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SYMMETRIC WEIGHTS AND S-REPRESENTATIONS

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Abstract

We study irreducible representations of compact Lie groups relating an algebraic condition (the highest weight λ is "symmetric", i.e., in any simple factor all non zero $\langle\lambda,\alpha\rangle$ are equal, for any positive root α and any invariant inner product) with a geometric one (for all orbits, the d-th osculating space coincides with the representation space).

We prove that, if $d=2$ and λ is symmetric, the irreducible representation with highest weight λ corresponds to the isotropy representation of a symmetric space.

1. Introduction

Let K be a compact Lie group and ϕ a faithful irreducible orthogonal representation. Our aim is to investigate the interplay between algebraic properties of the weight system of ϕ and geometric properties of the representation ϕ.

Among orthogonal representations, a crucial rôle in submanifold geometry is played by the isotropy representations of symmetric spaces, called *s-representations*. Indeed the principal orbits of s-representations are isoparametric and the singular ones are their focal manifolds. Moreover all orbits of s-representations are *taut* [2].

The s-representations are strictly related to an another class of orthogonal representations whose definition is geometrically more appealing: the *polar representations*. A representation of a compact Lie group K on vector space V is polar if there is a linear subspace $\Sigma \subset V$ that meets all orbits of K and every time it meets an orbit of K, it meets it perpendicularly. It is not difficult to see that any s-representation is polar. Moreover it is still true that any orbit of a polar representation is taut, as it follows from results of Conlon [4] together with ones of Bott and Samelson [2].

On the other hand, Dadok [6] classified all irreducible polar representations and observed that some of them are s-representations and that, those that are not, have the same orbits as s-representations. For his classification, Dadok associated to any irreducible representation with highest weight λ, an integer $k(\lambda)$.

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He proved that for a polar representation one has the upper bound $k(\lambda) \leq 4$, if it is of real type, and $k(\lambda) \leq 2$, otherwise. This result was crucial in his proof since it reduced considerably the list of possible polar representations.

We will give a geometric interpretation of this upper bound on $k(\lambda)$ for polar representations. This is done in Section 2, where we consider, like in [5] the class $(\mathcal{O}_2)$ of orthogonal representations. In general, ϕ belongs to $(\mathcal{O}_d)$ if the d-th osculating space coincides with V. Any irreducible polar representation belongs to $(\mathcal{O}_2)$ and we prove that the above upper bound for $k(\lambda)$ holds, more generally, for irreducible representations of class $(\mathcal{O}_2)$ (Theorem 2.1).

The next part of our work starts from the observation that, up to a few exceptions ($SU^*(2n)/Sp(n)$ and E_6/F_4), the irreducible polar representations that *are* s-representations (and not just orbit equivalent to them) and not transitive on the unit sphere are those for which $k(\lambda)$ assumes its maximal value mentioned above (i.e., $k(\lambda) = 4$ if ϕ is of real type and $k(\lambda) = 2$ otherwise) and whose highest weight is *symmetric*, i.e., all nonzero $\langle \lambda, \alpha \rangle$ are equal for any positive root α chosen in any simple factor of $\mathfrak{k}$, where $\langle \, , \rangle$ is any $\mathfrak{k}$-invariant inner product (cf. Theorem 9 (ii) and Theorem 10 (ii) in [6]).

Rather than giving a new proof of Dadok's Theorem (this was done by Eschenburg and Heintze in [8], using submanifold geometry) we aim to study the interplay between s-representations and representations with symmetric highest weight λ and for which $k(\lambda)$ assumes the maximal value allowed for the class $(\mathcal{O}_2)$. Our main result on one hand generalizes to some extent Theorems 9 and 10 in [6], since we do not assume the representation to be polar; on the other, it gives an alternative proof of them.

THEOREM 1.1. *Let $\phi_\lambda : K \to O(V)$ be a faithful irreducible complex representation of a semisimple, compact, connected Lie group K with highest weight λ, and let $\langle \, , \rangle$ be a $\mathfrak{k}$-invariant inner product on $\mathfrak{k}$.*

(a) *If ϕ_λ is of real type, $k(\lambda) = 4$ and λ is symmetric then $\phi_\lambda^{\boldsymbol{R}}$ is the isotropy representation of a compact, simply connected, irreducible symmetric space.*

(b) *If ϕ_λ is of complex type, $k(\lambda) = 2$ and λ is symmetric then $(K \cdot U(1), [\phi_\lambda \hat{\otimes} e^{i\theta}]_{\boldsymbol{R}})$ is the isotropy representation of a compact, simply connected, irreducible hermitian symmetric space.*

(c) *If ϕ_λ is of symplectic type, $k(\lambda) = 3$ (thus is this case ϕ_λ cannot belong to $(\mathcal{O}_2)$) and λ is symmetric then $(K \cdot Sp(1), \phi_\lambda \hat{\otimes} \nu_2)$ is the isotropy representation of a compact, simply connected, irreducible quaternionic symmetric space.*

Note that in case (c) $\phi_\lambda \hat{\otimes} \nu_2$ is of real type, its highest weight λ' has $k(\lambda') = 4$. Thus (c) is a special case of (a).

For the proof we state some properties of irreducible representations whose highest weight is symmetric and $k(\lambda)$ is maximal for $(\mathcal{O}_2)$ (Lemma 3.1 and 3.2). These properties generalize to the ones for which $k(\lambda)$ is maximal for $(\mathcal{O}_d)$. What turns out to be crucial in the case of class $(\mathcal{O}_2)$ is that in this case, up to a few special cases, λ is a sum of minuscule weights, each in any simple factor of K.

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2. Osculating spaces and weights

In this Section we want to give a geometric proof of the upper bound for $k(\lambda)$. To this purpose, we first recall some facts from representation theory, then we give the definition of $k(\lambda)$ and finally we relate $k(\lambda)$ with the decomposition of the representation space into osculating spaces.

Let ϕ be an irreducible representation of a compact Lie group K on a complex vector space V. V will be always endowed with a K-invariant inner product (which is uniquely defined up to a constant factor). Let $\mathfrak{k}^{\boldsymbol{C}}$ be the complexification of the Lie algebra $\mathfrak{k}$ of K and $\phi^{\boldsymbol{C}}$ the corresponding irreducible representation of $\mathfrak{k}^{\boldsymbol{C}}$ on the complex vector space V.

