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COMPACT EINSTEIN-WEYL FOUR-MANIFOLDS WITH COMPATIBLE ALMOST COMPLEX STRUCTURES

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1. Introduction

A Weyl manifold is a smooth conformal manifold (M, C) equipped with a torsion-free affine connection D preserving the conformal structure C. A Weyl manifold (M, C, D) is said to be Einstein-Weyl if its symmetrized Ricci tensor $r^{D(\mathrm{sym})}$ is proportional to a metric representative of C. The Levi-Civita connection ∇ of an Einstein manifold (M, g) gives an Einstein-Weyl structure $([g], \nabla)$ on M, where $[g]$ denotes the conformal structure determined by g. Thus the notion of Einstein-Weyl structures is a generalization of Einstein metrics, so there are many studies in this topic (see Pedersen-Swann [9], [10], Itoh [4], and their references).

An almost complex structure J on a conformal manifold (M, C) is said to be compatible if J preserves C. Let (M, C, J) be a conformal manifold with a compatible almost complex structure J. By making use of the Lee form β_g of each metric g in C, we can naturally define a unique Weyl connection D on (M, C, J), which is called the canonical Weyl connection associated with (C, J). In the 4-dimensional case, we shall call such a quadruple (M, C, D, J) an *almost Hermitian-Weyl* 4-manifold. It is known that for an almost Hermitian-Weyl 4-manifold (M, C, D, J), J is integrable if and only if J is parallel with respect to D. When J is D-parallel, (M, C, D, J) is called a Hermitian-Weyl manifold. Note that the definition of (almost) Hermitian-Weyl manifolds is very similar to that of (almost) Kähler manifolds. An almost Hermitian-Einstein-Weyl 4-manifold means an almost Hermitian-Weyl 4-manifold whose Weyl structure is Einstein-Weyl.

Sekigawa [6] showed that any compact almost Kähler-Einstein manifold with nonnegative scalar curvature must be Kähler-Einstein. Motivated by his result, we shall consider the integrability problem for almost Hermitian-Einstein-Weyl 4-manifolds. Our main result is as follows:

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THEOREM 1.1. *A compact almost Hermitian-Einstein-Weyl 4-manifold with nonnegative conformal scalar curvature must be Hermitian-Einstein-Weyl.*

2. Almost Hermitian-Einstein-Weyl structures

Let (M, C, D) be a 4-dimensional Weyl manifold. Then for any metric g in C, there exists a 1-form ω_g such that $Dg = \omega_g \otimes g$. We note that $d\omega_g$ is independent of the choice of $g \in C$. Indeed, for another metric $g' = e^f g$ in C, the corresponding 1-forms ω_g and $\omega_{g'}$ satisfy the following:

$$\omega_{g'} = \omega_g + df, \quad d\omega_g = d\omega_{g'} (=: d\omega). \tag{2.1}$$

Denote respectively by R^D, r^D and s_g^D the curvature tensor, the Ricci curvature and the conformal scalar curvature of D with respect to g in C:

$$R^D(X, Y)Z := D_X(D_Y Z) - D_Y(D_X Z) - D_{[X, Y]}Z,$$
$$r^D(X, Y) := \mathrm{tr}(V \mapsto R^D(V, Y)X), \quad s_g^D := \mathrm{tr}_g(r^D), \quad s^D := s_g^D g.$$

Note that the Ricci tensor r^D is not necessarily symmetric. We then denote by $r^{D(\mathrm{sym})}$ and $r^{D(\mathrm{skew})}$ the symmetric and skewsymmetric parts of r^D, respectively:

$$r^{D(\mathrm{sym})}(X, Y) := \frac{1}{2}(r^D(X, Y) + r^D(Y, X)),$$
$$r^{D(\mathrm{skew})}(X, Y) := \frac{1}{2}(r^D(X, Y) - r^D(Y, X)).$$

It is known that the skewsymmetric part $r^{D(\mathrm{skew})}$ is given by $r^{D(\mathrm{skew})} = -d\omega$.

The curvature tensor R^D decomposes as

$$R^D = W_+ \oplus W_- \oplus r_0^{D(\mathrm{sym})} \oplus r_+^{D(\mathrm{skew})} \oplus r_-^{D(\mathrm{skew})} \oplus s^D, \tag{2.2}$$

where $W_\pm$ are the self-dual and anti-self-dual parts of the Weyl conformal curvature tensor, $r_0^{D(\mathrm{sym})}$ is the traceless part of $r^{D(\mathrm{sym})}$, and $r_\pm^{D(\mathrm{skew})}$ are the self-dual and anti-self-dual parts of $r^{D(\mathrm{skew})}$ (see Pedersen-Swann [9]).

A Weyl manifold (M, C, D) is said to be *Einstein-Weyl* if the symmetric part $r^{D(\mathrm{sym})}$ of the Ricci tensor is proportional to a metric g in C:

$$r^{D(\mathrm{sym})} = \frac{s_g^D}{4} g.$$

Unlike the Einstein case, the conformal scalar curvature s_g^D is not constant in general; however, the sign of s_g^D is well-defined for compact Einstein-Weyl 4-manifolds (cf. Pedersen-Swann [10], Itoh [4]).

