\{s\}$ be a group of linear transformations on Σ : $f \to f' = sf$.

Axiom III (W.I, p.180) Σ is a Banach space w.r.t. the norm $|f|$, and $s\in\sigma$ is an isometric transformation on Σ : $|f| = |sf|$.

$f\in\Sigma$ is called almost periodic (a.p.) if $\{sf;\ s\in\sigma\}$ is a totally bounded subset of Σ. We denote by Σ_f^o the smallest invariant closed subspace of Σ which contains f. Clearly Σ_f^o is the closure of all the elements $\sum \xi_i(s_i f)$ (finite sum).

Let $\sigma \ni s \to \Omega(s) = \|\omega_{ij}(s)\|$ be a matric representation of σ with degree h. We call $\mathfrak{g} = (g_1,\dots,g_k)$ ($g_i\in\Sigma$) an Ω-row if

(1) $s(g_1,\dots,g_k)=(g_1,\dots,g_k)\Omega(s),\ s\in\sigma$

Especially $\mathfrak{g}$ is called a primitive row if Ω is a unitary irreducible representation of σ.

Theorem 1. (Theorem of strong approximation, W.I, p.199) "Let $f\in\Sigma$ be a.p. Then f can be approximated w.r.t. the norm with arbitrary accuracy by finite sums of the elements of primitive rows."

We shall first notice that the function $F_f(s) = sf$ with domain σ and with range in Σ is a.p. in the sense of Bochner-Neumann [1] (B.N.) if and only if $f\in\Sigma$ is a.p. We denote its mean value by $\int sf$. It is easily proved (W.I, p.182) that every $f'\in\Sigma_f^o$ is also a.p. if f is a.p.

Now let $f\in\Sigma$ be a.p. and $\Omega^\lambda(s)$ be a unitary irreducible representation of σ with degree n_λ. We call g_{ij}^λ defined by

(2) $$g_{ij}^\lambda = \int_t \overline{\omega_{ji}^\lambda(t)}(tf) = \int_t \omega_{ij}^\lambda(t^{-1})(tf) \quad (i,j = 1,\dots,n_\lambda)$$

the primitive expansion coefficient (p.e.c.) of f. g_{ij}^λ belongs to Σ_f^o. Theorem 1 follows then from Lemma 1, 2:

Lemma 1. "Let $f\in\Sigma$ be a.p. and g_{ij}^λ be p.e.c. of f. Then

(3) $$\mathfrak{g}_i^\lambda = (g_{i1}^\lambda,\dots,g_{in_\lambda}^\lambda)\quad (i=1,\dots,n_\lambda)$$

are primitive rows."

Proof. Let F(s) and $\varphi(s)$ be a.p. functions taking values in Σ and complex numbers respectively. We define $F\times\varphi(s) = \int_t \varphi(t)F(st^{-1}) = \int_t\varphi(t^{-1}s)F(t)$. We have then

$$g_{ij}^\lambda = \int_t \omega_{ij}^\lambda(t)(t^{-1}f) = F_f \times \omega_{ij}^\lambda(1).$$

Hence we have

$$\begin{aligned} sg_{ij}^\lambda &= \int_t \omega_{ij}^\lambda(t)(st^{-1}f) = F_f \times \omega_{ij}^\lambda(s) \\ &= \int_t \omega_{ij}^\lambda(t^{-1}s)(tf) \\ &= \sum_{k=1}^{n_\lambda}\Big(\int_t\omega_{ik}^\lambda(t^{-1})(tf)\Big)\,\omega_{kj}^\lambda(s) \\ &= \sum_{k=1}^{n_\lambda} g_{ik}^\lambda\,\omega_{kj}^\lambda(s), \quad \text{q.e.d.}\end{aligned}$$

Lemma 2. "Let $f\in\Sigma$ be a.p. Then f can be approximated w.r.t. the norm with arbitrary accuracy by finite sums of p.e.c. of f."

Proof. By Theorem 22 of B.N. we can find for any assigned $\varepsilon>0$ a special weight function $\varphi = \sum \gamma_{ij}^\lambda\,\omega_{ij}^\lambda(s)$ such that $|sf - F_f\times\varphi(s)| < \varepsilon$ ($s\in\sigma$). If we put s=1, then $F_f\times\varphi(1) = \sum\gamma_{ij}^\lambda F_f\times\omega_{ij}^\lambda(1) = \sum\gamma_{ij}^\lambda g_{ij}^\lambda$. Hence we have $|f - \sum\gamma_{ij}^\lambda g_{ij}^\lambda| < \varepsilon$, q.e.d.