One says that ϕ is of *real type* if $\phi^{\boldsymbol{C}}$ comes from a representation of $\mathfrak{k}^{\boldsymbol{C}}$ on a real vector space W by extension of scalars (i.e., $V = W \otimes_{\boldsymbol{R}} \boldsymbol{C}$). This is equivalent to the existence of an invariant real structure on V, i.e., a conjugate linear endomorphism $\mathscr{J}$ of V such that $\mathscr{J}^2 = id$. The representation ϕ is of *symplectic type* if it comes from a quaternionic representation by restriction of scalars, or equivalently if there exists an invariant symplectic structure, i.e., a conjugate linear endomorphism of V whose square is minus the identity. One says that a representation is of *complex type* if it is neither real nor symplectic. Note that real and symplectic representations share the property that there exists a non degenerate invariant bilinear form on V. For this reason they are also called *self dual*. On the other hand, an irreducible representation of a complex Lie algebra $\mathfrak{k}^{\boldsymbol{C}}$ on a real vector space is of complex or symplectic type if it comes from a complex representation by restriction of scalars. Otherwise it is of real type.

If $\phi : K \to O(V)$ is of real type we will consider the orthogonal representation $\phi^{\boldsymbol{R}}$ on the real part $V^{\boldsymbol{R}}$ of V, i.e. the $+1$-eigenspace of $\mathscr{J}$. In the complex or symplectic case we will regard V as a real vector space (and when we want to stress the difference between regarding V as a complex and a real vector space we will write $[\phi]_{\boldsymbol{R}}$ in the latter case).

For representations of real type the following lemma [5] describes the real part of the representation space explicitly

LEMMA 2.1. *Let ϕ_λ be an irreducible representation of real type of highest weight λ, $\{\rho\}$ the set of its positive weights and let $\{x_{\rho,\iota}\}$ be a union over unitary bases of the weight spaces of the positive weights. Then*

$$\{v_{\rho,\iota} = x_{\rho,\iota} + \mathscr{J} x_{\rho,\iota}, w_{\rho,\iota} = i(x_{\rho,\iota} - \mathscr{J} x_{\rho,\iota})\}$$

is a basis of $V^{\boldsymbol{R}}$, the $+1$-eigenspace of $\mathscr{J}$. Moreover $X_\alpha \mathscr{J} = -\mathscr{J} X_{-\alpha}$ and $\mathscr{J} v_\lambda = v_{-\lambda}$.

Recall that K is finitely covered by a compact Lie group $\tilde{K} = K_1 \times \cdots \times K_l \times T^m$, where any K_ι is simple and T^m is an m-dimensional torus. ϕ induces a representation $\tilde{\phi}$ of $\tilde{K}$ having the same orbits as ϕ. Thus without loss of generality we will assume that $K = K_1 \times \cdots \times K_l \times T^m$.

If ϕ is of real type then T^m lies in the kernel of ϕ. Thus, if ϕ is effective, then $K = K_1 \times \cdots \times K_l$.

If ϕ is of complex or symplectic type, then $\phi_{|T^m}$ has a one dimensional kernel. Hence, if ϕ is effective, then $K = K_1 \times \cdots \times K_l \times T^1$. In this case ϕ is the external tensor product of a representation $\bar{\phi}$ of the semisimple $K_s = K_1 \times \cdots \times K_l$ and a one dimensional representation of T^1 given by multiplication by $e^{i\theta}$ (which we will denote by $e^{i\theta}$). Hence $\phi = \bar{\phi} \hat{\otimes} e^{i\theta}$ and, if $\bar{\phi}$ (considered as a real representation) is irreducible, ϕ is also.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}^{\boldsymbol{C}}$ and denote by $\langle\,,\rangle$ an invariant inner product of $\mathfrak{k}$ (note that, on each simple factor of $\mathfrak{k}$, by Schur's Lemma, it coincides up to a constant with the negative of the Killing form of K). Let $\Sigma = \{\alpha\}$ be the set of roots of $\mathfrak{k}^{\boldsymbol{C}}$ with respect to $\mathfrak{t}$ and the coroot H_α be given by $\langle H_\alpha, H\rangle = \alpha(H)$, for any $H \in \mathfrak{t}$.

Recall that there exists a basis of $\mathfrak{k}^{\boldsymbol{C}}$, $\{H_\alpha, X_\alpha, X_{-\alpha}\}$, with

$$\langle X_\alpha, X_{-\alpha}\rangle = 1 \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = H_\alpha.$$

We will call such a basis a *Chevalley basis.*

One can obtain a description of the Lie algebra $\mathfrak{k}$ of the compact Lie group K in terms of root vectors. This can be done as follows. Let $\mathfrak{t}_0$ denote the real subspace of $\mathfrak{t}$ consisting of the real linear combinations of the coroots H_α. The Lie algebra $\mathfrak{k}$ of K is then spanned by $i\mathfrak{t}_0$, $X_\alpha - X_{-\alpha}$ and $i(X_\alpha + X_{-\alpha})$, where $X_\alpha \in \mathfrak{k}^{\boldsymbol{C}}$ is a (suitable) root vector (see for instance [13]).

Suppose now that ϕ has highest weight λ. In this case we will put an index λ, writing ϕ_λ and V_λ and often will denote the representation as well as the representation space by V_λ. Moreover we will often write $X \cdot v$ instead of $\phi_\lambda(X)v$, for $X \in \mathfrak{k}$, $v \in V_\lambda$.

Let v_λ be a highest weight vector of ϕ_λ and let $U(\mathfrak{k}^{\boldsymbol{C}})$ denote the universal enveloping algebra of $\mathfrak{k}^{\boldsymbol{C}}$. Recall that

$$V_\lambda = U(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda = U(\mathfrak{n}^-) \cdot v_\lambda,$$

where $\mathfrak{n}^-$ is the (nilpotent) subalgebra of $\mathfrak{k}^{\boldsymbol{C}}$ generated by $X_{-\gamma}, \gamma \in \Sigma^+$ (see, e.g. [10]).

We now come to the definition of $k(\lambda)$. Dadok [6, Proposition 7] proved that there exists a system $\mathcal{O} = \{\beta_1, \ldots, \beta_l\}$ of strongly orthogonal positive roots such that $s_0 = s_{\beta_1} \cdot s_{\beta_2} \cdots s_{\beta_l}$ is the Weyl group element mapping the positive Weyl chamber into its negative.

Note that $s_0(\lambda)$ is the smallest weight [1, Remark 2, p. 127]. Moreover if ϕ_λ is self dual (i.e. either real or symplectic) then $s_0\lambda = -\lambda$ [1, Chapitre VIII, Proposition 12, p. 132].

Let $v_{s_0(\lambda)}$ be a weight vector relative to $s_0(\lambda)$.

Definition. $k(\lambda)$ is the smallest integer such that $v_{s_0(\lambda)} \in U^{k(\lambda)}(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda$ and $v_{s_0(\lambda)} \notin U^r(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda$ for any $r < k(\lambda)$.