We next consider almost complex structures on Weyl manifolds. Let (M, C, D) be a 4-dimensional Weyl manifold and J an almost complex structure on M. Suppose that J preserves C, i.e., $g(JX, JY) = g(X, Y)$ for any metric g in

C. The fundamental form Ω_g of (g, J) is now defined by $\Omega_g(X, Y) := g(JX, Y)$. It follows from the peculiarity of the 4-dimensional case that there exists a 1-form β_g, called the Lee form of (M, g, J), satisfying

$$d\Omega_g = \beta_g \wedge \Omega_g. \tag{2.3}$$

In particular, the exterior derivative $d\beta_g$ of the Lee form is orthogonal to the fundamental form Ω_g:

$$d\beta_g \wedge \Omega_g \equiv 0. \tag{2.4}$$

For another metric $g' = e^f g$ in C, the Lee forms β_g and $\beta_{g'}$ satisfy the following:

$$\beta_{g'} = \beta_g + df, \quad d\beta_{g'} = d\beta_g. \tag{2.5}$$

Comparing (2.1) with (2.5), we see that $\beta_g - \omega_g$ is independent of the choice of g.

If D is the canonical Weyl connection, i.e., $\beta_g \equiv \omega_g$, then (M, C, D, J) is called an *almost Hermitian-Weyl* manifold. Furthermore, (M, C, D, J) is said to be *almost Hermitian-Einstein-Weyl* if (C, D) is also Einstein-Weyl. An almost Hermitian-Weyl manifold (M, C, D, J) is said to be *Hermitian-Weyl* if $DJ \equiv 0$.

PROPOSITION 2.1. *(M, C, D, J) is an almost Hermitian-Weyl 4-manifold if and only if (g, D, J) satisfies*

$$g((D_X J)Y, Z) + g((D_Y J)Z, X) + g((D_Z J)X, Y) \equiv 0, \tag{2.6}$$

where g is a metric in C. Furthermore, if (M, C, D, J) is an almost Hermitian-Weyl manifold, then the following holds

$$(D_{JX} J)JY + (D_X J)Y \equiv 0. \tag{2.7}$$

In particular, the g-trace $\mathrm{tr}_g(DJ)$ of $(X, Y) \mapsto (D_X J)Y$ is identically zero:

$$\mathrm{tr}_g(DJ) \equiv 0. \tag{2.8}$$

Proof. By definition, the covariant derivative $D\Omega_g$ of the fundamental form Ω_g satisfies

$$(D_X \Omega_g)(Y, Z) = g((D_X J)Y, Z) + \omega_g(X) g(JY, Z).$$

Since D is torsion-free, we have

$$\begin{aligned} d\Omega_g(X, Y, Z) &= \mathfrak{S}_{X,Y,Z}(D_X \Omega_g)(Y, Z) \\ &= \mathfrak{S}_{X,Y,Z}\{g((D_X J)Y, Z) + \omega_g(X)\Omega_g(Y, Z)\} \\ &= (\omega_g \wedge \Omega_g)(X, Y, Z) + \mathfrak{S}_{X,Y,Z}\, g((D_X J)Y, Z), \end{aligned}$$

where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic summation with respect to X, Y, Z. It then follows that (M, C, D, J) is an almost Hermitian-Weyl manifold if and only if (g, D, J) satisfies (2.6).

In order to show (2.7), we note that

$$(D_XJ)JY = -J(D_XJ)Y, \quad g((D_XJ)Y,Z) = -g(Y,(D_XJ)Z). \tag{2.9}$$

By using (2.6) and (2.9), we have

$$\begin{aligned} g((D_XJ)Y,Z) + g((D_YJ)Z,X) + g((D_ZJ)X,Y) &\equiv 0 \\ g((D_XJ)Y,Z) - g((D_{JY}J)JZ,X) - g((D_{JZ}J)X,JY) &\equiv 0 \\ g((D_{JX}J)Y,JZ) - g((D_YJ)Z,X) + g((D_{JZ}J)JX,Y) &\equiv 0 \\ g((D_{JX}J)JY,Z) + g((D_{JY}J)Z,JX) - g((D_ZJ)X,Y) &\equiv 0. \end{aligned}$$

Taking summation of these, we have

$$2g((D_XJ)Y + (D_{JX}J)JY,Z) \equiv 0.$$

This shows (2.7). By taking g-trace of (2.7), we immediately obtain (2.8). □

From Proposition 2.1, we may regard an almost Hermitian-Weyl manifold as a conformal geometric analogue to almost Kähler one. Indeed, our results for almost Hermitian-Weyl 4-manifolds can be proved by making use of arguments similar to those in almost Kähler geometry (cf. Sekigawa [6], Draghici [1]).

As in almost Hermitian geometry, we introduce the notion of the *$*$-Ricci tensor* r^{D*} and the *$*$-scalar curvature* s^{D*} of (C,D,J):

$$r^{D*}(X,Y) := \mathrm{tr}(V \mapsto R^D(Y,JV)JX), \quad s_g^{D*} := \mathrm{tr}_g(r^{D*}),$$

where g is a metric representative of C.