We add now

Lemma 3. "The p.e.c. $\tilde{g}^{\lambda}_{ij}$ of sf is a linear combination of the p.e.c. $\{g^{\lambda}_{ij}\}$ of f."

Proof. $\tilde{g}^{\lambda}_{ij} = \int_t \omega^{\lambda}_{ij}(t^{-1})(tsf) = \int_t \omega^{\lambda}_{ij}(st^{-1})(tf)$

$$= \sum_k \omega^{\lambda}_{ik}(s) \int_t \omega^{\lambda}_{kj}(t^{-1})(tf)$$

$$= \sum_k \omega^{\lambda}_{ik}(s) g^{\lambda}_{kj}, \qquad \text{q.e.d.}$$

Since any $f' \in \sum^{\circ}_f$ can be approximated by $\sum_i \gamma_i (s_i f)$, we can prove the following theorem from Lemma 3:

Theorem 2. "Any $f' \in \sum^{\circ}_f$ can be approximated with arbitrary accuracy by linear combinations of p.e.c. of f."

Now we assume further

Axiom II (W.I, p.179) With any two $f, g \in \sum$ there is associated a complex number (f,g) with the usual properties of the inner product, and

$$(sf, sg) = (f,g), \quad \text{and } \|f\| = \sqrt{(f,f)} \leqq |f|$$

We don't assume the completeness of $\sum$ w.r.t. $\|f\|$.

Let $g^{\lambda}_{\alpha} = (g^{\lambda}_{\alpha 1}, \cdots, g^{\lambda}_{\alpha n_\lambda})$ and $g^{\mu}_{\beta} = (g^{\mu}_{\beta 1}, \cdots, g^{\mu}_{\beta n_\mu})$ be two primitive rows. Then from the Schur's lemma follows that

$$(g^{\lambda}_{\alpha i}, g^{\mu}_{\beta j}) = 0 \quad \text{for} \quad \lambda \neq \mu$$

and

$$(g^{\lambda}_{\alpha i}, g^{\lambda}_{\beta j}) = \gamma^{\lambda}_{\alpha\beta} \delta_{ij}$$

$$(i,j = 1, \ldots, n_\lambda).$$

(Cf. W.I, p.194). Hence we can define the inner product of two primitive rows g^{λ}_{α}, g^{λ}_{β} by

(4) $$(g^{\lambda}_{\alpha}, g^{\lambda}_{\beta}) = \gamma^{\lambda}_{\alpha\beta}.$$

We consider now the primitive rows g^{λ}_{i} in Lemma 1, and we define

(5) $$\mathfrak{h}^{\lambda} = \left\{ \sum_{i=1}^{n_\lambda} \gamma_i g^{\lambda}_i \right\}.$$

Let m_λ be the rank of $\mathfrak{h}^{\lambda}$, then $m_\lambda \leqq n_\lambda$. We can take then

$$f^{\lambda}_k = \sum_{j=1}^{n_\lambda} \alpha_{kj} g^{\lambda}_j \qquad (k = 1, \cdots, m_\lambda)$$

in $\mathfrak{h}^{\lambda}$ such that

$$(f^{\lambda}_k, f^{\lambda}_j) = \delta_{kj} \qquad (k, j = 1, \ldots, m_\lambda).$$

Theorem 3. (W.I, p.196) Let $f \in \sum$ be a.p. There are at most a countable infinite set of p.e.c. $g^{\lambda}_{ij} \neq 0$. From each $\mathfrak{h}^{\lambda}$ take

$$f^{\lambda}_k = (h^{\lambda}_{k1}, \cdots, h^{\lambda}_{kn_\lambda}) \qquad (k = 1, \ldots, m_\lambda)$$

as above. Then the countable system

$$h^{\lambda}_{kj} \qquad (k = 1, \ldots, m_\lambda, \ j = 1, \ldots, n_\lambda)$$

forms a orthonormal system of $\sum^{\circ}_f$, i.e. for any $f' \in \sum^{\circ}_f$ the Parseval equation

(6) $$\|f'\|^2 = \sum_{\lambda k j} |(h^{\lambda}_{kj}, f')|^2$$

holds."