The system $\mathcal{O}$ of strongly orthogonal roots can be used to decide whether ϕ_λ is of real, symplectic or complex type. Moreover one can give a formula for $k(\lambda)$. Namely

PROPOSITION 2.1. (i) *ϕ_λ is complex if and only if $\lambda \notin \mathrm{span}_{\boldsymbol{R}}\{\beta_1, \dots, \beta_l\}$,*

(ii) *ϕ_λ is of real (resp. symplectic) type, if and only if $\lambda \in \mathrm{span}_{\boldsymbol{R}}\{\beta_1, \dots \beta_l\}$ and $k(\lambda)$ is an even (resp. odd) integer.*

Moreover

$$k(\lambda) = \sum_{i=1}^{l} \frac{\langle \beta_i, \lambda - s_0\lambda \rangle}{\langle \beta_i, \beta_i \rangle}, \tag{1}$$

and, in case ϕ_λ is real or symplectic,

$$k(\lambda) = \sum_{i=1}^{l} \frac{2\langle \beta_i, \lambda \rangle}{\langle \beta_i, \beta_i \rangle}, \tag{2}$$

where $\langle , \rangle$ is any $\mathfrak{k}$-invariant inner product on $\mathfrak{k}$.

Proof. The proof of the property that ϕ_λ is complex if and only if λ does not belong to $\mathrm{span}_{\boldsymbol{R}}\{\beta_1, \dots, \beta_l\}$ and real or symplectic elsewhere, can be found in [6, p. 131] (this is actually in the same vein as [1, Chapitre VIII, Proposition 12, p. 132] or [13, p. 142]).

We now prove (1) and (2). Since $s_0 = s_{\beta_1} \cdot s_{\beta_2} \cdots s_{\beta_l}$ we have that $s_0\lambda = \lambda - \sum_i b_i\beta_i$. Thus $v_{s_0\lambda} \in U^{\sum_i b_i}(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda$ and that $v_{s_0\lambda} \notin U^r(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda$ for any $r < \sum_i b_i$. Moreover

$$b_i = \frac{\langle \beta_i, \lambda - s_0\lambda \rangle}{\langle \beta_i, \beta_i \rangle},$$

thus by definition

$$k(\lambda) = \sum_{i=1}^{l} \frac{\langle \beta_i, \lambda - s_0\lambda \rangle}{\langle \beta_i, \beta_i \rangle},$$

and if λ is self dual (2) follows from the fact that $s_0\lambda = -\lambda$. q.e.d.

Next we recall some notions of submanifold geometry. Let M be a submanifold of $\boldsymbol{R}^n$. The *d-th osculating space of M at p* is the space $\mathcal{O}_p^d(M)$ spanned by the first d derivatives in 0 of curves $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$. Note that $\mathcal{O}_p^1(M) = T_pM$. Now let $\rho : K \to O(n)$ be a representation that we assume to be irreducible and let M be an orbit of K. Since ρ is irreducible, for any $p \in M$ there is a natural number h, called the degree of the orbit, such that $\mathcal{O}_p^h(M) = \boldsymbol{R}^n$. Remark that $\mathcal{O}_{\rho(g)}^h(M) = \rho(g)\mathcal{O}_p^h(M)$, hence if $\mathcal{O}_p^h(M) = \boldsymbol{R}^n$ for some $p \in M$, then $\mathcal{O}_q^h(M) = \boldsymbol{R}^n$ for all $q \in M$. Notice also that there is a natural number d such that $\mathcal{O}_p^d(M) = \boldsymbol{R}^n$ $(p \in M)$ for all orbits M of ρ, and let $d(\rho)$ denote the smallest of such numbers d. In other words $d(\rho)$ is the smallest number such that all orbits have degree $d(\rho)$.

We will denote by $(\mathcal{O}_d)$ the class of irreducible orthogonal representations ρ such that $d(\rho) = d$. Notice that the smallest d for which $(\mathcal{O}_d)$ is nonempty is

$d = 2$. From the point of view of submanifold geometry the complexity of the orbits of ρ grows as $d(\rho)$ gets larger.

The classes of representations we have mentioned in the Introduction (polar, s-representations, with all orbits taut) all belong to $(\mathcal{O}_2)$ if we restrict ourselves to irreducible representations. To see this one has to use the following result of Kuiper [12, Théorème 2]. *Let M be a taut submanifold in $\boldsymbol{R}^n$ that is full in the sense that it is not contained in any hyperplane. Then there is a point $p \in M$ such that $\mathcal{O}_p^2(M) = \boldsymbol{R}^n$.* This is for example true for every point $p \in M$ that is the maximum of a distance function. Thus, in particular, we get that *any irreducible polar representation belongs to $(\mathcal{O}_2)$.*

Recall that a properly embedded submanifold M of $\boldsymbol{R}^n$ is said to be *taut* if for almost all $x \in \boldsymbol{R}^n$ the distance function $d_x : M \to \boldsymbol{R}$; $p \to d(x,p)^2$ is a $\boldsymbol{Z}_2$ perfect Morse function (i.e., the Morse inequalities with respect to $\boldsymbol{Z}_2$ are equalities).

We now consider an irreducible representation $\phi_\lambda : K \to O(V)$. If ϕ_λ is of real type we will say that it belongs to $(\mathcal{O}_2)$ if $\phi_\lambda^{\boldsymbol{R}}$ does.

The definition of osculating space and a computation shows the following

Lemma 2.2. *Let λ be a highest weight, and v_λ a highest weight vector. Suppose ϕ_λ is of class $(\mathcal{O}_2)$.*

(i) *If V_λ is of real type, then $U^2(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda + U^2(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_{-\lambda} = V_\lambda$.*

(ii) *If V_λ is of complex or symplectic type, then $U^2(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda = V_\lambda$.*

The proof can be found in [5, Lemma 3]. For representations of real type, one needs Lemma 2.1.

More in general, if ϕ_λ is of class $(\mathcal{O}_d)$, then $U^d(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda + U^d(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_{-\lambda} = V_\lambda$, if V_λ is of real type, and $U^d(\mathfrak{k}^{\boldsymbol{C}}) \cdot v_\lambda = V_\lambda$, if V_λ is of complex or symplectic type.

Next we use Lemma 2.2 to show the following

Theorem 2.1. *Let ϕ_λ be an irreducible faithful orthogonal representation belonging to $(\mathcal{O}_2)$ (e.g. an irreducible polar representation). Then*

(i) *if ϕ_λ is of real type, $k(\lambda) \leq 4$;*

(ii) *if ϕ_λ is of complex or symplectic type, $k(\lambda) \leq 2$.*

Proof. First we give a geometric interpretation of the property expressed by the Theorem. Recall that, if α is a root, ρ is a weight, X belongs to the root space relative to α and v_ρ is a weight vector relative to ρ, $X \cdot v_\rho$ belongs to the weight space $V_{\alpha+\rho}$, if it is not zero. Said in another way, the action of a X in the root space relative to α can be interpreted as a translation in the weight diagram shifting each of the dots (corresponding to the weights in the weight diagram) over by α. Hence the geometric meaning of the Theorem is that one can reach $-\lambda$ from λ in at most 4 steps, in the real case (and in 2 steps in the other cases).