For a $(0,2)$-tensor field t on (M,C,D,J), we denote respectively by $t^{(\mathrm{sym})}$ and $t^{(\mathrm{skew})}$ the symmetric and skewsymmetric parts of t, and also denote by $t^{(\mathrm{inv})}$ and $t^{(\mathrm{anti})}$ the J-invariant and J-anti-invariant parts of t:

$$\begin{aligned} t^{(\mathrm{inv})}(X,Y) &:= \frac{1}{2}(t(X,Y) + t(JX,JY)), \\ t^{(\mathrm{anti})}(X,Y) &:= \frac{1}{2}(t(X,Y) - t(JX,JY)). \end{aligned}$$

On the space of 2-forms, we obtain the following orthogonal decomposition:

$$\textstyle\bigwedge^2 T^*M = \bigwedge_+ \oplus \bigwedge_-; \quad \bigwedge_+ = \boldsymbol{R}\Omega_g \oplus \bigwedge^{(\mathrm{anti})}, \quad \bigwedge_- = \bigwedge_0^{(\mathrm{inv})},$$

where $\bigwedge_\pm, \boldsymbol{R}\Omega_g, \bigwedge_0^{(\mathrm{inv})}$ and $\bigwedge^{(\mathrm{anti})}$ denote respectively self-dual and anti-self-dual 2-forms, multiples of the fundamental form Ω_g, the traceless J-invariant 2-forms and the J-anti-invariant 2-forms.

For simplicity, we set

$$t^{(\mathrm{sym.inv})}(X, Y) := \frac{1}{4}(t(X, Y) + t(Y, X) + t(JX, JY) + t(JY, JX)),$$

$$t^{(\mathrm{sym.anti})}(X, Y) := \frac{1}{4}(t(X, Y) + t(Y, X) - t(JX, JY) - t(JY, JX)),$$

$$t^{(\mathrm{skew.inv})}(X, Y) := \frac{1}{4}(t(X, Y) - t(Y, X) + t(JX, JY) - t(JY, JX)),$$

$$t^{(\mathrm{skew.anti})}(X, Y) := \frac{1}{4}(t(X, Y) - t(Y, X) - t(JX, JY) + t(JY, JX)).$$

If we define a tensor field τ associated with a given $(0,2)$-tensor field t by $\tau(X, Y) := t(JX, Y)$, then the following hold:

$$\tau^{(\mathrm{sym.inv})}(X, Y) = t^{(\mathrm{skew.inv})}(JX, Y), \tau^{(\mathrm{sym.anti})}(X, Y) = t^{(\mathrm{sym.anti})}(JX, Y),$$

$$\tau^{(\mathrm{skew.inv})}(X, Y) = t^{(\mathrm{sym.inv})}(JX, Y), \tau^{(\mathrm{skew.anti})}(X, Y) = t^{(\mathrm{skew.anti})}(JX, Y).$$

The J-invariant parts $r^{D*(\mathrm{sym.inv})}$ and $r^{D(\mathrm{sym.inv})}$ of the symmetrized $*$-Ricci and Ricci tensors of D satisfy the following relation:

PROPOSITION 2.2. *For an almost Hermitian-Weyl 4-manifold (M, C, D, J), we have the following formulae*:

$$r^{D*(\mathrm{sym.inv})}(X, Y) = r^{D(\mathrm{sym.inv})}(X, Y) + \frac{1}{2}B(X, Y) \tag{2.10}$$

$$s_g^{D*} = s_g^D + \frac{1}{2}|DJ|_g^2, \tag{2.11}$$

where B is defined by $B(X, Y) := \mathrm{tr}_g g((DJ)X, (DJ)Y)$.

Proof. We first recall the definition of the second covariant derivative of J:

$$(D_X D_Y J)Z := D_X((D_Y J)Z) - (D_{D_X Y} J)Z - (D_Y J)D_X Z.$$

By definition, we have the following formulae:

LEMMA 2.3. *Let (M, C, D) be a Weyl manifold with a compatible almost complex structure J and g a metric representative of C. Then (g, D, J) satisfies*

$$g((D_X D_Y J)U, V) + g(U, (D_X D_Y J)V) \equiv 0, \tag{2.12}$$

$$(D_X D_Y J)JV + J(D_X D_Y J)V = -(D_X J)(D_Y J)V - (D_Y J)(D_X J)V. \tag{2.13}$$

Furthermore, if (M, C, D, J) is almost Hermitian-Weyl, then we have

$$g((D_V D_X J)Y - (D_V D_Y J)X, U) = -g((D_V D_U J)X, Y). \tag{2.14}$$

We next recall the following curvature identity, so-called the Ricci identity:

$$R^D(X, Y)JV - JR^D(X, Y)V = (D_X D_Y J)V - (D_Y D_X J)V. \tag{2.15}$$

By the definitions of r^D and r^{D*} and the Ricci identity (2.15), we have the following:

$$r^{D*}(X, Y) = r^D(X, Y) - \sum_A g((D_A D_Y J - D_Y D_A J)e_A, JX), \tag{2.16}$$

where $\{e_A\}$ is a g-orthonormal frame field and where D_A denotes the covariant derivative by e_A.