Proof. From the inequality of Bessel follows $\alpha = \|f'\|^2 - \sum |(h^{\lambda}_{kj}, f')|^2 \geqq 0$. If $\alpha > 0$, we can apply Theorem 2 for $\sqrt{\alpha}$ so that we can choose a linear combination of p.e.c. $g = \sum \gamma^{\lambda}_{ij} g^{\lambda}_{ij} = \sum \beta^{\lambda}_{kj} h^{\lambda}_{kj}$ with $|f' - g| < \sqrt{\alpha}$. Hence we have $\alpha \leqq \|f' - g\|^2 \leqq |f' - g|^2 < \alpha$, which is a contradiction, q.e.d.

We shall consider then the characterization of $\mathfrak{h}^{\lambda}$ in W.I. In the following we fix an irreducible unitary representation Ω^{λ} with degree $r = n_\lambda$ and omit the index λ. We correspond now to every primitive row g_α a vector

(7) $$a_\alpha = ((g_{\alpha 1}, f), \cdots, (g_{\alpha r}, f)).$$

g_α is called a hidden row if $a_\alpha = 0$ and g_β is called upright if $(g_\alpha, g_\beta) = 0$ for every hidden row g_α (W.I, p.194). We define

(8) $$\mathfrak{g}^{\lambda} = \{ g_\alpha = (g_{\alpha 1}, \cdots, g_{\alpha r});\ g_{\alpha i} \in \textstyle\sum^{\circ}_f,\ i = 1, \cdots, r \}.$$

Clearly we have

(9) $$\mathfrak{h}^{\lambda} \subset \mathfrak{g}^{\lambda}.$$

Lemma 4. "A necessary and sufficient condition for a primitive row g_α be hidden is that $(g_{\alpha i}, f') = 0$ $(i = 1, \cdots, r)$ for every $f' \in \sum^{\circ}_f$."

Proof.

$$a_\alpha = 0$$

$$\rightleftarrows (g_{\alpha i}, f)=0, \quad i=1,\dots,r$$
$$\rightleftarrows (g_{\alpha i}, sf)=(s^{-1} g_{\alpha i}, f)$$
$$= \sum_j \omega_{ji}(s)(g_{\alpha j}, f)=0$$
$$\rightleftarrows (g_{\alpha i}, f')=0 \quad \text{for every } f' \in \Sigma_f^{\circ}$$
$$(i=1,\dots,r), \text{q.e.d.}$$

From Lemma 4 follows immediately

Lemma 5. "If $g_\alpha \in \mathfrak{G}^\lambda$, then g_α is upright."

Lemma 6. "If g_λ is upright, then $g_\alpha \in \mathfrak{h}^\lambda$."

Proof. If $g_\alpha \notin \mathfrak{h}^\lambda$, we can decompose g_α such that

$$g_\alpha = g_\beta + g_\gamma, \quad g_\beta \neq 0, \quad g_\gamma \in \mathfrak{h}^\lambda$$
$$(g_\beta, g_i^\lambda)=0, \quad i=1,\dots,r.$$

Since g_α and g_γ are upright, so is g_β. Hence $\alpha_\beta \neq 0$ and so $(g_{\beta i}, f) \neq 0$ for some i. Now take $\|g_{\beta i}\| = 1$ and $|(g_{\beta i}, f)| > \varepsilon > 0$. By Theorem 1 we can choose $h = \sum \gamma_{ij} g_{ij}^\lambda \in \mathfrak{h}^\lambda$ so that $|f-h| < \varepsilon$. Then

$$0 < \varepsilon < |(g_{\beta i}, f)|$$
$$\leq |(g_{\beta i}, f-h)| + |(g_{\beta i}, h)| \leq \varepsilon \|g_{\beta i}\| = \varepsilon$$

which is a contradiction. Hence we have $g_\alpha \in \mathfrak{h}^\lambda$, q.e.d.

From (9), Lemma 5, Lemma 6 we have

Theorem 4. "We can characterize $\mathfrak{h}^\lambda$ in Theorem 3 as $\mathfrak{G}^\lambda$ or as the set of all the upright Ω^λ-rows."

§2. We shall give here a simple proof of the following Lemma of W.II. Let G be an open set in 3-space. A function $\varphi(x)$ is called of class Γ_2 if $\varphi(x)$ vanishes outside some compact subset G^* of G and continuous with its derivatives up to the second order.