We shall now give the proof of (i). The other part is similar and actually easier. Recall that if ϕ_λ is real, $k(\lambda)$ is even. Thus if at least 6 steps would be necessary to go from $-\lambda$ to λ, by symmetry reasons there would exist a weight μ lying on the wall of s_0 which is 3 steps far from both $-\lambda$ and λ. So any $v_\mu \in V_\mu$ could not belong to $U^2(\mathfrak{k}^C) \cdot v_\lambda + U^2(\mathfrak{k}^C) \cdot v_{-\lambda}$. But by Lemma 2.2, $U^2(\mathfrak{k}^C) \cdot v_\lambda + U^2(\mathfrak{k}^C) \cdot v_{-\lambda} = V_\lambda$. q.e.d.

Remark. Theorem 2.1 implies that if $\phi_\lambda : K \to O(V)$ is any representation of real type belonging to $(\mathcal{O}_2)$, then K has at most 4 simple factors and that if ϕ_λ is symplectic or complex, then K has at most 2 simple factors. These results were proved in [5].

Observe also that Theorem 2.1 generalizes to (irreducible, faithful) orthogonal representations of class $(\mathcal{O}_d)$. In that case we have that, if ϕ_λ is of real type, $k(\lambda) \leq 2d$ and, if ϕ_λ is of complex or symplectic type, $k(\lambda) \leq d$.

3. Symmetric weights

Wang and Ziller found very strong connections between the isotropy representations of irreducible symmetric spaces and their highest weights [16], cf. also [14]. In this Section we want to relate them with the conditions of symmetry of the highest weight and with the maximality of $k(\lambda)$.

As a start we give the following

Definition [6, p. 128]. A highest weight λ of a simple Lie algebra $\mathfrak{k}^C$ is called *symmetric* if all non zero $\langle \lambda, \alpha \rangle$, $\alpha \in \Sigma^+$ are equal. Here $\langle\, , \rangle$ is a (uniquely defined up to a simple factor) $\mathfrak{k}$-invariant inner product on $\mathfrak{k}$. A highest weight of a semisimple $\mathfrak{k}^C$ is called symmetric if it is symmetric for each of its constant factors.

Note that, for semisimple $\mathfrak{k}^C$ one can rescale on each simple factor a $\mathfrak{k}$ invariant inner product $\langle\, , \rangle$ on $\mathfrak{k}$ so that all non zero $\langle \lambda, \alpha \rangle$, $\alpha \in \Sigma^+$ are equal.

Next, we recall the results in [16] we refer to.

The first establishes a condition for the highest weight of the (complex) isotropy representation of a compact irreducible symmetric space.

Theorem 3.1 [16]. *Let $M = G/K$ be a compact irreducible symmetric space with G the identity component of the full isometry group of M and without euclidean factors. Let B be the negative of the Killing form of G and λ the highest weight of its (complex) isotropy representation. Then*

$$B(\lambda, \lambda) = 2B(\lambda, \alpha), \tag{3}$$

for any positive root α of $\mathfrak{k}$ such that $B(\lambda, \alpha) \neq 0$ and $2\lambda - \alpha$ is not a root.

As a consequence we get that, if $2\lambda - \alpha$ is not a root for any root α such that $B(\lambda, \alpha) \neq 0$, then the highest weight of an s-representation λ is symmetric.

Remark. If ϕ is of real type and $k(\lambda)$ is 4, then $2\lambda - \alpha$ is never a root, [6, proof of Lemma 11, p. 131].

If ϕ is of complex type and $k(\lambda)$ is 2, then $\lambda + \lambda^* = \lambda - s_0(\lambda)$ is never a root. This follows immediately since $\lambda + \lambda^* = \beta_i + \beta_j$ for some $\beta_i, \beta_j \in \mathcal{O}$ and the sum of two strongly orthogonal roots is never a root.

Wang and Ziller obtained a refinement of the above formula (3) for Hermitian and quaternionic symmetric spaces. Namely:

(i) If $G/H \cdot U(1)$ is Hermitian symmetric other than the complex projective spaces, with isotropy representation $[\phi_\lambda]_{\boldsymbol{R}} = [\phi_{\lambda'} \hat{\otimes} e^{i\theta}]_{\boldsymbol{R}}$, then $B(\lambda, \lambda^*) = 0$ and $B(\lambda', \lambda') + B(\lambda', \lambda'^*) = 2B(\lambda', \alpha)$, where α is any simple root of H with $B(\lambda', \alpha) \neq 0$.

(ii) If $G/H \cdot Sp(1)$ is quaternionic symmetric other than the quaternionic projective spaces, with isotropy representation $\phi \otimes \boldsymbol{C} = \phi_\lambda = \phi_{\lambda'} \hat{\otimes} v_2$ or $\phi \otimes \boldsymbol{C} = \phi_\lambda \oplus \phi_\lambda^* = [\phi_{\lambda'} \oplus \phi_{\lambda'}^*] \hat{\otimes} v_2$, then $B(\lambda', \lambda') = (3/2)B(\lambda', \alpha)$, where α is any simple root of H with $B(\lambda', \alpha) \neq 0$.

Here and below v_2 is the two dimensional representation of $Sp(1)$.

Conversely Wang and Ziller proved that the above identities actually characterize the isotropy representations of irreducible symmetric spaces, among all irreducible representations of compact Lie groups. Namely

THEOREM 3.2 [16]. *Let $\phi_\lambda : K \to O(V)$ be a faithful irreducible complex representation of a semisimple, compact, connected Lie group K with highest weight λ, and let $\langle\, , \rangle$ be a $\mathfrak{k}$-invariant inner product on $\mathfrak{k}$.*

(a) *If ϕ_λ is of real type and $\langle \lambda, \lambda \rangle = 2\langle \lambda, \alpha \rangle$, for every simple root α of K with $\langle \lambda, \alpha \rangle \neq 0$ then $\phi_\lambda^{\boldsymbol{R}}$ is the isotropy representation of a compact, simply connected, irreducible symmetric space, except in the case (G_2, ϕ_7), where ϕ_7 is the 7-dimensional representation of G_2.*

(b) *If ϕ_λ is of complex type (hence $\phi_\lambda \neq \phi_\lambda^*$) and $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = 2\langle \lambda, \alpha \rangle$, for every simple root α of K with $\langle \lambda, \alpha \rangle \neq 0$ then $(K \cdot U(1), [\phi_\lambda \hat{\otimes} e^{i\theta}]_{\boldsymbol{R}})$ is the isotropy representation of a compact, simply connected, irreducible hermitian symmetric space.*

(c) *If ϕ_λ is of symplectic type and $\langle \lambda, \lambda \rangle = 3/2\langle \lambda, \alpha \rangle$, for every simple root α of K with $\langle \lambda, \alpha \rangle \neq 0$ then $(K \cdot Sp(1), \phi_\lambda \hat{\otimes} v_2)$ is the isotropy representation of a compact, simply connected, irreducible quaternionic symmetric space.*

We suppose now that λ is symmetric and $k(\lambda)$ assumes the maximal value for representations of class $(\mathcal{O}_2)$.