Notice that the g-trace of a $(1, 2)$-tensor field T and the covariant derivative DT of T satisfy

$$D\,\mathrm{tr}_g T = \mathrm{tr}_g DT + \omega_g \otimes \mathrm{tr}_g T.$$

By (2.8), we thus obtain

$$\sum_A g((D_Y D_A J)e_A, JX) \equiv 0. \tag{2.17}$$

Applying (2.14) to the term $\sum_A g((D_A D_Y J)e_A, JX)$, we see that

$$r^{D*}(X, Y) = r^D(X, Y) + \sum_A g((D_A D_A J)JX - (D_A D_{JX} J)e_A, Y). \tag{2.18}$$

On the other hand, it follows from (2.16) and (2.17) that

$$r^{D*}(JY, JX) = r^D(JY, JX) + \sum_A g((D_A D_{JX} J)e_A, Y). \tag{2.19}$$

From (2.18) and (2.19), we obtain the following:

$$r^{D*}(X, Y) + r^{D*}(JY, JX) = r^D(X, Y) + r^D(JY, JX) + \sum_A g((D_A D_A J)JX, Y).$$

By using (2.13), we obtain (2.10) as follows:

$$\begin{aligned}
4(r^{D*(\mathrm{sym.inv})}&(X, Y) - r^{D(\mathrm{sym.inv})}(X, Y)) \\
&= \sum_A \{g((D_A D_A J)JX, Y) + g((D_A D_A J)JY, X)\} \\
&= \sum_A g(J(D_A D_A J)Y + (D_A D_A J)JY, X) \\
&= -2\sum_A g((D_A J)(D_A J)Y, X) \\
&= 2\sum_A g((D_A J)X, (D_A J)Y) \\
&= 2\,\mathrm{tr}_g\, g((DJ)X, (DJ)Y) = 2B(X, Y).
\end{aligned}$$

Taking g-trace of (2.10), we immediately obtain (2.11). □

In the rest of this section, we always assume that (M, C, D, J) is a compact almost Hermitian-Weyl 4-manifold.

We now consider the first Chern class $c_1(M)$ of such a manifold (M, C, D, J). We first define an affine connection D' by

$$D'_X Y := D_X Y - \frac{1}{2} J(D_X J) Y. \tag{2.20}$$

Then D' preserves J and C, i.e., $D'J \equiv 0, D'g = \omega_g \otimes g$. The curvature tensors $R' = R^{D'}$ and R^D satisfy the following relation:

PROPOSITION 2.4.

$$\begin{aligned} R'(X, Y)V = &\frac{1}{2}(R^D(X, Y)V - JR^D(X, Y)JV) \\ &- \frac{1}{4}((D_X J)(D_Y J) - (D_Y J)(D_X J))V. \end{aligned} \tag{2.21}$$

Let $T^{1,0}M$ denote the $\sqrt{-1}$-eigenspace of J in the complexified tangent bundle $TM \otimes \boldsymbol{C}$. Then we can identify TM with $T^{1,0}M$, as complex vector bundle over M. The cohomology class of a closed 2-form $\gamma' := \mathrm{Re}(\sqrt{-1}\, \mathrm{tr}_{\boldsymbol{C}}(R'))$ determines the first Chern class $c_1(M)$ of (M, J), namely, $2\pi c_1(M) = [\gamma']$ in $H^2(M; \boldsymbol{R})$, the second cohomology group with real coefficient. By (2.21), we can rewrite γ' as

$$\gamma'(X, Y) = \rho^{D*(\mathrm{skew})}(X, Y) - \frac{1}{4}\mathscr{D}(X, Y),$$

where ρ^{D*} and $\mathscr{D}$ are defined respectively by

$$\rho^{D*}(X, Y) := r^{D*}(JX, Y), \quad \mathscr{D}(X, Y) := \mathrm{tr}(V \mapsto (D_X J)(D_Y J)JV).$$

From (2.7) in Proposition 2.1, $\mathscr{D}$ is a J-invariant 2-form on M satisfying

$$\mathscr{D} \wedge \Omega_g = \frac{1}{4}|DJ|_g^2 \Omega_g^2. \tag{2.22}$$

By making use of (2.10), we can express γ' as

$$\gamma' = \rho^{D(\mathrm{skew.inv})} + \frac{1}{2}\mathscr{B} - \frac{1}{4}\mathscr{D} + \rho^{D*(\mathrm{skew.anti})},$$

where $\mathscr{B}$ is defined by $\mathscr{B}(X, Y) := B(JX, Y)$. Note that $\mathscr{B}$ is a J-invariant 2-form satisfying

$$\mathscr{B} \wedge \Omega_g = \frac{1}{4}|DJ|_g^2 \Omega_g^2.$$

In our 4-dimensional case, we can verify the following:

LEMMA 2.5.