Weyl's Lemma. "Let $\eta(x)$ be a measurable function on G with $\int_G |\eta(x)|^2 < \infty$. If $\eta(x)$ satisfies the equation

$$(10) \qquad \int \eta(x) \Delta \zeta(x) = 0$$

for every $\zeta(x)$ of class Γ_2, then there is a harmonic function $\eta^*(x)$ which is equal to $\eta(x)$ almost everywhere on G."

Proof. If $\eta(x)$ is of class C^2, i.e. continuous with its derivatives up to the second order, then $\Delta\eta = 0$ follows from (10) by the Green's formula.

Now let $\varphi(x)$ be any measurable function which is integrable on every closed sphere $S(x,r)$ with centre x and with radius r $(S \subset G)$. Let G_r be the set of all the points $x \in G$ such that $S(x,r) \subset G$. Then we define

$$(11) \qquad M_r\varphi(x) = \frac{1}{|S_r|}\int_{S(x,r)} \varphi(y)dy \quad \left(|S_r| = \tfrac{4}{3}\pi r^3\right)$$

for $x \in G_r$. It is well known that if $\varphi(x)$ is measurable, then $M_r\varphi$ is continuous; if $\varphi(x)$ is continuous then $M_r\varphi$ is of class C^1 and if $\varphi(x)$ is of class C^1 then $M_r\varphi$ is of class C^2.

If $\zeta(x)=0$ on $G - G^*$ (G^* compact), take $r < \rho(G^*, G^c)$. Since $\eta(x+y)$ is measurable w.r.t. two variables x,y we can apply Fubini's theorem:

$$\int_{G^*} M_r\eta(x)\Delta\zeta(x)dx = \frac{1}{|S_r|}\int_{G^*}\int_{|y|<r} \eta(x+y)\Delta\zeta(x)dxdy$$
$$= \frac{1}{|S_r|}\int\left(\int_{G^*}\eta(x+y)\Delta\zeta(x)dx\right)dy$$
$$= \frac{1}{|S_r|}\int\left(\int_{G^*-y}\eta(x)\Delta\zeta(x-y)dx\right)dy = 0.$$

Hence $M_r\eta(x)$ satisfies also the equation (11). Repeating this process we can conclude that

$$(12) \qquad \eta^*(x) = M_r M_s M_t \eta(x)$$

also satisfies the equation (11) in G_{r+s+t}. Since $\eta^*(x)$ is of class C^2, $\eta^*(x)$ is a harmonic function on G_{r+s+t}.

Now we fix s and t. Let us write $\eta_2(x) = M_s M_t \eta(x)$ and $\eta_r^*(x) = M_r \eta_2(x)$. By a well known theorem on the derivative of the indefinite integral we have

$$(13) \qquad \eta_2(x) = \lim_{r\to 0} M_r\eta_2(x) = \lim_{r \to 0}\eta_r^*(x)$$
almost everywhere.

On the other hand the harmonic function $\eta_r^*(x)$ satisfies the relation

$$M_s\,\eta_r^*(x) = \eta_r^*(x) \quad \text{for every } s > 0,$$

i.e. $M_s M_r \eta_2(x) = M_r \eta_2(x)$. Since

$$M_s M_r \eta_2(x) = \frac{1}{|S_r|}\frac{1}{|S_s|}\int_{|y|<r}\int_{|z|<s} \eta_2(x+y+z)\,dy\,dz = M_r M_s \eta_2(x)$$

follows from Fubini's theorem, we have

$$\eta_r^*(x) = M_r \eta_2(x) = M_s \eta_2(x) = \eta_s^*(x)$$

for every $r > 0$, $s > 0$. Hence we can write $\eta^*(x) = \eta_r^*(x)$ without index r and of course $\eta^*(x) = \lim_{r\to 0} \eta_r^*(x)$. Thus we have

$$\eta^*(x) = \eta_2(x)$$

almost everywhere on G_{r+s+t}.

Since $\eta_2(x)$ is continuous, $\eta^*(x) = \eta_2(x)$ everywhere on G_{r+s+t}. Repeating this method we have $\eta^*(x) = \eta_1(x) = M_t\eta(x)$ on G_{r+s+t} and $\eta^*(x) = \eta(x)$ almost everywhere on G_{r+s+t}. Since we can take $r+s+t$ arbitrarily small, lemma 2 is completely proved.

(*) Received June 4, 1949.

S. Bochner - J. von Neumann [1], Almost periodic functions in a group II, Trans. Amer. Math. Soc., 37 (1935).

Tokyo Bunrika Daigaku.