The following Lemma, together with Theorem 3.2 above, yields a first proof of Theorem 1.1.

LEMMA 3.1. *Let ϕ_λ be in the same hypothesis as in Theorem 3.2.*

(a) *If ϕ_λ is of real type, $k(\lambda) = 4$ and λ is symmetric then $\langle \lambda, \lambda \rangle = 2\langle \lambda, \alpha \rangle$, for every positive root α of K with $\langle \lambda, \alpha \rangle \neq 0$.*

(b) *If ϕ_λ is of complex type, $k(\lambda) = 2$ and λ is symmetric then $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = 2\langle \lambda, \alpha \rangle$, for every positive root α of K with $\langle \lambda, \alpha \rangle \neq 0$.*

(c) *If ϕ_λ is of symplectic type, $k(\lambda) = 3$ (thus in this case $\phi_\lambda \notin (\mathcal{O}_2)$!) and λ is symmetric then $\langle \lambda, \lambda \rangle = 3/2\langle \lambda, \alpha \rangle$, for every positive root α of K with $\langle \lambda, \alpha \rangle \neq 0$.*

Note that in case (c) $\phi_\lambda \hat{\otimes} v_2$ is of real type, its highest weight λ' has $k(\lambda') = 4$. Thus (c) is a special case of (a).

Remark. We have the following generalization of Lemma 3.1. If ϕ_λ is of real type, $k(\lambda) = 2d$ and λ is symmetric then $\langle \lambda, \lambda \rangle = d\langle \lambda, \alpha \rangle$, for every positive root α of K with $\langle \lambda, \alpha \rangle \neq 0$. If ϕ_λ is of complex type, $k(\lambda) = d$ and λ is symmetric then $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = d\langle \lambda, \alpha \rangle$, for every positive root α of K with $\langle \lambda, \alpha \rangle \neq 0$.

Proof. (a) If $k(\lambda) = 4$, then $2\lambda = \beta_1 + \beta_2 + \beta_3 + \beta_4$, for $\beta_i \in \mathcal{O}$ not necessarily distinct. If λ is symmetric, then $\langle \lambda, \alpha \rangle = \langle \lambda, \beta_i \rangle$ for any positive root α such that $\langle \lambda, \alpha \rangle \neq 0$ and any $i = 1, \dots, 4$ fixed. Then

$$\langle \lambda, \lambda \rangle = \frac{1}{2} 4\langle \lambda, \beta_i \rangle = 2\langle \lambda, \beta_i \rangle = 2\langle \lambda, \alpha \rangle.$$

(b) We remark first that $\lambda^* = -s_0(\lambda)$ [1, Chapitre VIII, Proposition 11, p. 13]. Thus, if $k(\lambda) = 2$, then $\lambda + \lambda^* = \lambda - s_0(\lambda) = \beta_1 + \beta_2$ for $\beta_i \in \mathcal{O}$ not necessarily distinct. Thus, since λ is symmetric,

$$\langle \lambda, \lambda \rangle + \langle \lambda^*, \lambda \rangle = \langle \lambda + \lambda^*, \lambda \rangle = \langle \lambda, \beta_1 + \beta_2 \rangle = 2\langle \lambda, \beta_1 \rangle = 2\langle \lambda, \alpha \rangle,$$

for any positive α with $\langle \lambda, \alpha \rangle \neq 0$.

(c) If $k(\lambda) = 3$, $2\lambda = \beta_1 + \beta_2 + \beta_3$, for $\beta_i \in \mathcal{O}$ not necessarily distinct. If λ is symmetric, then

$$\langle \lambda, \lambda \rangle = \frac{1}{2} 3\langle \lambda, \beta_i \rangle = \frac{3}{2}\langle \lambda, \beta_i \rangle = \frac{3}{2}\langle \lambda, \alpha \rangle,$$

for any positive α with $\langle \lambda, \alpha \rangle \neq 0$. q.e.d.

Observe that in case (a) the highest weight μ of the representation ϕ_7 of G_2 has $k(\lambda) = 2$. Hence the conditions of Theorem 1.1 do not hold.

Thus Lemma 3.1 gives a proof of Theorem 1.1, using the results in [16]. However the proof in [16], when ϕ_λ is orthogonal and K is not simple, is case by case and uses the classification of symmetric spaces. Thus we will give a conceptual proof for this case. If ϕ_λ is complex, we will show that K is simple. Thus we will use Theorem 4.6 in [16] which has a conceptual proof (cf. also [9]).

As a start, we prove the following

LEMMA 3.2. *Let ϕ_λ be an irreducible representation, not of symplectic type, of a semisimple compact Lie algebra $\mathfrak{k}$ with λ symmetric. Assume that $k(\lambda) = 4$, if ϕ_λ is of real type and $k(\lambda) = 2$, if ϕ_λ is of complex type. Then*

(a) *for any simple factor $\mathfrak{k}_\iota$ of $\mathfrak{k}$ there exists a unique simple root $\alpha^\iota_{j_\iota}$ which is not orthogonal to λ;*

(b) *any positive root of $\mathfrak{k}_\iota$ has either 0 or 1 as coefficient of $\alpha^\iota_{j_\iota}$ in its expression as a linear combination of simple roots;*

(c) *if λ_ι is the highest weight of $\phi_{\lambda|\mathfrak{k}_\iota}$ then λ_ι is an integral multiple of a minuscule weight of $\mathfrak{k}_\iota$. Moreover the coefficient of proportionality equals $\lambda(H_{\alpha^\iota_{j_\iota}})$.*

Remark. More in general the present Lemma holds if $k(\lambda) = 2d$ and ϕ_λ is of real type or $k(\lambda) = d$ and ϕ_λ is of complex type.

Proof. (a) Let α and β be simple roots (chosen in the same simple factor) such that $\langle\lambda,\alpha\rangle \neq 0$ and $\langle\lambda,\beta\rangle \neq 0$. Then, by [7, p. 266] there exists a minimal chain of simple roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r = \beta$ connecting α and β. Let $\gamma = \alpha_1 + \cdots + \alpha_r$. Then γ is a root of $\mathfrak{k}$ and $\langle\lambda,\gamma\rangle \neq 0$. In the real case we have that, since λ is symmetric

$$\langle\lambda,\lambda\rangle = 2\langle\lambda,\alpha\rangle = 2\langle\lambda,\gamma\rangle.$$

On the other hand, $\langle\lambda,\lambda\rangle = 2\langle\lambda,\gamma\rangle = 2\langle\lambda,\alpha_1\rangle + \cdots + 2\langle\lambda,\alpha_r\rangle \geq 2\langle\lambda,\alpha\rangle + 2\langle\lambda,\beta\rangle = 2\langle\lambda,\lambda\rangle$, which is clearly a contradiction.