$$\rho^{D*(\mathrm{skew.inv})} - \rho^{D(\mathrm{skew.inv})} \left(= \frac{1}{2}\mathscr{B} \right) = \frac{1}{4}(s_g^{D*} - s_g^D)\Omega_g.$$

From Proposition 2.2 and the lemma above, we can also rewrite γ' as

$$\gamma' = \rho_0^{D(\mathrm{skew.inv})} + \frac{1}{4}\left(s_g^D + \frac{1}{4}|DJ|_g^2\right)\Omega_g - \frac{1}{4}\mathscr{D}_0 + \rho^{D*(\mathrm{skew.anti})},$$

where $\rho_0^{D(\mathrm{skew.inv})}$ and $\mathscr{D}_0$ denote the components of $\rho^{D(\mathrm{skew.inv})}$ and $\mathscr{D}$ orthogonal to Ω_g, respectively.

The squared first Chern class $c_1^2(M)$ is given by $4\pi^2 c_1^2(M) = [\gamma' \wedge \gamma']$ in $H^4(M : \boldsymbol{R})$. Identifying $H^4(M; \boldsymbol{R})$ with $\boldsymbol{R}$ via the integration, we obtain the following formula:

PROPOSITION 2.6.

$$4\pi^2 c_1^2(M) = \int_M \left\{ \frac{1}{8}(s_g^D)^2 + \frac{s_g^D}{16}|DJ|_g^2 + \frac{1}{128}|DJ|_g^4 - \left|\rho_0^{D(\mathrm{skew.inv})} - \frac{1}{4}\mathscr{D}_0\right|_g^2 + |\rho^{D*(\mathrm{skew.anti})}|_g^2 \right\}\sigma_g, \tag{2.23}$$

where σ_g denotes the volume form of (M, g) (i.e., $\sigma_g = (1/2)\Omega_g^2$).

3. Main result

In this section, we prove the following result, which is a conformal analogue to the result due to Sekigawa [6]:

THEOREM 3.1. *Let (M, C, D, J) be a compact almost Hermitian-Einstein-Weyl 4-manifold. If the conformal scalar curvature s^D is nonnegative, then J must be integrable, i.e., (M, C, D, J) is a Hermitian-Einstein-Weyl manifold.*

We first recall the following (see Pedersen-Poon-Swann [8]):

PROPOSITION 3.2. *Let (M, C, D) be a compact oriented Einstein-Weyl 4-manifold. Then the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ satisfy the following*:

$$2\chi(M) + 3\tau(M) = \frac{1}{4\pi^2}\int_M \left\{ 2|W_+|_g^2 + \frac{1}{24}(s_g^D)^2 + \frac{1}{4}|d\omega|_g^2 \right\}\sigma_g. \tag{3.1}$$

Notice that for a compact almost complex 4-manifold M, the squared first Chern class $c_1^2(M)$ coincides with the characteristic number $2\chi(M) + 3\tau(M)$.

Let (M, C, D, J) be an almost Hermitian-Weyl 4-manifold. If it is also Einstein-Weyl, then

$$\rho_0^{D(\mathrm{skew.inv})} \equiv 0. \tag{3.2}$$

Hence the formula (2.23) leads us to another expression of $c_1^2(M)$:

$$4\pi^2 c_1^2(M) = \int_M \left\{ \frac{1}{8}(s_g^D)^2 + \frac{s_g^D}{16}|DJ|_g^2 + \frac{1}{128}|DJ|_g^4 - \frac{1}{16}|\mathscr{D}_0|_g^2 + |\rho^{D*(\text{skew.anti})}|_g^2 \right\} \sigma_g. \tag{3.3}$$

The squared norm of $\mathscr{D}_0$ can be calculated as follows. At each point p on M, we define a subspace $\mathscr{N}_p$ of T_pM by $\mathscr{N}_p := \{X \in T_pM \mid D_XJ = 0\}$. It is immediate from Proposition 2.7 that $\mathscr{N}_p$ is J-invariant and hence has even real dimension. Note that $g((D_XJ)Y, V)$ is J-anti-invariant and skewsymmetric with respect to Y, V. Since the real dimension of $\bigwedge^{(\text{anti})}$ is two, we can write $g((D_XJ)Y, V)$, at least locally, as

$$g((D_XJ)Y, V) = \alpha_2(X)\Phi_2(Y, V) + \alpha_3(X)\Phi_3(Y, V),$$

where α_2, α_3 are local 1-forms and $\{\Phi_2, \Phi_3\}$ is a local basis for $\bigwedge^{(\text{anti})}$. Then $X \in \mathscr{N}_p$ if and only if $\alpha_1(X) = \alpha_2(X) = 0$. Counting the dimensions, we see that the real dimension of $\mathscr{N}_p$ is not less than two. Take a g-orthonormal basis $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ for T_pM satisfying $e_1, e_2 \in \mathscr{N}_p$. We then obtain $\mathscr{D}(e_i, e_A) = 0$ $(i = 1, 2;\ A = 1, 2, 3, 4)$. From J-invariance of $\mathscr{D}$ and (2.22), the squared norm of $\mathscr{D}_0$ is given by $|\mathscr{D}_0|_g^2 = (1/8)|DJ|_g^4$. Summarizing these, we obtain the following:

$$c_1^2(M) = \frac{1}{4\pi^2}\int_M \left\{ \frac{1}{8}(s_g^D)^2 + \frac{s_g^D}{16}|DJ|_g^2 + |\rho^{D*(\text{skew.anti})}|_g^2 \right\} \sigma_g. \tag{3.4}$$