In the complex case one gets similarly a contradiction from $\langle\lambda+\lambda^*,\lambda\rangle = 2\langle\lambda+\lambda^*,\lambda\rangle$.

(b) We know by (a) that for any simple factor $\mathfrak{k}_\iota$ of $\mathfrak{k}$ there exists a unique simple root $\alpha^\iota_{j_i}$ not orthogonal to λ. On the other hand, if we denote by $\tilde{\alpha}^\iota = \sum a_k\alpha^\iota_k$ the maximal root of $\mathfrak{k}_\iota$, we have

$$\langle\lambda,\alpha^\iota_{j_i}\rangle = 2\langle\lambda,\lambda\rangle = \langle\lambda,\tilde{\alpha}^\iota\rangle.$$

Then $\tilde{\alpha}^\iota$ has coefficient 1 in $\alpha^\iota_{j_i}$ in its expression as a linear combination of simple roots. But, in general, for any root of $\mathfrak{k}_\iota$, $\beta^\iota = \sum b_k\alpha^\iota_k$ we have $b_k \leq a_k$, thus we get (b).

(c) By (a) $\lambda_\iota = m_i\omega_{j_i}$, where m_i is an integer and ω_{j_i} is the fundamental weight of $\mathfrak{k}_\iota$ corresponding to the simple root $\alpha^\iota_{j_i}$. On the other hand, by (b) the maximal root $\tilde{\alpha}^\iota$ has coefficient 1 in $\alpha^\iota_{j_i}$. Thus by [1, Chapitre VIII, Proposition 8, p. 128] ω_{j_i} is a minuscule weight. Moreover $\lambda(H_{\alpha^\iota_{j_i}}) = \lambda_i(H_{\alpha^\iota_{j_i}}) = m_i$. q.e.d.

NOTATION. In any simple factor $\mathfrak{k}_\iota$ the system of simple roots will be denoted by $\alpha^\iota_1, \ldots, \alpha^\iota_{l_i}$, with l_i the rank of $\mathfrak{k}_\iota$. The unique simple root not orthogonal to λ will be denoted by $\alpha^\iota_{j_i}$.

Observe that $k(\lambda_\iota) = m_i k(\omega_{j_i})$.

Next we deal with the case of representations of real type.

If $k(\lambda) = 4$, $\lambda(H_{\alpha^\iota_{j_i}}) \leq 4$ and λ_ι is a sum of $k(\lambda_\iota)$ strongly orthogonal positive roots, so we have the following cases:

(1) $m_i = 1$ *for any* $i = 1, \ldots, l$. Then λ is a sum of minuscule weights (each for any simple factor).

(2) $m_i = 2$ *for one* i. Then if $k(\omega_{j_i}) = 2$, $\mathfrak{k}$ is simple of types B_l or D_l (and $\lambda = 2\omega_1$; here and below we use the notation of [1] for the fundamental weights). If $k(\omega_{j_i}) = 1$, then, if $\mathfrak{k}$ has 2 simple factors, $\mathfrak{k}$ is of type $A_1 + B_l$ or $A_1 + D_l$ and $\lambda = 2\omega_1 + \omega_1$, if $\mathfrak{k}$ has 3 simple factors it is of type $A_1 + A_1 + A_1$ and $\lambda = 2\omega_1 + \omega_1 + \omega_1$.

(3) $m_i = 3$ *for one* i. Then $k(\omega_{j_i}) = 1$ and $\mathfrak{k}$ is of type $A_1 + A_1$ with $\lambda = 3\omega_1 + \omega_1$.

(4) $m_i = 4$ *for a* (*unique*) i. Then $\mathfrak{k}$ is simple and of type A_1, $\lambda = 4\omega_1$.

Like in [16], we give the following decomposition of the second exterior power of ϕ_λ:

$$\Lambda^2\phi_\lambda = ad_{\mathfrak{k}} \oplus \chi,$$

where χ is the isotropy representation of $SO(V_\lambda)/\phi_\lambda(K)$. We will compute the irreducible components of χ (with respect to K).

Our goal is to prove that there exists a non trivial $\mathfrak{k}$-invariant curvature tensor R on V_λ with values in $\phi_\lambda(\mathfrak{k}) \subset \mathfrak{so}(V_\lambda) \cong \Lambda^2 V_\lambda$. Then, as we will explain below, the classical Cartan's construction yields a conceptual proof of Theorem 1.1.

By a $\mathfrak{k}$-invariant curvature tensor R on V_λ we mean a $(4,0)$ tensor on V_λ which is invariant by the naturally induced action of $\mathfrak{k}$ on the space of $(4,0)$ tensors and has the same algebraic properties as the Riemannian curvature tensor. It can be easily seen that such an object identifies with a $\mathfrak{k}$-invariant element of the second symmetric power of $\Lambda^2 V_\lambda$, $S^2\Lambda^2 V_\lambda$. More precisely, if $\mathscr{R}$ denotes the space of such curvature tensors we have the decomposition

$$S^2\Lambda^2 V_\lambda = \Lambda^4 V_\lambda \oplus \mathscr{R},$$

with $\mathscr{R}$ the space of $\mathfrak{k}$-invariant curvature tensors.

The Cartan's construction can be summarized as follows (cf. [16, Lemma 4.1]). *Let R be a non trivial $\mathfrak{k}$-invariant curvature tensor with values in $\phi_\lambda(\mathfrak{k}) \subset \mathfrak{so}(V_\lambda) \cong \Lambda^2 V_\lambda$. Then the vector space $\mathfrak{g} = \mathfrak{k} \oplus V_\lambda$ can be made into a Lie algebra by defining*

$$[X,v] = -[v,X] = \phi_\lambda(X)(v), \quad X \in \mathfrak{k}, v \in V_\lambda,$$
$$[v,w] = Y \in \mathfrak{k}, \quad \textit{where } \phi_\lambda(Y) = -R(v,w), \quad u,v \in V_\lambda.$$

Our strategy is then to show that our strong assumptions on λ (i.e. λ symmetric and $k(\lambda) = 4$) imply that there exists such a curvature tensor. To do this we deduce from the decomposition of $\Lambda^2 V_\lambda$ into irreducible summands, the dimensions of the space of $\mathfrak{k}$-invariant elements in $S^2\Lambda^2 V_\lambda = \Lambda^4 V_\lambda \oplus \mathscr{R}$. Roughly, we show that $\dim \mathscr{R}$ is big enough so that a linear combination of $\mathfrak{k}$-invariant curvature tensors yields a non trivial one having values in $\phi_\lambda(\mathfrak{k})$.

This technique was first used by Kostant if the space of $\mathfrak{k}$-invariant tensors in $\Lambda^4 V_\lambda$, $(\Lambda^4 V_\lambda)^{\mathfrak{k}}$, vanishes [11], and in case χ is irreducible in [16].

The following lemma gives the decomposition of $\Lambda^2\phi_\lambda$ into irreducible components, by describing the highest weights of the irreducible components of χ.

LEMMA 3.3. *The highest weights of the irreducible components of χ are $2\lambda - \alpha^i_{j_i}$, where $\alpha^i_{j_i}$ is the unique simple root in the simple factor $\mathfrak{k}_i$ which is not orthogonal to λ.*

Note that, as a consequence, we have that, if $\mathfrak{k}$ has $l \leq 4$ simple factors then $ad_{\mathfrak{k}}$ has l irreducible components and χ has l irreducible components.

Proof. If λ is not a sum of minuscule weights (each for any simple factor), one can check the Lemma directly. This corresponds to the few cases mentioned above ((2), (3) and (4)).

Next we consider the case of λ sum of minuscule weights. Recall that this implies that, for any simple factor $\mathfrak{k}_\iota$, the weights of $\phi_{\lambda|\mathfrak{k}_\iota}$ are a single orbit of the Weyl group of $\mathfrak{k}_\iota$. In particular, all weights have multiplicity one and have the same length.

If $\langle\lambda,\alpha\rangle \neq 0$, then $\lambda-\alpha$ is a weight and $\langle\lambda,\lambda\rangle = \langle\lambda-\alpha,\lambda-\alpha\rangle$. Thus $\langle\lambda,\lambda\rangle = \langle\alpha,\alpha\rangle$. Moreover $\lambda - 2\alpha$ is never a weight. If, in addition, $\lambda-\alpha-\beta$ is a weight with $\alpha\neq\beta$ in the same simple factor, $\alpha+\beta$ not a root and $\langle\lambda,\alpha\rangle = \langle\lambda,\beta\rangle \neq 0$, then an easy computation shows that $\langle\alpha,\beta\rangle = 0$, i.e., that α and β are strongly orthogonal. So we may assume that $\alpha,\beta\in\mathcal{O}$.

Hence, if $2\lambda = \beta_1+\beta_2+\beta_3+\beta_4$, the possible highest weights of χ can be of the types

$$2\lambda-\alpha^\iota_{j_\iota},\quad 2\lambda-\beta_r,\quad 2\lambda-\beta_r-\beta_s.$$

On the other hand, $2\lambda-\alpha^\iota_{j_\iota}$ has clearly multiplicity one in χ. We are thus left to show that for β_r,β_s both in $\mathfrak{k}_\iota$, $2\lambda-\beta_r$ and $2\lambda-\beta_r-\beta_s$ belong to the irreducible component $V_{2\lambda-\alpha^\iota_{j_i}}$ of χ having $2\lambda-\alpha^\iota_{j_\iota}$ as highest weight. But this is clear, since they have the same multiplicity in $\Lambda^2\phi_\lambda$ and in $V_{2\lambda-\alpha^\iota_{j_i}}$. q.e.d.

Next we observe that since $ad_{\mathfrak{k}}\oplus\chi$ has $2l$ irreducible components, by Schur's lemma, the space of $\mathfrak{k}$-invariant tensors on $S^2\Lambda^2V_\lambda = S^2(ad_{\mathfrak{k}}\oplus\chi)$ has dimension $2l$, i.e.

$$\dim(S^2(ad_{\mathfrak{k}}\oplus\chi))^{\mathfrak{k}} = 2l.$$

Thus a $\mathfrak{k}$-invariant curvature tensor $R\in\mathscr{R}^{\mathfrak{k}}$ can be written as a matrix of the following form with respect to the decomposition of $\Lambda^2V_\lambda = ad_{\mathfrak{k}}\oplus\chi$ into irreducible summands

$$(R)\qquad R=\begin{pmatrix} a_1\mathbf{1} & & & & & 0\\ & \ddots & & & & \\ & & a_l\mathbf{1} & & & \\ & & & b_1\mathbf{1} & & \\ & & & & \ddots & \\ 0 & & & & & b_l\mathbf{1}\end{pmatrix},$$

where the a_i are in the $ad_{\mathfrak{k}}$ component and the b_j in the χ component. (Actually, we identify an element of $S^2\Lambda^2V_\lambda$ with the corresponding symmetric endomorphism of Λ^2V_λ.)

Hence, in order to have a non trivial invariant curvature tensor with values in $\mathfrak{k}$ we need to have at least $l+1$ independent invariant curvature tensors in $\mathscr{R}$, i.e., we have to prove

$$\dim\mathscr{R}^{\mathfrak{k}}\geq l+1\quad\text{or}\quad\dim(\Lambda^4V_\lambda)^{\mathfrak{k}}\leq l-1.$$

To this purpose we proceed like in [16, pp. 308–309]. We choose in each simple factor a Chevalley basis, take weight vectors v_λ and $v_{-\lambda}$ such that $\langle v_\lambda, v_{-\lambda}\rangle = 1$ and set

$$v_{\lambda-\alpha^i_{j_i}} = \phi_\lambda(X_{-\alpha^i_{j_i}})v_\lambda, \quad v_{\alpha^i_{j_i}-\lambda} = \phi_\lambda(X_{\alpha^i_{j_i}})v_\lambda.$$

Then (by [16, p. 308]) we have $\langle v_{\lambda-\alpha^i_{j_i}}, v_{\alpha^i_{j_i}-\lambda}\rangle = -\langle \lambda, \alpha^i_{j_i}\rangle$. We denote by $pr_{\mathfrak{k}}(pr_{\mathfrak{k}_i})$ the orthogonal projection from $\Lambda^2 V_\lambda$ to $\mathfrak{k}(\mathfrak{k}_i)$ via the embedding ϕ_λ. As in [16] one can prove that

$$pr_{\mathfrak{k}_i}(v_\lambda \wedge v_{-\lambda}) = H^i_\lambda,$$

$$pr_{\mathfrak{k}_i}(v_{\lambda-\alpha^i_{j_i}} \wedge v_{\alpha^i_{j_i}-\lambda}) = -\langle \lambda, \alpha^i_{j_i}\rangle X_{\lambda-\alpha^i_{j_i}},$$

$$pr_{\mathfrak{k}_i}(v_\lambda \wedge v_{\alpha^i_{j_i}-\lambda}) = -\langle \lambda, \alpha^i_{j_i}\rangle X_{\alpha^i_{j_i}},$$

$$pr_{\mathfrak{k}_i}(v_{\lambda-\alpha^i_{j_i}} \wedge v_{-\lambda}) = \langle \lambda, \alpha^i_{j_i}\rangle X_{-\alpha^i_{j_i}},$$

$$pr_{\mathfrak{k}_i}(v_\lambda \wedge v_{\lambda-\alpha^i_{j_i}}) = 0.$$

Let $\eta \in (\Lambda^4 V_\lambda)^{\mathfrak{k}}$, then it can be written as a matrix in the same form as (R).