We can simplify the term $|\rho^{D*(\text{skew.anti})}|_g^2$ as follows. Let $(R^D)_g$ denote the curvature operator on $\bigwedge^2 T^*M$. Namely, it is defined by raising indices of the curvature tensor R^D with respect to g:

$$(R^D)_g(\alpha)(X, Y) := \frac{1}{2}\sum_{A,B,I,J} \alpha_{AB} g^{AI} g^{BJ} g(R^D(X, Y)e_J, e_I),$$

where α_{AB} are the components of a 2-form α with respect to a local frame field $\{e_A\}$ and where (g^{AB}) denotes the inverse matrix of $g = (g_{AB}) = (g(e_A, e_B))$. It should be noted that the 2-form $(R^D)_g(\Omega_g)$ is independent of the choice of g in C (i.e., $(R^D)_g(\Omega_g) = (R^D)_{g'}(\Omega_{g'})$ for $g, g' \in C$). Setting $R^D(\Omega) := (R^D)_g(\Omega_g)$, we can show the following:

Proposition 3.3. *Let (M, C, D) be a Weyl manifold with a compatible almost complex structure J. Then we have*

$$R^D(\Omega)^{(\text{inv})} = \rho^{D*(\text{skew.inv})}, \quad R^D(\Omega)^{(\text{anti})} = \rho^{D*(\text{skew.anti})}.$$

If (M, C, D, J) is an almost Hermitian-Einstein-Weyl 4-manifold, then we obtain

$$R^D(\Omega)^{(\mathrm{inv})} = \frac{1}{4}\left(s_g^D + \frac{1}{2}|DJ|_g^2\right)\Omega_g, \quad R^D(\Omega)_0^{(\mathrm{inv})} \equiv 0, \tag{3.5}$$

$$\rho^{D*(\mathrm{skew.anti})} = R^D(\Omega) - \frac{1}{4}\left(s_g^D + \frac{1}{2}|DJ|_g^2\right)\Omega_g. \tag{3.6}$$

The formulae (3.5) and (3.6) can be seen from (3.2), Proposition 2.2 and Lemma 2.5.

Suppose that (M, C, D, J) is an almost Hermitian-Einstein-Weyl 4-manifold. Taking account of (3.6), we have

$$|\rho^{D*(\mathrm{skew.anti})}|_g^2 = |R^D(\Omega)|_g^2 - \frac{1}{8}\left(s_g^D + \frac{1}{2}|DJ|_g^2\right)^2.$$

If M is compact, we can then rewrite (2.23) as follows:

$$c_1^2(M) = \frac{1}{4\pi^2}\int_M \left\{-\frac{s_g^D}{16}|DJ|_g^2 - \frac{1}{32}|DJ|_g^4 + |R^D(\Omega)|_g^2\right\}\sigma_g. \tag{3.7}$$

Comparing (3.7) with (3.1), we therefore obtain the following integral formula:

$$\begin{aligned}\int_M \left\{-\frac{s_g^D}{16}|DJ|_g^2 - \frac{1}{32}|DJ|_g^4\right\}\sigma_g \\ = \int_M \left\{2|W_+|_g^2 + \frac{1}{24}(s_g^D)^2 - |R^D(\Omega)|_g^2 + \frac{1}{4}|d\omega|_g^2\right\}\sigma_g.\end{aligned} \tag{3.8}$$

The following is sufficient to prove our main theorem:

PROPOSITION 3.4. *For any compact almost Hermitian-Einstein-Weyl 4-manifold (M, C, D, J), the following inequality holds*:

$$\int_M \left\{2|W_+|_g^2 + \frac{1}{24}(s_g^D)^2 - |R^D(\Omega)|_g^2 + \frac{1}{4}|d\omega|_g^2\right\}\sigma_g \geq 0.$$

If the conformal scalar curvature s_g^D is nonnegative, then the left hand side of (3.8) is nonpositive; however, from Proposition 3.4, the right hand side of (3.8) is nonnegative. It therefore follows that $|DJ|_g^2 \equiv 0$, i.e., J is integrable.

Before proving Proposition 3.4, we first recall that the decomposition (2.2) of R^D for an Einstein-Weyl 4-manifold (M, C, D) is given explicitly by

$$\begin{aligned}g(R^D(X,Y)V,U) = g(W(X,Y)V,U) + \frac{s_g^D}{24} g \owedge g(X,Y,V,U) \\ + \frac{1}{4} d\omega \owedge g(X,Y,V,U) - \frac{1}{2} d\omega \otimes g(X,Y,V,U),\end{aligned} \tag{3.9}$$

where $\owedge$ denotes the Kulkarni-Nomizu product:

$$
\begin{aligned}
(t \owedge g)(X, Y, V, U) := {} & t(X, U)g(Y, V) - t(Y, U)g(X, V) \\
& + t(Y, V)g(X, U) - t(X, V)g(Y, U),
\end{aligned}
$$

for any $(0,2)$-tensor field t. By (3.9), we can show the following:

LEMMA 3.5. *Let (M, C, D, J) be an almost Hermitian-Einstein-Weyl 4-manifold. Then*

$$
R^D(\Omega) = W(\Omega) + \frac{s_g^D}{12}\Omega_g - \frac{1}{2}J(d\omega)^{(\mathrm{anti})}, \tag{3.10}
$$

where $J(d\omega)^{(\mathrm{anti})}(X, Y) := (d\omega)^{(\mathrm{anti})}(X, JY)$ and $W(\Omega)$ is defined by replacing R^D of $R^D(\Omega)$ with W.

Proof of Proposition 3.4. Let $\{\Phi_1, \Phi_2, \Phi_3\}$ be a local orthonormal frame field for $\bigwedge_+$, the space of self-dual 2-forms, such that $\Phi_1 := \Omega_g/\sqrt{2}$ and that $\{\Phi_2, \Phi_3\}$ forms an orthonormal basis for $\bigwedge^{(\mathrm{anti})}$. We may express the self-dual Weyl tensor W_+ as

$$
W_+ = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{12} & w_{22} & w_{23} \\ w_{13} & w_{23} & w_{33} \end{pmatrix}.
$$

By definition, the trace of W_+ vanishes:

$$
\operatorname{tr} W_+ = w_{11} + w_{22} + w_{33} \equiv 0. \tag{3.11}
$$

The squared norm $|W_+|_g^2$ of W_+ is given by

$$
|W_+|_g^2 = w_{11}^2 + w_{22}^2 + w_{33}^2 + 2(w_{12}^2 + w_{13}^2 + w_{23}^2).
$$

Noting that $W(\Omega) = W_+(\Omega)$ and $g(J(d\omega)^{(\mathrm{anti})}, \Omega_g) \equiv 0$, we can rewrite (3.10) as

$$
\begin{aligned}
R^D(\Omega) &= W(\Omega) + \frac{s_g^D}{12}\Omega_g - \frac{1}{2}J(d\omega)^{(\mathrm{anti})} \\
&= \sqrt{2}\left(w_{11} + \frac{s_g^D}{12}\right)\Phi_1 + \left(\sqrt{2}w_{12} - \frac{1}{2}g(J(d\omega)^{(\mathrm{anti})}, \Phi_2)\right)\Phi_2 \\
&\quad + \left(\sqrt{2}w_{13} - \frac{1}{2}g(J(d\omega)^{(\mathrm{anti})}, \Phi_3)\right)\Phi_3.
\end{aligned}
$$

From (3.11), we have

$$
\begin{aligned}
2|W_+|_g^2 &+ \frac{1}{24}(s_g^D)^2 - |R^D(\Omega)|_g^2 \\
&= 2(w_{11}^2 + w_{22}^2 + w_{33}^2) + 4(w_{12}^2 + w_{13}^2 + w_{23}^2) + \frac{1}{24}(s_g^D)^2 - 2\left(w_{11} + \frac{1}{12}s_g^D\right)^2 \\
&\quad - \left(\sqrt{2}w_{12} - \frac{1}{2}g(J(d\omega)^{(\mathrm{anti})}, \Phi_2)\right)^2 - \left(\sqrt{2}w_{13} - \frac{1}{2}g(J(d\omega)^{(\mathrm{anti})}, \Phi_3)\right)^2 \\
&= 2\left\{\left(w_{22} + \frac{1}{12}s_g^D\right)^2 + \left(w_{33} + \frac{1}{12}s_g^D\right)^2\right\} + \left(\sqrt{2}w_{12} + \frac{1}{2}g(J(d\omega)^{(\mathrm{anti})}, \Phi_2)\right)^2 \\
&\quad + \left(\sqrt{2}w_{13} + \frac{1}{2}g(J(d\omega)^{(\mathrm{anti})}, \Phi_3)\right)^2 + 4w_{23}^2 - \frac{1}{2}|J(d\omega)^{(\mathrm{anti})}|_g^2 \\
&\geq -\frac{1}{2}|J(d\omega)^{(\mathrm{anti})}|_g^2 = -\frac{1}{2}|(d\omega)^{(\mathrm{anti})}|_g^2 \\
&= -\frac{1}{2}|(d\omega)_+|_g^2.
\end{aligned}
$$