Since η is a 4-form we have that the quantities

$$\eta(v_\lambda, v_{\lambda-\alpha^i_{j_i}}, v_{-\lambda}, v_{\alpha^i_{j_i}-\lambda}) = -b_i\langle \lambda, \alpha^i_{j_i}\rangle,$$

$$\eta(v_\lambda, v_{\alpha^i_{j_i}-\lambda}, v_{\lambda-\alpha^i_{j_i}}, v_{-\lambda}) = -a_i\langle \lambda, \alpha^i_{j_i}\rangle^2 + b_i\langle \lambda, \alpha^i_{j_i}\rangle(1 + \langle \lambda, \alpha^i_{j_i}\rangle),$$

$$\eta(v_\lambda, v_{-\lambda}, v_{\alpha^i_{j_i}-\lambda}, v_{\lambda-\alpha^i_{j_i}}) = \sum_J a_j\langle H^J_\lambda, H_{\lambda-\alpha^i_{j_i}}\rangle\langle \lambda, \alpha^i_{j_i}\rangle$$

are all equal. The equality between the first and the second gives rise to the equations

$$b_i = a_i\langle \lambda, \alpha^i_{j_i}\rangle - b_i(1 + \langle \lambda, \alpha^i_{j_i}\rangle), \quad i = 1, \dots, l$$

i.e., to l independent equations.

The equality between the second and the third yields at least one more linearly independent condition. This proves that $\dim(\Lambda^4 V_\lambda)^{\mathfrak{k}} \le l - 1$.

Now we deal with the case of representations of complex type.

Let ϕ_λ be an irreducible representation of complex type of a compact semisimple Lie algebra $\mathfrak{k}$. Consider the splitting

$$\phi_\lambda \otimes \phi^*_\lambda = \mathbf{1} \oplus ad_{\mathfrak{k}} \oplus \chi,$$

where $\mathbf{1}$ is the trivial representation and χ is the isotropy representation of the homogeneous space $SU(V_\lambda)/\phi_\lambda(K)$.

Lemma 3.4. *Let ϕ_λ be an irreducible representation of complex type of a compact semisimple Lie algebra $\mathfrak{k}$ with λ symmetric and $k(\lambda) = 2$. Then*

(a) *either $\mathfrak{k}$ is simple or $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(m')$ and ϕ_λ is the external tensor product of the standard representation of $\mathfrak{su}(m)$ on $\mathbf{C}^m$ and the dual of the standard representation of $\mathfrak{su}(m')$ on $\mathbf{C}^{m'}$;*

(b) *if* $\mathfrak{k}$ *is simple, then* $\chi = \phi_{\lambda+\lambda^*}$ (*in particular*, χ *is irreducible*), $\Lambda^2\phi_\lambda = \phi_{2\lambda-\alpha_i}$ (*with* α_i *is the unique simple root not orthogonal to* λ).

Proof. (a) We already know that either $\mathfrak{k}$ is simple or it has two simple factors. In the latter case, ϕ_λ is the external tensor product of two representations ϕ_{λ_i} of the two simple factors $\mathfrak{k}_1$ and $\mathfrak{k}_2$, with $k(\lambda_i) = 1$, λ_i symmetric and $\lambda + \lambda^* = \beta_1 + \beta_2$. Thus, by the classification of simple Lie algebras, we must have $\mathfrak{k}_i$ of type A_l.

(b) By Lemma 3.2 (c), if $\mathfrak{k}$ is simple, then $\lambda = m_i\omega_i$ is a integral multiple of the minuscule weight ω_i corresponding to the unique simple root α_i not orthogonal to λ. Moreover $m_i = \lambda(H_{\alpha_i}) = 1$. Indeed, if $\lambda - 2\alpha_i$ would be a weight we would have that $\lambda + \lambda^* = 2\alpha_i$ and thus the only possibility would be that $\mathfrak{k}$ is of type A_1 and $\lambda = 2\omega_1$, but in this case ϕ_λ is not of complex type. Thus $\lambda = \omega_i$ and all weights have multiplicity one and the same length.

If $\langle\lambda, \alpha\rangle \neq 0$, then $\lambda - \alpha$ is a weight and $\langle\lambda, \lambda\rangle = \langle\lambda - \alpha, \lambda - \alpha\rangle$. Thus $\langle\lambda + \lambda^*, \lambda\rangle = \langle\alpha, \alpha\rangle$. Moreover $\lambda - 2\alpha$ is never a weight. If, in addition, $\lambda - \alpha - \beta$ is a weight with $\alpha + \beta$ not a root and $\langle\lambda, \alpha\rangle = \langle\lambda, \beta\rangle \neq 0$, then an easy computation shows that $\langle\alpha, \beta\rangle = 0$, i.e., that α and β are strongly orthogonal. So we may assume that $\alpha, \beta \in \mathcal{O}$.

Then if $\lambda + \lambda^* = \beta_1 + \beta_2$, the possible highest weights of $\phi_\lambda \otimes \phi_\lambda^*$ are of the form

$$\lambda + \lambda^* \quad \text{or} \quad \lambda + \lambda^* - \beta_r \quad \text{or} \quad \lambda + \lambda^* - \beta_r - \beta_s.$$

On the other hand $\lambda + \lambda^*$ is a highest weight of χ with multiplicity one in $\phi_\lambda \otimes \phi_\lambda^*$ and we have that $\lambda + \lambda^* - \beta_1 = \beta_2$ and $\lambda + \lambda^* - \beta_1 - \beta_2 = 0$ are highest weights of $\phi_\lambda \otimes \phi_\lambda^*$ which belong to $ad_{\mathfrak{k}}$ and $\mathbf{1}$ respectively. Then, proceeding like in the proof of Lemma 3.3 it is possible to prove that χ is irreducible with highest weight $\lambda + \lambda^*$.

The highest weight $2\lambda - \alpha_i$ has multiplicity one in $\Lambda^2\phi_\lambda$. Hence, like in the proof of Lemma 3.3, one gets that $\Lambda^2\phi_\lambda = \phi_{2\lambda-\alpha_i}$. q.e.d.

We can now apply Theorem 4.6 in [16]. This completes our proof of Theorem 1.1.

Final remark. As already observed many properties of irreducible representations with λ symmetric and $k(\lambda) = 2d$, in the real case and $k(\lambda) = d$ otherwise, are similar to the ones in the special case $d = 2$ (Lemma 3.1 and 3.2). It would be interesting to give a geometric characterization of these representations in the general case.

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