Here we notice that the last equality can be seen by using (2.4): $d\omega \wedge \Omega_g \equiv 0$. Thus we obtain

$$
\begin{aligned}
\int_M &\left\{2|W_+|^2 + \frac{1}{24}(s_g^D)^2 - |R^D(\Omega)|_g^2 + \frac{1}{4}|d\omega|_g^2\right\}\sigma_g \\
&\geq \int_M \left\{-\frac{1}{2}|(d\omega)_+|_g^2 + \frac{1}{4}|d\omega|_g^2\right\}\sigma_g \\
&= -\frac{1}{4}\int_M \{|(d\omega)_+|_g^2 - |(d\omega)_-|_g^2\}\sigma_g \\
&= -\frac{1}{4}\int_M d\omega \wedge d\omega = -\frac{1}{4}\int_M d(\omega \wedge d\omega) = 0.
\end{aligned}
$$

This shows the proposition. □

4. Remarks

It is well-known that for a compact Einstein-Weyl manifold (M, C, D), there exists a metric g in C such that the dual vector field $\omega_g^{\#}$ of ω_g is a Killing vector field on (M, g). Such a metric g is unique up to homothety and hence called the *standard metric* (see Gauduchon [2], Pedersen-Swann [9]). It is also well-known that for a compact Einstein-Weyl manifold (M, C, D), the 1-form ω_g of the standard metric g must vanish if $s^D < 0$. Thus any compact almost Hermitian-Einstein-Weyl 4-manifold with negative conformal scalar curvature is determined by an almost Kähler-Einstein structure.

By virtue of Theorem 3.1, any compact almost Hermitian-Einstein-Weyl 4-manifold with nonnegative conformal scalar curvature must be Hermitian-Einstein-Weyl (i.e., the almost complex structure is integrable). Gauduchon-Ivanov [3] studied such manifolds and obtained the following:

PROPOSITION 4.1. *Let g be the standard metric for a compact Hermitian-Einstein-Weyl 4-manifold (M, C, D, J). Then the following two cases occur:*
(i) *(M, g, J) is Kähler-Einstein, or*
(ii) *(M, g) is locally isometric to $\boldsymbol{R} \times S^3$, the Lee form β_g is ∇-parallel, the Weyl structure D is flat, and (M, J) is a Hopf surface, where ∇ denotes the Levi-Civita connection of (M, g).*

From Theorem 3.1 and Proposition 4.1, we obtain

COROLLARY 4.2. *A compact almost Hermitian-Einstein-Weyl 4-manifold, which is not determined by any almost Kähler-Einstein structure, must be a Hermitian-Einstein-Weyl manifold of type* (ii) *in Proposition* 4.1.

We finally remark on higher dimensional cases. Let (M, C) be a compact conformal manifold of real dimension $2n(>4)$ with a compatible almost complex structure J, and D the canonical Weyl connection of (M, C, J). Suppose that the condition (2.3) is satisfied (i.e., $d\Omega_g = \beta_g \wedge \Omega_g$). Then the Lee form β_g is automatically closed, and hence (M, C, D, J) is determined by a locally conformal almost Kähler (l.c.a.K.) structure, and vice versa (see Vaisman [11]).

If (M, C, D) is also Einstein-Weyl, then (M, C, D, J) is determined by a locally conformal almost Kähler-Einstein structure. Let g be the standard metric for (M, C, D, J). By the closedness of the Lee form β_g, the dual vector field $\beta_g^\#$ of β_g is parallel with respect to the Levi-Civita connection ∇ of (M, g). Then the conformal scalar curvature s_g^D is constant, since s_g^D is a harmonic function on (M, g). In particular, the sign of s^D is well-defined (see Pedersen-Swann [10]). We further suppose that s^D is nonnegative. If $\beta_g \equiv 0$, then (M, g, J) is an almost Kähler-Einstein manifold with nonnegative scalar curvature. From Sekigawa's result [7], (M, g, J) is in fact Kähler-Einstein.

In the case where $s^D > 0$, the Ricci curvature of (M, g) is strictly positive. From Myers' theorem, the fundamental group $\pi_1(M)$ is finite, and hence the first Betti number $b_1(M)$ vanishes. We therefore obtain $\beta_g \equiv 0$, since β_g is a harmonic 1-form on (M, g). In the case where $s^D \equiv 0$, we may assume that $\beta_g \neq 0$. Then the standard argument tells us $b_1(M) = 1$ (see Pedersen-Swann [10]).

On the other hand, Kashiwada [5] studied the integrability problem for an *almost generalized Hopf manifold*, which means a locally conformal almost Kähler manifold (M, g, J) with parallel Lee form β_g satisfying that $J\beta_g^\#$ is a Killing vector field on (M, g). If J is also integrable, then (M, g, J) is called a *generalized Hopf manifold.* Notice that for a locally conformal Kähler manifold (M, g, J), the vector field $J\beta_g^\#$ is automatically a Killing vector field on (M, g) if

β_g is parallel. For convenience, we regard (almost) Kähler-Einstein manifolds as (almost) generalized Hopf manifolds with vanishing Lee forms.

The following is an immediate consequence from a result due to Kashiwada [5]:

PROPOSITION 4.3. *Let (M, g, J) be a compact almost generalized Hopf manifold of dimension grater than four. Suppose that its canonical Weyl structure (C, D) is Einstein-Weyl. If the conformal scalar curvature s^D is nonnegative, then J must be integrable, i.e., (M, g, J) is a generalized Hopf manifold.*

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